CHAPTER 6
QUOMORPHISMS

The notion of homomorphism, introduced in §§2E and 4C, has a number of useful refinements. Their origin lies in the circumstance that a homomorphism of *total* algebras automatically has certain pleasant properties; these do not obtain in general, but they are of some interest all the same.

Indeed, a similar, and related, phenomenon occurs with the subalgebra concept: if $\mathbf{A} \subseteq_w \mathbf{B}$, and $\mathbf{A}$ is total, then in fact $\mathbf{A} \leq \mathbf{B}$.

Suppose f: $\mathbf{A} \longrightarrow \mathbf{B}$ is a homomorphism, and again $\mathbf{A}$ is total. Then if $f(a_0) = Q^{\mathbf{B}}(f(a_1),\dots, f(a_n))$, there must be $a'_1,\dots, a'_n$ such that $f(Q^{\mathbf{A}}(a'_1,\dots, a'_n)) = f(a_0)$; in fact we might take $a'_i = a_i$. Or in a subtly different vein, if $Q^{\mathbf{B}}(f(a_1)\dots,f(a_n))$ exists, so does $Q^{\mathbf{A}}(a_1,\dots, a_n)$. For total algebras, all this is trivial; for partial algebras, the stated properties of *weak reflectivity* and *closedness* may, but need not, obtain. These properties are naturally viewed as strengthenings of the concept of homomorphism. We shall consider them in the context of a *weakening* that will turn out to be rather natural in the theory of partial algebras.

Let $\mathbf{A}$ and $\mathbf{B}$ be algebras. A *quasi-homomorphism*, for short *quomorphism*, from $\mathbf{A}$ to $\mathbf{B}$ is a homomorphism from a relative subalgebra of $\mathbf{A}$ to $\mathbf{B}$. We write

$$\phi\colon \mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$$

if ϕ is a quomorphism from $\mathbf{A}$ to $\mathbf{B}$. If $\phi\colon \mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$, then $\phi\colon (\mathrm{Dom}\,\phi)_{\mathbf{A}} \longrightarrow \mathbf{B}$.

Examples

i. Relative subalgebras may retain very little of the structure of their containing algebra, and the restrictions on quomorphism are accordingly weak. Thus, let $\mathbf{N} = \langle \mathbb{N}, S\rangle$ be the algebra of natural numbers with the successor operation. *Any* map from the set $2\mathbb{N}$ of even numbers into $\mathbb{N}$ is a quomorphism from $\mathbf{N}$ to $\mathbf{N}$. Accordingly, few of these maps can be extended to homomorphisms.

ii. Indeed, for any algebras $\mathbf{A}$ and $\mathbf{B}$, $\emptyset$ is a quomorphism from $\mathbf{A}$ to $\mathbf{B}$.

Let $\mathbf{A}$ be an algebra; we call a set $B \subseteq A$ *grounded* (relative to $\mathbf{A}$) if for any basic operation Q of $\mathbf{A}$, $Q(a_1,\dots, a_n) \in B$ implies $a_1,\dots, a_n \in B$. A fortiori a [weak, relative] subalgebra of $\mathbf{A}$ is grounded if its universe is a grounded set.

Definition. Let $\mathbf{A}$ and $\mathbf{B}$ be algebras. A quomorphism from $\mathbf{A}$ to $\mathbf{B}$ is *grounded*, or a *growmorphism*, if its domain is grounded relative to $\mathbf{A}$.

Examples

— Homomorphisms are growmorphisms.

i′. The domain of a growmorphism ϕ from $\mathbf{N}$ to $\mathbf{N}$ must be an initial segment of $\mathbf{N}$, and $\mathrm{Dom}\,\phi$ and — if $\mathrm{Dom}\,\phi \neq \emptyset$ — $\phi(0)$ completely determine ϕ.

ii′. The void set is grounded, so for any algebras $\mathbf{A}$ and $\mathbf{B}$, $\emptyset$ is a growmorphism from $\mathbf{A}$ to $\mathbf{B}$.

Here is a simple characterization of growmorphisms:

Lemma. Let **A** and **B** be algebras. A partial function ϕ: $A \circ\!\!\longrightarrow B$ is a growmorphism from **A** to **B** if and only if for each operation symbol Q,

(1) for all $a_0,\ldots, a_{n-1} \in A$, $\phi(Q^{\mathbf{A}}(a_0,\ldots, a_{n-1})) \simeq Q^{\mathbf{B}}(\phi(a_0),\ldots, \phi(a_{n-1}))$.

Proof. Let ϕ: $A \circ\!\!\longrightarrow B$ be given.
(⇒) Assume ϕ is a growmorphism from **A** to **B**. Take $a_0,\ldots, a_{n-1} \in A$, and assume $\phi(Q^{\mathbf{A}}(a_0,\ldots, a_{n-1}))$ exists. In other words, $Q^{\mathbf{A}}(a_0,\ldots, a_{n-1}) \in \mathrm{Dom}\,\phi$, so by groundedness $a_0,\ldots, a_{n-1} \in \mathrm{Dom}\,\phi$. Since ϕ is a homomorphism from $(\mathrm{Dom}\,\phi)_{\mathbf{A}}$ to **B**, we must have $\phi(Q^{\mathbf{A}}(a_0,\ldots, a_{n-1})) = Q^{\mathbf{B}}(\phi(a_0),\ldots, \phi(a_{n-1}))$.
(⇐) Assume (1), for each operation symbol Q. Then $Q^{\mathbf{A}}(a_0,\ldots, a_{n-1}) \in \mathrm{Dom}\,\phi$ implies $a_0,\ldots, a_{n-1} \in \mathrm{Dom}\,\phi$, so $\mathrm{Dom}\,\phi$ is grounded in **A**. And finally, ϕ is a homomorphism from $(\mathrm{Dom}\,\phi)_{\mathbf{A}}$ to **B**, since if $a_0,\ldots, a_{n-1}$ and $Q^{\mathbf{A}}(a_0,\ldots, a_{n-1})$ all belong to $\mathrm{Dom}\,\phi$, $\phi(Q^{\mathbf{A}}(a_0,\ldots, a_{n-1}))$ exists, and by (1) must equal $Q^{\mathbf{B}}(\phi(a_0),\ldots, \phi(a_{n-1}))$. ⊠

Proposition. Let **A**, **B** and **C** be algebras; ϕ: $\mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$ and ψ: $\mathbf{B} \circ\!\!\longrightarrow \mathbf{C}$.
(i) 1_A is a growmorphism from **A** to **A**.
(ii) The composite $\psi \circ \phi$ is a quomorphism from **A** to **C**.
(iii) If ϕ and ψ are grounded, then so is $\psi \circ \phi$.

