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1. Consider a measure space (S, Σ, μ). Prove the following statements.

   (a) The measure μ is finitely additive.

   (b) If $A \subset B$ $(A, B \in \Sigma)$, then $\mu(A) \leq \mu(B)$.

   (c) If $A \subset B$ $(A, B \in \Sigma)$ and μ is a finite measure, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

2. Prove the following statements.

   (a) The intersection of an arbitrary family of d-systems is again a d-system.

   (b) The intersection of an arbitrary family of σ-algebras is again a σ-algebra. Characterize $\sigma(\mathcal{C})$ for a given collection $\mathcal{C} \subset 2^{\Omega}$.

   (c) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are collections of subsets of Ω with $\mathcal{C}_1 \subset \mathcal{C}_2$, then $d(\mathcal{C}_1) \subset d(\mathcal{C}_2)$.

3. Let $\mathcal{G}$ and $\mathcal{H}$ be two σ-algebras on Ω. Let $\mathcal{C} = \{G \cap H : G \in \mathcal{G},\, H \in \mathcal{H}\}$. Show that $\mathcal{C}$ is a π-system and that $\sigma(\mathcal{C}) = \sigma(\mathcal{G} \cup \mathcal{H})$.

4. Let $\mathcal{I}$ be a collection of subsets of a given set S and $\mathcal{D} = \{B \in d(\mathcal{I}) : B \cap C \in d(\mathcal{I}), \forall C \in \mathcal{I}\}$. Show that $\mathcal{D}$ is a d-system. It $\mathcal{I}$ is a π-system, then moreover $\mathcal{D} = d(\mathcal{I}) = \sigma(\mathcal{I})$.

5. Let $h : S \to \mathbb{R}$, Σ a σ-algebra on S and $\mathcal{B}$ the Borel σ-algebra on $\mathbb{R}$. Consider the collection $\mathcal{C} = \{B \in \mathcal{B} : h^{-1}[B] \in \Sigma\}$. Show that $\mathcal{C}$ is a σ-algebra.

6. If h_1 and h_2 are measurable functions, then $h_1 h_2$ is measurable too.

7. Let Ω be a countable set. Let $\mathcal{F} = 2^{\Omega}$ and let $p : \Omega \to [0, 1]$ satisfy $\sum_{\omega \in \Omega} p(\omega) = 1$. Put $\mathbb{P}(A) = \sum_{\omega \in A} p(\omega)$ for $A \in \mathcal{F}$. Show that $\mathbb{P}$ is a probability measure.

8. Let Ω be a countable set. Let $\mathcal{A}$ be the collection of $A \subset \Omega$ such that A or its complement has finite cardinality. Show that $\mathcal{A}$ is an algebra. What is $d(\mathcal{A})$?

9. Let $E_n, n \geq 1$ be subsets of some set S. Let $X_n(\omega) = 1_{E_n}(\omega)$. Show that $\limsup_n X_n(\omega) = 1_{\limsup_n E_n}(\omega)$ for all $\omega \in S$.

10. Let X be a random variable. Show that $\Pi(X) := \{X^{-1}(-\infty, x] : x \in \mathbb{R}\}$ is a π-system and that it generates $\sigma(X)$.

11. Consider an infinite sequence of coin tossing. We take $\Omega = \{H, T\}^{\infty}$, a typical element ω is an infinite sequence $(\omega_1, \omega_2, \ldots)$ with each $\omega_n \in \{H, T\}$, and $\mathcal{F} = \sigma(\{\omega \in \Omega : \omega_n = w\}, w \in \{H, T\}, n \in \mathbb{N})$. Define functions X_n by $X_n(\omega) = 1$ if $\omega_n = H$ and $X_n(\omega) = 0$ if $\omega_n = T$.

    (a) Show that all X_n are random variables, i.e. everyone of them is measurable.

(b) Let $S_n = \sum_{i=1}^n X_i$. Show that also S_n is a random variable.

