Let (X, Y) be bivariate normal with parameters $\mu_X, \mu_Y, \sigma_X, \sigma, Y$ and assume that X and Y are independent. Then (X, Y) has density

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left(-\frac{1}{2}(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2})\right).$$

Consider now the transformation $U = X + Y$, $V = X - Y$. So we can represent U as $g_1(X, Y)$ with $g_1(x, y) = x+y$ and V as $g_2(X, Y)$ with $g_2(x, y)$. Conversely, given (U, V) we have $(X, Y) = h(U, V)$, with $h_1(u, v) = \frac{1}{2}(u+v)$ and $h_2(u, v) = \frac{1}{2}(u - v)$.

We compute the joint density $f_{U,V}$ of (U, V) by using the transformating rule. To that end we need the determinant that appears in that rule, which has in our case the value -2 (you check). With the proper substitutions we then get that

$$f_{U,V}(u,v) = \frac{1}{4\pi\sigma_X\sigma_Y} \exp\left(-\frac{1}{2}(\frac{(\frac{1}{2}(u+v)-\mu_X)^2}{\sigma_X^2} + \frac{(\frac{1}{2}(u-v)-\mu_Y)^2}{\sigma_Y^2})\right). \quad (1)$$

This expression looks quite horrible, so let's see if we can rewrite this such that it gets a better looking appearence. Introduce the following new parameters. $\mu_U = \mu_X + \mu_Y$, $\mu_V = \mu_X - \mu_Y$, $\sigma_U^2 = \sigma_V^2 = \sigma_X^2 + \sigma_Y^2$ and $\rho = \frac{\sigma_X^2 - \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$. Then we can rewrite (if you want, you may check this, but see also below!) the expression in the exponential as

$$-\frac{1}{2(1-\rho^2)}(\frac{(u-\mu_U)^2}{\sigma_U^2} + \frac{(v-\mu_V)^2}{\sigma_V^2} - 2\rho\frac{(u-\mu_U)(v-\mu_V)}{\sigma_U\sigma_V}).$$

The term in front of the exponential can be rewritten as

$$\frac{1}{2\pi\sigma_U\sigma_V\sqrt{1-\rho^2}}.$$

From these considerations we conclude that (U, V) is bivariate normal with the parameters as we just defined them.

The computations simplify considerably if we take $\sigma_X^2 = \sigma_Y^2 = \sigma^2$. In that case we have $\sigma_U^2 = \sigma_V^2 = 2\sigma^2$ and $\rho = 0$ and equation (1) reduces to

$$f_{U,V}(u,v) = \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{2}(\frac{(\frac{1}{2}(u+v)-\mu_X)^2}{\sigma^2} + \frac{(\frac{1}{2}(u-v)-\mu_Y)^2}{\sigma^2}).\right).$$

Now it is a good exercise for you (do it!) to verify that this expression can be rewritten as

$$f_{U,V}(u,v) = \frac{1}{4\pi\sigma^2} \exp\left(-\frac{1}{2}(\frac{(u-\mu_U)^2}{2\sigma^2} + \frac{(v-\mu_V)^2}{2\sigma^2})\right).$$

Now you compute $f_U(u)$ and $f_V(v)$ (the marginal densities of U and V) and check that $f_{U,V}(u, v)$ can be written as the product of $f_U(u)$ and $f_V(v)$, which implies that U and V are independent.