Proof. (ii) Let $B_0 = \mathrm{Dom}\,\psi$, $\mathbf{B}_0 = (B_0)_{\mathbf{B}}$, $A_0 = \phi^{-1}[B_0]$, $f = \phi\restriction A_0$, and $\mathbf{A}_0 = (A_0)_{\mathbf{A}}$. Then f is a homomorphism from $\mathbf{A}_0$ to $\mathbf{B}_0$, so by proposition 4C2, $\psi \circ f$ is a homomorphism from $\mathbf{A}_0$ to **C**, and since $\psi \circ \phi = \psi \circ f$, $\psi \circ \phi$ is a quomorphism from **A** to **C**.
(iii) If $Q^{\mathbf{A}}(a_0,\ldots, a_{n-1}) \in \mathrm{Dom}\,\psi\phi$, then by the Lemma, since ϕ is grounded,

$$Q^{\mathbf{B}}(\phi(a_0),\ldots, \phi(a_{n-1})) = \phi(Q^{\mathbf{A}}(a_0,\ldots, a_{n-1})) \in \mathrm{Dom}\,\psi.$$

Then since ψ is grounded, $\phi(a_0),\ldots, \phi(a_{n-1}) \in \mathrm{Dom}\,\psi$, hence $a_0,\ldots, a_{n-1} \in \mathrm{Dom}\,\psi\phi$. ⊠

By the Proposition, we have concrete categories **Quom**, of quomorphisms, and **Grom**, of growmorphisms. We shall use the notation

$$\phi\colon \mathbf{A} \overset{g}{\longrightarrow} \mathbf{B}$$

for 'ϕ is a growmorphism from **A** to **B**'.

§A Reflectivity

Let **A** and **B** be algebras. A partial mapping ϕ: $A \circ\!\!\longrightarrow B$ *weakly reflects* **B** in **A** if for every operation symbol Q, whenever

$$Q^{\mathbf{B}}(\phi(a_1),\ldots, \phi(a_n)) \in \mathrm{Ran}\,\phi,$$

there are $a_1',\ldots, a_n' \in A$ such that

$$\phi(a_1') = \phi(a_1),\ldots, \phi(a_n') = \phi(a_n), \text{ and } Q^{\mathbf{A}}(a_1',\ldots, a_n') \in \mathrm{Dom}\,\phi.$$

A quomorphism ϕ: $\mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$ is *weakly reflective* if the underlying partial map ϕ: $A \circ\!\!\longrightarrow B$ weakly reflects **B** in **A**. If ϕ is a weakly reflective and surjective, **B** is a *partial image* of **A**. We say **B** is a *total image*, or simply an *image*, of **A**, if there exists a weakly reflective homomorphism f: $\mathbf{A} \twoheadrightarrow \mathbf{B}$.

Weakly reflective homomorphisms were called *full* by Grätzer [1978] and Burmeister [2002].

A **0 Lemma**. Let **A** and **B** be algebras. A quomorphism ϕ: **A** $\circ\!\!\longrightarrow$ **B** is weakly reflective if and only if for every operation symbol Q, for all $a_0, \dots, a_n$ such that $\phi(a_0) = Q^{\mathbf{B}}(\phi(a_1), \dots, \phi(a_n))$, there are $a_0', \dots, a_n' \in A$ such that

$$\phi(a_0') = \phi(a_0), \dots, \phi(a_n') = \phi(a_n), \text{ and } a_0' = Q^{\mathbf{A}}(a_1', \dots, a_n').$$

Proof. By the definition of weak reflectivity, there are $a_1', \dots, a_n' \in A$ such that $\phi(a_1') = \phi(a_1), \dots, \phi(a_n') = \phi(a_n)$, and $Q^{\mathbf{A}}(a_1', \dots, a_n') \in \mathrm{Dom}\,\phi$. Take $a_0' = Q^{\mathbf{A}}(a_1', \dots, a_n')$. Then $\phi(a_0') = \phi(a_0)$ since ϕ is a quomorphism. ☒

Examples (vind betere)
i. Let k be a positive natural number, $\mathbf{N} = \langle \mathbb{N}, +, -\rangle$, and $\mathbf{Z}_k = \langle Z_k, +', -'\rangle$, where $Z_k = \{0, \dots, k-1\}$, with $x +' y$ defined as $x + y$ if $x + y < k$, and $x + y - k$ otherwise, and $x -' y = z$ if and only if $x = y +' z$. Define f on $\mathbb{N}$ by

$$f(n) \text{ is the unique } x \in N_k \text{ for which } y \in \mathbb{N} \text{ exists with } n = yk + x.$$

(So $f(n)$ is the remainder of n in division by k. In the programming languages C and Java, this x is denoted by $n\%k$.) Then f is a weakly reflective homomorphism from $\langle \mathbb{N}, +, -\rangle$ onto $\langle Z_n, +, -\rangle$.
ii. The embedding of **N** into $\mathbf{Z} = \langle \mathbb{Z}, +, -\rangle$ is weakly reflective. In particular, if $x - y \in \mathrm{Ran}\, f$, for $x, y \in \mathbb{N}$, then $x - y \in \mathbb{N}$ as well.

a1 Proposition. Let **A**, **B** and **C** be algebras, and $A \overset{\phi}{\circ\!\!\longrightarrow} B \overset{\psi}{\circ\!\!\longrightarrow} C$ partial maps. If ϕ weakly reflects **B** in **A** and ψ weakly reflects **C** in **B**, and $\mathrm{Ran}\,\phi \supseteq \mathrm{Dom}\,\psi$, then $\psi \circ \phi$ weakly reflects **C** in **A**.