(c) Let $p \in [0,1]$ and $E_p = \{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} S_n(\omega) = p\}$. Show that E_p is an $\mathcal{F}$-measurable set.

12. Let $\{Y_\gamma : \gamma \in C\}$ be an arbitrary collection of random variables and $\{X_n : n \in \mathbb{N}\}$ be a countable collection of random variables, all defined on the same probability space.

    (a) Show that $\sigma\{Y_\gamma : \gamma \in C\} = \sigma\{Y_\gamma^{-1}(B) : \gamma \in C, B \in \mathcal{B}\}$.

    (b) Let $\mathcal{X}_n = \sigma\{X_1, \ldots, X_n\}$ ($n \in \mathbb{N}$) and $\mathcal{A} = \bigcup_{n=1}^\infty \mathcal{X}_n$. Show that $\mathcal{A}$ is an algebra and that $\sigma(\mathcal{A}) = \sigma\{X_n : n \in \mathbb{N}\}$.

13. Let $\mathcal{F}$ be a σ-algebra on Ω with the property that for all $F \in \mathcal{F}$ it holds that $\mathbb{P}(F) \in \{0,1\}$. Let $X : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Show that for some $c \in \mathbb{R}$ one has $\mathbb{P}(X = c) = 1$. (*Hint:* $\mathbb{P}(X \leq x) \in \{0,1\}$ for all x.)

14. Find the λ-sets of Σ_0 in the following cases.

    (a) $S = \mathbb{N}$, $\Sigma_0 = 2^S$, $\lambda(E) = |E|^2$ ($E \subset S$), where $|E|$ is the number of elements of E if E is a finite set and $|E| = \infty$ otherwise. Is λ an outer measure? Same question for $\lambda(E) = |E|^{1/2}$.

    (b) The setting is that of exercise 8. Consider $p : \Omega \to [0,1]$ and define λ on $\Sigma_0 = \mathcal{A}$ by $\lambda(A) = \sum_{\omega\in A} p(\omega)$ if A is finite and $\lambda(A) = \sum_{\omega\in\Omega} p(\omega) - \sum_{\omega\in A^c} p(\omega)$ if A has a finite complement. Is λ countably additive on Σ_0? Describe explictely the (unique?) extension of λ (if it exists) to $\sigma(\Sigma_0)$. Under what condition is the extension a probability measure?

15. Let $\mathcal{G}_0$ be an algebra on a set S, $\lambda : \mathcal{G}_0 \to [0,\infty]$ with $\lambda(\emptyset) = 0$ and $\mathcal{L}_0$ the sub-algebra of $\mathcal{G}_0$ consisting of the λ-sets. Show that for disjoint $L_k \in \mathcal{L}_0$ ($k = 1, \ldots, n$) and $G \in \mathcal{G}_0$ it holds that
$$\lambda\big((\bigcup_{k=1}^n L_k) \cap G\big) = \sum_{k=1}^n \lambda(L_k \cap G).$$

16. Consider the collection Σ_0 of subsets of $\mathbb{R}$ that can be written as a *finite* union of *disjoint* intervals of type $(a,b]$ with $-\infty \leq a \leq b < \infty$ or (a,∞). Show that Σ_0 is an algebra and that $\sigma(\Sigma_0) = \mathcal{B}(\mathbb{R})$.

17. Show that a finitely additive map $\mu : \Sigma_0 \to [0,\infty]$ is countably additive if $\mu(H_n) \to 0$ for every decreasing sequence of sets $H_n \in \Sigma_0$ with $\bigcap_n H_n = \emptyset$. If μ is countably additive, do we necessarily have $\mu(H_n) \to 0$ for every decreasing sequence of sets $H_n \in \Sigma_0$ with $\bigcap_n H_n = \emptyset$?

