Complements to Øksendal, section 12.3

1 Theorem 12.3.3

Let Y be given under a probability measure P by the SDE

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t), Y(0).$$

Here Y is more dimensional process, B is a more dimensional P-Brownian motion, $b(\cdot)$ vector valued and $\sigma(\cdot)$ matrix valued, all with the appropriate dimensions. Suppose that there exists a probability meaasure Q, equivalent to P on $\mathcal{F}_T$ (T some fixed terminal time), such that we can also write with a Q-Brownian motion $\tilde{B}$

$$dY(t) = \sigma(Y(t))d\tilde{B}(t), Y(0).$$

We know from Girsanov's theorem when we can make this assumption.

Let h be a given real valued function (assume sufficient regularity) and define $g(t,y) = E^y_Q h(Y(T-t))$. Denote by $g'(t,y) = \frac{\partial}{\partial y} g(t,y)$ the row vector of partial derivatives of g with respect to y. Then we have (similar to (12.3.24) but in a slightly different notation)

$$h(Y(T)) = E^y_Q h(Y(T)) + \int_0^T \phi(t) d\tilde{B}(t),$$

with $\phi(t) = g'(t, Y(t))\sigma(Y(t))$.

2 Example 12.3.5

The correct answer to $\phi(t)$ seems to be

$$\phi(t) = \sigma \exp(Y(t) - \rho(T-t) + \frac{1}{2}\sigma^2 T).$$

Exercise Verify this.

3 Theorem 12.3.6 b

This part of the book is rather messy. A lot of confusion arises because Øksendal messes up arguments that pertain to discounted price processes with representations and equations for the price processes themselves. As a result his formula (12.3.39) should be replaced with

$$\begin{aligned}\theta_1(t) &= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} f'(X_1(t) \exp(\rho(T-t) + \beta x - \frac{1}{2}\beta^2(T-t))) \times \\ &\quad \exp(\beta x - \frac{1}{2}\beta^2(T-t)) \exp(-\frac{x^2}{2(T-t)}) dx.\end{aligned}$$

We follow an approach similar to the proof in Øksendal, but we use our version of the results needed (but be careful, when reading this, because I found plenty of places to make mistakes and I have been very succesful in making a lot of them, when I prepared these notes).
Under the conditions of the theorem there exists a probability measure Q (the *equivalent martingale measure*), which is equivalent to P on $\mathcal{F}_T$ and a Q-Brownian motion $\tilde{B}$ such that we have for the discounted price process $\bar{X}_1(t) = X_1(t)/X_0(t)$

$$d\bar{X}_1(t) = \beta \bar{X}_1(t) d\tilde{B}(t), \bar{X}_1(0) = x_1.$$

Solving this equation yields

$$\bar{X}_1(t) = x_1 \exp(\beta \tilde{B}(t) - \frac{1}{2}\beta^2 t).$$

Define $h(y) = e^{-\rho T} f(y e^{\rho T})$. Then the discounted claim $\bar{F}$ can be written as $\bar{F} = h(\bar{X}_1(T))$. Then we can compute $g(t,y) = E_Q^y h(\bar{X}_1(T-t))$, which becomes

$$\begin{aligned} g(t,y) &= e^{-\rho T} E_Q f(y \exp(\rho T + \beta \tilde{B}(T-t) - \frac{1}{2}\beta^2 (T-t))) \\ &= \frac{e^{-\rho T}}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} f(y \exp(\rho T + \beta x - \frac{1}{2}\beta^2(T-t))) \exp(-\frac{x^2}{2(T-t)}) dx, \end{aligned}$$

since $\tilde{B}(T-t)$ has a $N(0, T-t)$ distribution.
We can now also compute

$$\begin{aligned} g'(t,y) &= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} f'(y \exp(\rho T + \beta x - \frac{1}{2}\beta^2(T-t))) \times \\ &\qquad \exp(\beta x - \frac{1}{2}\beta^2(T-t)) \exp(-\frac{x^2}{2(T-t)}) dx. \end{aligned}$$

But then the hedge strategy is determined by

$$\theta_1(t) = g'(t, \bar{X}_1(t)),$$

which yields the announced formula.