Proof. Suppose $Q^{\mathbf{C}}(\psi\phi(a_1), \dots, \psi\phi(a_n)) \in \psi\phi[A]$. Since ψ weakly reflects **C** in **B**, there are $b_1, \dots, b_n \in B$ such that $\psi\phi(a_1) = \psi(b_1), \dots, \psi\phi(a_n) = \psi(b_n)$, and $Q^{\mathbf{B}}(b_1, \dots, b_n) \in \mathrm{Dom}\,\psi$. Since $\mathrm{Ran}\,\phi \supseteq \mathrm{Dom}\,\psi$, there are $d_1, \dots, d_n \in A$ such that $\phi(d_1) = b_1, \dots, \phi(d_n) = b_n$, and $Q^{\mathbf{B}}(\phi(d_1), \dots, \phi(d_n)) \in \mathrm{Ran}\,\phi$. Since ϕ weakly reflects **B** in **A**, there are $e_1, \dots, e_n \in A$ such that $\phi(e_1) = \phi(d_1), \dots, \phi(e_n) = \phi(d_n)$, and $Q^{\mathbf{A}}(e_1, \dots, e_n) \in \mathrm{Dom}\,\phi$. Then $\psi\phi(e_i) = \psi\phi(d_i) = \psi(b_i) = \psi\phi(a_i)$ $(1 \le i \le n)$. ☒

a1.1 Definition. Let $\mathcal{T}$ be a nominator. A quomorphism ϕ: **A** $\circ\!\!\longrightarrow$ **B** is *weakly $\mathcal{T}$-reflective* if it weakly reflects $\mathbf{B}\restriction\mathcal{T}$ in **A**.

Corollary. If f: **A** $\twoheadrightarrow$ **B** and g: **B** $\longrightarrow$ **C** are weakly $\mathcal{T}$-reflective, then
$g \circ f$: **A** $\longrightarrow$ **C** is weakly $\mathcal{T}$-reflective.

Let **A** and **B** be algebras. A partial mapping ϕ: $A \circ\!\!\longrightarrow B$ *reflects* **B** in **A** if for every operation symbol Q,

$$\phi(a_0) = Q^{\mathbf{B}}(\phi(a_1), \dots, \phi(a_n)) \Rightarrow a_0 = Q^{\mathbf{A}}(a_1, \dots, a_n).$$

A quomorphism ϕ: **A** $\circ\!\!\longrightarrow$ **B** is *reflective* if the underlying partial map reflects **B** in **A**.

a2 Definition. Let $\mathcal{T}$ be a nominator. A quomorphism ϕ: **A** $\circ\!\!\longrightarrow$ **B** is *$\mathcal{T}$-reflective* if it reflects $\mathbf{B}\restriction\mathcal{T}$ in **A**.

Of course, $\mathcal{T}$-reflective quomorphisms are also weakly $\mathcal{T}$-reflective. Conversely, weakly $\mathcal{T}$-reflective *injections* are $\mathcal{T}$-reflective *tout court*.

Examples. **ii′**. The canonical embedding of $\langle \mathbb{N}, +, -\rangle$ into $\langle \mathbb{Z}, +, -\rangle$ is reflective.

iii. Isomorphisms are reflective.

a3 Proposition. If ϕ: **A** $\circ\!\!\longrightarrow$ **B** and ψ: **B** $\circ\!\!\longrightarrow$ **C** are $\mathcal{T}$-reflective, then $\psi \circ \phi$: **A** $\circ\!\!\longrightarrow$ **C** is $\mathcal{T}$-reflective.

Proof. Suppose $\psi\phi(a_0) = Q^{\mathbf{C}}(\psi\phi(a_1),\ldots, \psi\phi(a_n))$, with $Q \in \mathcal{T}$. Since g is $\mathcal{T}$-reflective, then $\phi(a_0) = Q^{\mathbf{B}}(\phi(a_1),\ldots, \phi(a_n))$. Since ϕ is $\mathcal{T}$-reflective, we get $a_0 = Q^{\mathbf{A}}(a_1,\ldots, a_n)$. ☒

§b Closed quomorphisms

Let $\mathcal{T}$ be a nominator. A quomorphism ϕ: **A** $\circ\!\!\longrightarrow$ **B** is *weakly $\mathcal{T}$-closed* if for every operation symbol $Q \in \mathcal{T}$, whenever $Q^{\mathbf{B}}(\phi(a_0),\ldots, \phi(a_{n-1}))\!\downarrow$, there are $c_0, \ldots, c_{n-1} \in A$ such that

$$\phi(a_0) = \phi(c_0),\ldots, \phi(a_{n-1}) = \phi(c_{n-1}), \text{ and } Q^{\mathbf{A}}(c_0,\ldots, c_{n-1}) \in \mathrm{Dom}\,\phi.$$

If $\mathcal{T} = \mathrm{Nom}\mathbf{A}$, we also say ϕ is *weakly* **A**-*closed*. If $\mathcal{T} = \mathrm{Nom}\mathbf{B}$, we call ϕ *weakly closed*.

Observe that every weakly $\mathcal{T}$-closed quomorphism is weakly $\mathcal{T}$-reflective.

Examples **i**. (cf. §a.i) Let k be a positive natural number, $\mathbf{N} = \langle \mathbb{N}, +, - \rangle$, and $\mathbf{Z}_k = \langle Z_k, +', -' \rangle$, where $Z_k = \{0,\ldots, k-1\}$, with $x +' y$ defined as $x + y$ if $x + y < k$, and $x + y - k$ otherwise, and $x -' y = z$ if and only if $x = y +' z$. Define f on $\mathbb{N}$ by

$$f(n) = n\%k.$$

Then f is a weakly closed, surjective homomorphism, and $\mathbf{Z}_k$ is a closed image of **N**.

ii. The embedding in §a Example ii is not weakly closed, since 0 – 1 does not belong to $\mathbb{N}$, whereas 0 and 1 do.

b1 Proposition. If ϕ: **A** $\circ\!\!\longrightarrow$ **B** and ψ: **B** $\circ\!\!\longrightarrow$ **C** are weakly $\mathcal{T}$-closed, and $\mathrm{Ran}\,\phi \supseteq \mathrm{Dom}\,\psi$, then $\psi \circ \phi$: **A** $\circ\!\!\longrightarrow$ **C** is weakly $\mathcal{T}$-closed.

Proof. Suppose $Q^{\mathbf{C}}(\psi\phi(a_0),\ldots, \psi\phi(a_{n-1}))\!\downarrow$, with $Q \in \mathcal{T}$. Since ψ is weakly $\mathcal{T}$-closed, there are $b_0, \ldots, b_{n-1} \in B$ such that

$$\psi\phi(a_0) = \psi(b_0),\ldots, \psi\phi(a_{n-1}) = \psi(b_{n-1}), \text{ and } Q^{\mathbf{B}}(b_0, \ldots, b_{n-1}) \in \mathrm{Dom}\,\psi.$$

Since $\mathrm{Ran}\,\phi \supseteq \mathrm{Dom}\,\psi$, there are $d_0,\ldots, d_{n-1} \in A$ such that

$$\phi(d_0) = b_0, \ldots, \phi(d_{n-1}) = b_{n-1}.$$

Then $Q^{\mathbf{B}}(\phi(d_0),\ldots, \phi(d_{n-1}))\!\downarrow$, and since ϕ is weakly $\mathcal{T}$-closed, there are $e_0,\ldots, e_{n-1} \in A$ such that

$$\phi(e_0) = \phi(d_0),\ldots, \phi(e_{n-1}) = \phi(d_{n-1}), \text{ and } Q^{\mathbf{A}}(e_1,\ldots, e_{n-1}) \in \mathrm{Dom}\,\phi.$$

But $\psi\phi(e_i) = \psi\phi(d_i) = \psi(b_i) = \psi\phi(a_i)$, for all $i < n$. ☒

Corollary. If f: **A** $\twoheadrightarrow$ **B** and g: **B** $\longrightarrow$ **C** are weakly $\mathcal{T}$-closed, with f surjective, then $g \circ f$: **A** $\longrightarrow$ **C** is weakly $\mathcal{T}$-closed.

b2 Definition. Let $\mathcal{T}$ be a nominator. A quomorphism ϕ: **A** $\circ\!\!\longrightarrow$ **B** is *$\mathcal{T}$-closed* if for every sequence $a_0,\ldots, a_{n-1}$ of elements of A and every operation symbol $Q \in \mathcal{T}$,

$$\langle \phi(a_0),\ldots, \phi(a_{n-1})\rangle \in \mathrm{Dom}(Q^{\mathbf{B}}) \text{ implies } Q^{\mathbf{A}}(a_0,\ldots, a_{n-1}) \in \mathrm{Dom}\,\phi.$$

If $\mathcal{T}$ = Nom**A**, ϕ is **A**-*closed*. If $\mathcal{T}$ = Nom**B**, ϕ is *closed*.

Of course, $\mathcal{T}$-closed quomorphisms are also weakly $\mathcal{T}$-closed. Closed homomorphisms were called *strong* by Grätzer [1978].

Examples. **iii**. Let f: $\mathbf{N} \longrightarrow \mathbf{Z}_k$ be as in Example i. This f is not closed:
$n - (n + 1)$ does not exist in **N**.
iv. The mapping $\langle Y, f\rangle \longmapsto \langle Y, f, \mathrm{Dom}(f)\rangle$ is a closed embedding of the category **Set** into the category **Rel**.
v. If **A** is a total algebra, then any homomorphism of **A** is **A**-closed.
vi. Homomorphisms between total algebras of the same type are always closed.

b3 Proposition. Let $\mathcal{T}$ be a nominator. Composites of $\mathcal{T}$-closed quomorphisms are $\mathcal{T}$-closed.

Proof. Let **A**, **B** and **C** be algebras, and ϕ: $\mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$ and ψ: $\mathbf{B} \circ\!\!\longrightarrow \mathbf{C}$ $\mathcal{T}$-closed.
Suppose $Q \in \mathcal{T}$, and

$$\langle \psi\phi(a_0),\dots, \psi\phi(a_{n-1})\rangle \in \mathrm{Dom}(Q^{\mathbf{C}}).$$

Then since g is $\mathcal{T}$-closed, $\langle \phi(a_0),\dots, \phi(a_{n-1})\rangle \in \mathrm{Dom}(Q^{\mathbf{B}})$. Again since ϕ is $\mathcal{T}$-closed, $Q^{\mathbf{A}}(a_0,\dots, a_{n-1}) \in \mathrm{Dom}\,\phi$. ⊠

If we suppress all mention of $\mathcal{T}$, the proof shows:

Corollary. If ϕ: $\mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$ and ψ: $\mathbf{B} \circ\!\!\longrightarrow \mathbf{C}$ are closed, then $\psi \circ \phi$: $\mathbf{A} \circ\!\!\longrightarrow \mathbf{C}$ is closed.

Example **vii**. If f: $\mathbf{A} \longrightarrow \mathbf{B}$ is **A**-closed and g: $\mathbf{B} \longrightarrow \mathbf{C}$ is **B**-closed, then $g \circ f$ need not be **A**-closed. Let c be a nullary operation symbol, and consider **A**, **B** and **C** with the same singleton universe $\{*\}$, where $I_{\mathbf{A}} = \{\emptyset \longleftarrow\!\!\dashv c\}$, $I_{\mathbf{B}} = \emptyset$, and $I_{\mathbf{C}} = \{\{* \longleftarrow\!\!\dashv \emptyset\} \longleftarrow\!\!\dashv c\}$. Then $\{* \longleftarrow\!\!\dashv *\}$ is a homomorphism, **A**-closed from **A** to **B** and **B**-closed from **B** to **C**, but not **A**-closed from **A** to **C**.

b4 Proposition. (a) Let $\mathcal{T}$ be a nominator, and ϕ: $\mathbf{A} \circ\!\!\longrightarrow\!\!\!\!\rightarrow \mathbf{B}$. 1° If ϕ is weakly $\mathcal{T}$-reflective, then ϕ is weakly $\mathcal{T}$-closed; 2° if ϕ is $\mathcal{T}$-reflective, then ϕ is $\mathcal{T}$-closed.
(b) Let $\mathcal{T}$ be a nominator, and ϕ: $\mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$ injective. 1° If ϕ is weakly $\mathcal{T}$-closed, then ϕ is $\mathcal{T}$-closed; 2° if ϕ is weakly $\mathcal{T}$-reflective, then ϕ is $\mathcal{T}$-reflective; 3° if ϕ is $\mathcal{T}$-closed, then ϕ is $\mathcal{T}$-reflective.