4 A trivial example

We take the function f to be the identity, so we look at the claim $F = X_1(T)$. Obviously, the hedging portfolio is such that $\theta_1(t) \equiv 1$ and $\theta_0(t) \equiv 0$. Let us see whether this also follows from the formula above.
Since $f' = 1$ we get

$$\theta_1(t) = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} \exp(\beta x - \frac{1}{2}\beta^2(T-t)) \exp(-\frac{x^2}{2(T-t)}) dx.$$

Working out the integral or using an expression for the moment generating function of the normal distribution, we see that indeed the result is $\theta_1(t) = 1$ for all t.

Exercise Show that for the claim $F = X_1(T)^2$ in the classical Balck-Scholes framework the hedging strategy is determined by $\theta_1(t) = 2X_1(t)e^{(\rho+\beta^2)(T-t)}$.

5 Interpretation of the hedging strategy

We follow the generalized Black-Scholes model with deterministic functions ρ and β, see page 274. Suppose we want to price a contingent claim $F = f(X_1(T))$. With $h(y) = \frac{f(yX_0(T))}{X_0(T)}$ we know from the above discussion how to obtain the hedge portfolio and we have the martingale representation formula for $h(\bar{X}_1(T))$. Consider now the conditional expectation

$$E_Q[h(\bar{X}_1(T))|\mathcal{F}_t] = X_0(T)^{-1}E_Q[f(X_1(T))|\mathcal{F}_t].$$

We can rewrite this as $E_Q^{\bar{X}_1(t)}h(\bar{X}_1(T-t))$, so it is nothing else but $g(t, \bar{X}_1(t))$. From the computations (involving η) on page 271 we obtain

$$\begin{aligned} g(t, \bar{X}_1(t)) &= g(0, X_1(0)) + \int_0^t g'(s, \bar{X}_1(s))\sigma(\bar{X}_1(s))d\tilde{B}(s) && (1) \\ &= g(0, X_1(0)) + \int_0^t g'(s, \bar{X}_1(s))d\bar{X}_1(s). && (2) \end{aligned}$$

Comparing this with the above expression for the portfolio we again recognize

$$\theta_1(t) = g'(t, \bar{X}_1(t)).$$

The interpretation of $g(t, \bar{X}_1(t))$ is that it is the discounted fair price at time t of the claim. Let $V(t) = v(t, X_1(t))$ be the value at time t of the claim. Clearly, we have $V(T) = F$ and $V(t) = X_0(t)g(t, \bar{X}_1(t))$ so that $v(t, x) = X_0(t)g(t, \frac{x}{X_0(t)})$ and consequently $v'(t, x) = g(t, \frac{x}{X_0(t)})$, implying that $v'(t, X_1(t)) = g(t, \bar{X}_1(t)) = \theta_1(t)$. The hedge portfolio is thus such that the amount that has to be invested in the risky asset is just a partial derivative of the value process of the claim!
If we look again at the trivial example above ($F = X_1(T)$), then obviously we get $V(t) = X_1(t)$ and $v(t, x) = x$ and $v'(t, x) = 1$, perfectly in agreement with what we found before.

6 Black-Scholes PDE

In exercise 8.6 we already encountered the Black-Scholes partial differential equation and you have (?) solved it, using the Feynman-Kac formula. Here we derive this equation based on the fact that $g(t, x)$ is a solution to

$$g_t(t, x) + \frac{1}{2}\beta^2 x^2 g_{xx}(t, x) = 0$$

with boundary condition $g(T,x) = h(x)$ (see pages 270-271). Of course we use the classical Black-Scholes framework as in the sections above and the notation we used there and we consider a claim $F = f(X_1(T))$. Take $X_0(t) = e^{\rho t}$ above and thus $v(t,x) = e^{\rho t} g(t, xe^{-\rho t})$. It is an elementary exercise (but do it, if you have never done it before) to deduce from the PDE for g by computing the partial derivatives of v that v satisfies

$$v_t(t,x) - \rho v(t,x) + \rho x v_x(t,x) + \frac{1}{2}\beta^2 x^2 v_{xx}(t,x) = 0$$

with boundary condition $v(T,x) = f(x)$.

Exercise Let v satisfy the above PDE and set $V(t) = v(t, X_1(t))$. Use Itô's formula to find a stochastic differential equation for $V(T)$.
Let $F = f(X_1(T))$ be a contingent claim and V its value process. Deduce from the SDE for V the replicating portfolio for F and give an expression for the fair price of the claim at $t = 0$.