Proof. Assume $Q \in \mathcal{T}$.
(a) Suppose $\langle \phi(a_1),\dots, \phi(a_n)\rangle \in \mathrm{Dom}(Q^{\mathbf{B}})$. Since ϕ is surjective,

$$Q^{\mathbf{B}}(\phi(a_1),\dots, \phi(a_n)) \in \mathrm{Ran}\,\phi.$$

Then there are $c_1,\dots, c_n \in A$ such that $\phi(c_i) = \phi(a_i)$ for $1 \le i \le n$ and

$$Q^{\mathbf{A}}(c_1,\dots, c_n) \in \mathrm{Dom}\,\phi,$$

since ϕ is weakly $\mathcal{T}$-reflective. If ϕ is $\mathcal{T}$-reflective, we may take $c_i = a_i$.
(b) The first two statements are trivial. 3° Suppose

$$\phi(a_0) = Q^{\mathbf{B}}(\phi(a_1),\dots, \phi(a_n)).$$

Since ϕ is $\mathcal{T}$-closed, we must have $Q^{\mathbf{A}}(a_1,\dots, a_n) \in \mathrm{Dom}\,\phi$. Then

$$\phi(Q^{\mathbf{A}}(a_1,\dots, a_n)) = Q^{\mathbf{B}}(\phi(a_1),\dots, \phi(a_n)) = \phi(a_0).$$

Since ϕ is injective, this implies $a_0 = Q^{\mathbf{A}}(a_1,\dots, a_n)$. ⊠

§c Quomorphisms and subalgebras

Let $\mathbf{A} = \langle A, I\rangle$ and $\mathbf{B} = \langle B, J\rangle$ be algebras, and ϕ: $\mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$ a quomorphism. Then we define $\phi[\mathbf{A}]$, the *partial image of* **A** *through* ϕ, to be $\langle \phi[A], K\rangle$, where $\mathrm{Dom}(K) = \mathrm{Dom}(J)$, and for $Q \in \mathrm{Dom}(J)$,

$$K(Q) = \{\langle \phi(a_0), \langle \phi(a_1),\dots, \phi(a_{n_Q})\rangle\rangle \mid \langle a_0, \langle a_1,\dots, a_{n_Q}\rangle\rangle \in I(Q)\}.$$

Example **i**. Take $\mathbf{A} = \mathbb{N}$, with void interpretation, and $\mathbf{B} = \langle \mathbb{N}, +\rangle$. Then $1_{\mathbb{N}}[\mathbf{A}] = \langle \mathbb{N}, \emptyset\rangle$, with + interpreted as the void binary operation.

The definition of $\phi[\mathbf{A}]$ implies

(1) $$\phi(Q^{\mathbf{A}}(a_1,\dots, a_n)) \simeq Q^{\phi[\mathbf{A}]}(\phi(a_1),\dots, \phi(a_n)).$$

c1 Theorem. Let **A** and **B** be algebras, and ϕ: $\mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$ a quomorphism. Then
(a) $\phi[\mathbf{A}]$ is a weak subalgebra of **B**; and if $\mathrm{Dom}\,\phi = A$, $\phi[\mathbf{A}]$ is an image of **A**;
(b) ϕ is weakly reflective if and only if $\phi[\mathbf{A}] \subseteq \mathbf{B}$;
(c) ϕ is weakly closed if and only if $\phi[\mathbf{A}] \leq \mathbf{B}$.

Proof. Let $\mathbf{A} = \langle A, I\rangle$, $\mathbf{B} = \langle B, J\rangle$, and $\phi[\mathbf{A}] = \langle \phi[A], K\rangle$.
(a) By definition, $\mathrm{Dom}(K) = \mathrm{Dom}(J)$. Suppose $Q \in \mathrm{Dom}(K)$. If

$$\langle b_0, \langle b_1,\dots, b_n\rangle\rangle \in K(Q),$$

then by definition there must be $a_i \in A$, $0 \leq i \leq n$, such that $b_i = \phi(a_i)$, and

$$a_0 = Q^{\mathbf{A}}(a_1,\dots, a_n).$$

Since ϕ is a quomorphism, this implies $b_0 = Q^{\mathbf{B}}(b_1,\dots, b_n)$, i.e.

$$\langle b_0, \langle b_1,\dots, b_n\rangle\rangle \in J(Q).$$

So $K(Q) \subseteq J(Q)$. So $\phi[\mathbf{A}]$ is a weak subalgebra of **B**.

Now assume $\mathrm{Dom}\,\phi = A$.

If $a_0 = Q^{\mathbf{A}}(a_1,\dots, a_n)$, then $\phi(a_0) = Q^{\mathbf{B}}(\phi(a_1),\dots, \phi(a_n))$, since ϕ: $\mathbf{A} \longrightarrow \mathbf{B}$ is a homomorphism. Then in particular $Q \in \mathrm{Dom}(J)$. So by definition

$$\langle \phi(a_0), \langle \phi(a_1),\dots, \phi(a_n)\rangle\rangle \in K(Q),$$

that is, $\phi(a_0) = Q^{\phi[\mathbf{A}]}(\phi(a_1),\dots, \phi(a_n))$. So ϕ: $\mathbf{A} \longrightarrow \phi[\mathbf{A}]$.

If $\phi(a_0) = Q^{\phi[\mathbf{A}]}(\phi(a_1),\dots, \phi(a_n))$, then by the definition of $K(Q)$ there must be $c_0,\dots, c_n \in A$ such that $\phi(a_0) = \phi(c_0),\dots, \phi(a_n) = \phi(c_n)$, and $c_0 = Q^{\mathbf{A}}(c_1,\dots, c_n)$. So ϕ: $\mathbf{A} \longrightarrow \phi[\mathbf{A}]$ is weakly reflective. Since it also is surjective, $\phi[\mathbf{A}]$ is an image of **A**.
(b) ($\Rightarrow$) Suppose ϕ: $\mathbf{A} \circ\!\!\longrightarrow \mathbf{B}$ is weakly reflective. To prove that $\phi[\mathbf{A}]$ is a relative subalgebra of **B**, we must prove, for every operation symbol Q, that

$$\phi(a_0) = Q^{\mathbf{B}}(\phi(a_1),\dots, \phi(a_{n_Q})) \text{ implies } \phi(a_0) = Q^{\phi[\mathbf{A}]}(\phi(a_1),\dots, \phi(a_{n_Q})).$$

Let $n = n_Q$. Now if $\phi(a_0) = Q^{\mathbf{B}}(\phi(a_1),\dots, \phi(a_n))$, since ϕ is weakly reflective, there are $a'_1,\dots, a'_n \in A$ such that $\phi(a_1) = \phi(a'_1),\dots, \phi(a_n) = \phi(a'_n)$, and $Q^{\mathbf{A}}(a'_1,\dots, a'_n) \in \mathrm{Dom}\,\phi$. Then $\phi(a_0) = Q^{\mathbf{B}}(\phi(a'_1),\dots, \phi(a'_n))$, hence by (1):

$$\phi(a_0) = \phi(Q^{\mathbf{A}}(a'_1,\dots, a'_n)) = Q^{\phi[\mathbf{A}]}(\phi(a'_1),\dots, \phi(a'_n)) = Q^{\phi[\mathbf{A}]}(\phi(a_1),\dots, \phi(a_n)).$$

($\Leftarrow$) Suppose $\phi[\mathbf{A}]$ is a relative subalgebra of **B**. Then

$$\phi(a_0) = Q^{\mathbf{B}}(\phi(a_1),\ldots, \phi(a_n))$$

implies

$$\phi(a_0) = Q^{\phi[\mathbf{A}]}(\phi(a_1),\ldots, \phi(a_n)),$$

which by definition implies there are $c_0,\ldots, c_n \in A$ such that $\phi(a_0) = \phi(c_0)$, …, $\phi(a_n) = \phi(c_n)$, and $c_0 = Q^{\mathbf{A}}(c_1,\ldots, c_n)$.
(c) (⇒) Suppose ϕ is weakly closed. Then by (b) $\phi[\mathbf{A}]$ is a relative subalgebra of **B**, so to prove that $\phi[\mathbf{A}]$ is a subalgebra of **B**, by 5a2(ii) it will suffice to show that $\phi[A]$ is closed under the basic operations of **B**. So suppose

$$\langle\phi(a_1),\ldots, \phi(a_n)\rangle \in \mathrm{Dom}(Q^{\mathbf{B}}).$$

Since ϕ is weakly closed, this implies there are $a'_1,\ldots, a'_n \in A$ such that $\phi(a'_1) = \phi(a_1),\ldots, \phi(a'_n) = \phi(a_n)$, and $Q^{\mathbf{A}}(a'_1,\ldots, a'_n) \in \mathrm{Dom}\,\phi$. Hence

$$\phi(Q^{\mathbf{A}}(a'_1,\ldots, a'_n)) = Q^{\mathbf{B}}(\phi(a'_1),\ldots, \phi(a'_n)) = Q^{\mathbf{B}}(\phi(a_1),\ldots, \phi(a_n)),$$

which shows $Q^{\mathbf{B}}(\phi(a_1),\ldots, \phi(a_n)) \in \phi[A]$.
(⇐) Suppose $\phi[\mathbf{A}] \leq \mathbf{B}$. Then $Q^{\mathbf{B}}(\phi(a_1),\ldots, \phi(a_n)) \simeq Q^{\phi[\mathbf{A}]}(\phi(a_1),\ldots, \phi(a_n))$; so if $Q^{\mathbf{B}}(\phi(a_1),\ldots, \phi(a_n))\downarrow$, by the definition of K there are $a'_1,\ldots, a'_n \in A$ such that $\phi(a'_1) = \phi(a_1),\ldots, \phi(a'_n) = \phi(a_n)$, and $Q^{\mathbf{A}}(a'_1,\ldots, a'_n) \in \mathrm{Dom}\,\phi$. ☒

Corollary 1. Let **A** and **B** be algebras, and ϕ: **A** ⟜⟶ **B** injective. Then
(a) if $\mathrm{Dom}\,\phi = A$, ϕ is an isomorphism of **A** onto $\phi[\mathbf{A}]$;
(b) ϕ: **A** ⟜⟶ **B** is reflective if and only if $\phi[\mathbf{A}] \subseteq \mathbf{B}$;
(c) ϕ: **A** ⟜⟶ **B** is closed if and only if $\phi[\mathbf{A}] \leq \mathbf{B}$.

Proof. For (b) and (c), apply Proposition b4. ☒

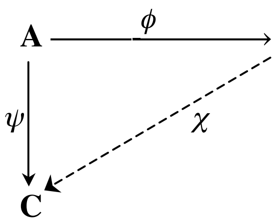
Corollary 2. Every quomorphism is the composite of an embedding and a weakly reflective and surjective quomorphism, and every homomorphism is the composite of an embedding and a weakly reflective and surjective homomorphism.

Proof. Roughly speaking, ϕ: **A** ⟜⟶ **B** factorizes as $\mathbf{A} \overset{\phi}{\circ\!\!\twoheadrightarrow} f[\mathbf{A}] \subseteq_{\mathrm{w}} \mathbf{B}$. ☒

We finish with two triangle completion lemmas.

c2 Lemma (*triangle completion lemma for quomorphisms*). Let **A**, **B** and **C** be algebras, ψ: **A** ⟜⟶ **C** a quomorphism, and ϕ: **A** ⟜⟶ **B** a weakly reflective quomorphism.
(i) There exists a quomorphism χ: **B** ⟜⟶ **C** such that $\chi \circ \phi = \psi$ if and only if $\Delta_{\mathrm{Dom}\,\psi} \subseteq \ker\phi \subseteq \ker\psi \cup \nabla_{A-\mathrm{Dom}\,\psi}$.
(ii) If such quomorphisms exist, there is a least one.
(iii) This least quomorphism is total if and only if $\phi[\mathrm{Dom}\,\psi] = B$,
injective if and only if $\ker\phi = \ker\psi$, and
surjective if and only if ψ is surjective.

The situation is sketched in the diagram below.



Proof.
(i) ($\Rightarrow$) If $\chi \circ \phi = \psi$, then $\mathrm{Dom}\,\psi \subseteq \mathrm{Dom}\,\phi$, hence $\Delta_{\mathrm{Dom}\,\psi} \subseteq \ker\phi$; and if $a \in \mathrm{Dom}\,\psi$, and $\phi(a) = \phi(x)$, then $\psi(a) = \chi(\phi(a)) = \chi(\phi(x)) = \psi(x)$.
($\Leftarrow$) Assume

$$\Delta_{\mathrm{Dom}\,\psi} \subseteq \ker\phi \subseteq \ker\psi \cup \nabla_{A-\mathrm{Dom}\,\psi}.$$

Then by the triangle completion lemma of §1h1, there is a unique mapping χ: $\phi[\mathrm{Dom}\,\psi] \longrightarrow C$ satisfying $\chi \circ \phi = \psi$. We shall prove that this χ is a homomorphism from $(\phi[\mathrm{Dom}\,\psi])_{\mathbf{B}}$ into $\mathbf{C}$. Assume $a_0,\dots, a_n \in \mathrm{Dom}\,\psi$ and $\phi(a_0) = Q^{\mathbf{B}}(\phi(a_1),\dots, \phi(a_n))$. Then since ϕ is weakly reflective, there are $a_0',\dots, a_n' \in A$ such that $a_0' = Q^{\mathbf{A}}(a_1',\dots, a_n')$ and $\phi(a_i') = \phi(a_i)$ for all $i \leq n$. Then since $\chi \circ \phi = \psi$, and ψ is a quomorphism,

$$\chi(\phi(a_0)) = \psi(a_0') = Q^{\mathbf{C}}(\psi(a_1'),\dots, \psi(a_n')) = Q^{\mathbf{C}}(\chi(\phi(a_1)),\dots, \chi(\phi(a_n))).$$

(ii) Any quomorphism χ': $\mathbf{B} \circ\!\!\longrightarrow \mathbf{C}$ such that $\chi' \circ \phi = \psi$ must contain the χ defined in (i).
(iii) Observe that $\chi \circ \phi = (\chi\restriction \phi[\mathrm{Dom}\,\psi]) \circ \phi$. ☒

c3 Corollary (*triangle completion lemma for homomorphisms*). Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be algebras, f: $\mathbf{A} \twoheadrightarrow \mathbf{B}$ weakly reflective, and g: $\mathbf{A} \longrightarrow \mathbf{C}$.
(i) There exists a homomorphism h: $\mathbf{B} \longrightarrow \mathbf{C}$ such that $h \circ f = g$ if and only if $\ker(f) \subseteq \ker(g)$.
(ii) If it exists, this homomorphism h is uniquely determined by the condition $h \circ f = g$.
(iii) Moreover, h is injective if and only if $\ker(f) = \ker(g)$, and
(iv) surjective if and only if g is surjective.

Proof. This is just the triangle completion lemma for mappings (§1h1), except for the claim that h is a homomorphism. ☒

§d Some categories of algebras

Let us take stock of the observations on composition in the preceding sections.

d1 Theorem. (a) The triples $\langle \mathbf{B}, \phi, \mathbf{A}\rangle$, where $\mathbf{A}$ and $\mathbf{B}$ are algebras and ϕ is a quomorphism from $\mathbf{A}$ to $\mathbf{B}$, form a large category, with composition defined by

$$\langle \mathbf{D}, \psi, \mathbf{C}\rangle \circ \langle \mathbf{B}, \phi, \mathbf{A}\rangle \text{ exists if and only if } \mathbf{C} = \mathbf{B}\text{, and then it is } \langle \mathbf{D}, \psi \circ \phi, \mathbf{A}\rangle,$$

and identities all triples of the form $\langle \mathbf{X}, 1_X, \mathbf{X}\rangle$.
(b) Statement (a) continues to hold if we replace the word ‘quomorphism’ by ‘closed quomorphism’, or ‘reflective quomorphism’.

(c) Statement (a) continues to hold if we replace the word 'quomorphism' by 'g-morphism', or 'homomorphism'; and for these modifications of (a), the analogs of (b) hold.

Proof. (a) By the proposition in the introduction.
(b) See propositions a3 and b3.
(c) Cf. the introduction to this chapter and §4c. ☒

We agreed to refer to the category described in (a) as **Quom**. The categories in (b) will be called **Clq** and **Reflq**, respectively. These are subcategories of **Quom**. The corresponding categories of g-morphisms are **Gmor** (< **Quom**), **Clg** and **Reflg**; of homomorphisms, **Alg** (< **Quom**), **Clh** and **Reflh**. The category $\mathbf{TAlg}_{\mathcal{T}}$ of homomorphisms of total algebras of type $\mathcal{T}$ is a full subcategory of the categories $\mathbf{Clh}_{\mathcal{T}}$ and $\mathbf{Reflh}_{\mathcal{T}}$ of, respectively, closed homomorphisms and reflective homomorphisms between algebras of type $\mathcal{T}$.

d2 Theorem. A quomorphism is an isomorphism (in **Quom**) if and only if it is total, bijective and closed.

Proof. Let ϕ: $\mathbf{A} \multimap \mathbf{B}$ be a quomorphism.
(⇒) If ϕ is an isomorphism, then there must be ψ: $\mathbf{B} \multimap \mathbf{A}$ such that $\psi\phi = 1$ and $\phi\psi = 1$. So ϕ must be a bijective mapping from A to B (cf. §2d, Example v). Suppose $\langle\phi(a_0),\ldots, \phi(a_{n-1})\rangle \in \mathrm{Dom}(Q^{\mathbf{B}})$. Since ψ is a quomorphism, this implies

$$\langle a_0,\ldots, a_{n-1}\rangle = \langle\psi\phi(a_0),\ldots, \psi\phi(a_{n-1})\rangle \in \mathrm{Dom}(Q^{\mathbf{A}}).$$

(⇐) Assume ϕ is total, bijective and closed. We show that ϕ^{-1} is a homomorphism, by calculating

$$\begin{aligned}\phi^{-1}(Q^{\mathbf{B}}(b_0,\ldots, b_{n-1})) &= \phi^{-1}(Q^{\mathbf{B}}(\phi\phi^{-1}(b_0),\ldots, \phi\phi^{-1}(b_{n-1}))) \\ &= \phi^{-1}\phi(Q^{\mathbf{A}}(\phi^{-1}(b_0),\ldots, \phi^{-1}(b_{n-1}))) \\ &= Q^{\mathbf{A}}(\phi^{-1}(b_0),\ldots, \phi^{-1}(b_{n-1})).\end{aligned}$$ ☒

d3 Corollary. A homomorphism is an isomorphism (in **Alg**) if and only if it is bijective and closed.

d4 Corollary. A closed or reflective homomorphism is an isomorphism if and only if it is bijective.

Proof. In the reflective case, use Proposition b4(a). ☒

The category of small categories. The class of small categories determines a full subcategory of **Alg**; we denote it by **Cat**.

Exercises

1. Let $\mathbf{N} = \langle\mathbb{N}, S\rangle$ be the algebra of natural numbers with the successor operation. Show that a quomorphism ϕ: $\mathbf{N} \multimap \mathbf{N}$ can be extended to an endomorphism of **N** if and only if (i) if n_0 is the least element of $\mathrm{Dom}\,\phi$, then $n_0 \leq \phi(n_0)$;
(ii) if $n, n + k \in \mathrm{Dom}\,\phi$, then $\phi(n + k) = \phi(n) + k$.

Conclude that even quasi-endomorphisms of $\langle\mathbb{N}, \mathrm{Clo}\,\mathbf{N}\rangle$ may not be contained in an endomorphism.

§A

1.

2. Are there algebras **A** such that **Sub**(**A**) is not distributive?

3. Verify that a set lattice is indeed a lattice, and that a field of sets is a Boolean algebra.

4. Show that a subalgebra of a category $\mathbf{C} = \langle C, \circ, \mathsf{d}, \mathsf{b} \rangle$ is a category.

5. Prove Proposition 2. Prove that the relative subalgebra relation and the weak subalgebra relation are orderings of the class of all algebras as well.

6. Prove (i) and (ii) of Proposition 4.

7. Let **N** be the algebra with universe $\mathbb{N}$ and for each $m \in \mathbb{N}$ a single m-ary basic operation Q_m, defined by: $Q_m(n_0,\dots, n_{m-1}) = m$ if $n_0,\dots, n_{m-1}$ are all distinct,
0 otherwise.

Prove that **N** is a minimal algebra.

§B

1. Show by example that closed homomorphisms need not be reflective, and that reflective homomorphisms need not be closed. Conclude that weakly reflective homomorphisms need neither be closed nor reflective.

2.

3.

4.

5M.

6.

7. Show by example that a composite of weakly reflective/closed homomorphisms need not be weakly reflective/closed.

9. Let **A** and **B** be algebras; suppose f: A --› B is injective. Prove that f is weakly reflective if and only if f^{-1} is a weakly reflective homomorphism from **B** to **A**.

10. Let f: $\mathbf{A} \twoheadrightarrow \mathbf{B}$ be weakly reflective. Is there necessarily a homomorphism $g : \mathbf{B} \to \mathbf{A}$ such that $f \circ g = 1_B$?

11. Let $\mathbf{A} \xrightarrow{f} \mathbf{B} \xrightarrow{g} \mathbf{C}$ be homomorphisms, and $\mathcal{T}$ a nominator. Prove:

(a) If $g \circ f$ is $\mathcal{T}$- closed, then f is $\mathcal{T}$- closed, and if $g \circ f$ is $\mathcal{T}$- reflective, then f is $\mathcal{T}$-reflective.

(b) If f is surjective, then if $g \circ f$ is $\mathcal{T}$- closed, g is $\mathcal{T}$- closed, and if $g \circ f$ is $\mathcal{T}$-reflective, g is $\mathcal{T}$- reflective.

(c) If $g \circ f$ is weakly $\mathcal{T}$-reflective and injective, then f is weakly $\mathcal{T}$-reflective and injective.

(d) If $g \circ f$ is weakly $\mathcal{T}$-reflective and injective, and f is surjective, then g is weakly $\mathcal{T}$-reflective and injective.

§25

1. Prove the Theorem.

2. Prove the Proposition.

3. Construct an example to show that forgetful functors are not necessarily full.

§

1. Let **A** and **B** be algebras. Prove:

(a) $\mathbf{Alg}(\mathbf{A} \leftarrow \mathbf{B}) \subseteq \mathrm{Sub}(\mathbf{A} \times \mathbf{B}) \cap A^B$;

(b) $\mathbf{Alg}(\mathbf{A} \leftarrow \mathbf{B}) \supseteq \mathrm{Sub}(\mathbf{A} \times \mathbf{B}) \cap A^B$ if for every operation symbol Q, $Q^{\mathbf{B}} \neq \emptyset$ implies $Q^{\mathbf{A}}$ is total.

Afval

Only in this case, the strongest notion of subalgebra sets the standard; whereas

(§D, kategorieën) More in general, let $\mathcal{T}$ be a nominator and **K** a subcategory of **Alg**. We denote the full subcategory of **K** determined by the algebras with nominator included in $\mathcal{T}$ by $\mathcal{T}$❑ **K**. In this notation, **Set** = ∅ ❑ **Alg**.
*Dit is helemaal fout. Voor een klasse **K** van algebra's is*

$$\mathcal{T} ❑ \boldsymbol{K} = \{\mathbf{A}\lceil\mathcal{T} \mid \mathbf{A} \in \boldsymbol{K}\}.$$

Daar kun je een kategorie van maken, maar het ligt niet per se voor de hand dat dat een volle subkategorie is van een gegeven kategorie **K** *waarvan **K** de objectklasse is.*

Varianten