18. Let F be a distribution function on $\mathbb{R}$. Then there exists a (probability) measure μ on $(\mathbb{R}, \mathcal{B})$ such that $F(x) = \mu(-\infty, x]$. Show this by proving it first for the case where $F(0) = 0$, $F(1) = 1$, then for the case where $F(-N) = 0$ and $F(N) = 1$ for some $N > 0$ and finally for the general case.

19. Let (S, Σ, μ) be a measure space. Call a subset N of S a (μ, Σ)-null set if there exists a set $N' \in \Sigma$ with $N \subset N'$ and $\mu(N') = 0$. Denote by $\mathcal{N}$ the collection of all (μ, Σ)-null sets. Let Σ^* be the collection of subsets E of S for which there exist $F, G \in \Sigma$ such that $F \subset E \subset G$ and $\mu(G \setminus F) = 0$. For $E \in \Sigma^*$ and F, G as above we define $\mu^*(E) = \mu(F)$.

   (a) Show that Σ^* is a σ-algebra and that $\Sigma^* = \sigma(\mathcal{N} \cup \Sigma)$.

   (b) Show that μ^* restricted to Σ coincides with μ and that $\mu^*(E)$ doesn't depend on the specific choice of F in its definition.

   (c) Show that the collection of (μ^*, Σ^*)-null sets is $\mathcal{N}$.

20. Let X be a (real) random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $\Lambda(B) = \mathbb{P}(X^{-1}[B])$ for every $B \in \mathcal{B}(\mathbb{R})$ and $F(x) = \Lambda((-\infty, x])$, $x \in \mathbb{R}$. Prove the following.

   (a) Λ is a probability measure on $\mathcal{B}(\mathbb{R})$.

   (b) F is increasing with $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$ and F is rightcontinuous.

   (c) For every $d \in \mathbb{R}$ we have $\mathbb{P}(X = d) = F(d) - F(d-)$ (where $F(d-) = \lim_{x\uparrow d} F(x)$). Show that the set $D = \{d \in \mathbb{R} : \mathbb{P}(X = d) > 0\}$ is at most countable.

21. Let (S, Σ, μ) be a measure space and let f be a nonnegative simple function, $f = \sum_{k=1}^{n} a_k 1_{A_k}$ say, where the A_k are measurable sets. If f has the alternative representation $f = \sum_{k=1}^{m} a'_k 1_{A'_k}$, then $\sum_{k=1}^{n} a_k \mu(A_k) = \sum_{k=1}^{m} a'_k \mu(A'_k)$.

22. Let f and g be nonnegative simple functions on the measure space (S, Σ, μ). Show that $\mu_0(f+g) = \mu_0(f) + \mu_0(g)$. (*Hint:* wite $f = \sum_k f_k 1_{F_k}$ with the F_k disjoint, $g = \sum_j g_j 1_{G_j}$ with the G_j disjoint and look at what happens on the intersections $F_k \cap G_j$).

23. Consider the measurable space (S, Σ). Let $f : S \to [0, \infty]$ be measurable. Put $E_k^n = \{k2^{-n} \le f < (k+1)2^{-n}\}$ for $n, k \ge 0$ and $F^n = \{f \ge n\}$. Define

$$f_n = \sum_{k=0}^{n2^n - 1} k2^{-n} 1_{E_k^n} + n 1_{F^n}.$$

   Show that the f_n are simple functions and that $f_n(s) \uparrow f(s)$ for all $s \in S$.

24. Consider the measure space (S, Σ, μ). Let $f : S \to \mathbb{R}$ be measurable with $\mu(|f|) < \infty$. Show that $|\mu(f)| \le \mu(|f|)$.

25. Consider the measure space (S, Σ, μ). Show that the mapping $f \mapsto \mu(f)$ is linear on $\mathcal{L}^1(S, \Sigma, \mu)$.

26. Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is the counting measure, i.e. $\mu(E) \le \infty$ is equal to the number of elements of $E \subset \mathbb{N}$. Interpret the integrals $\mu(f)$ in 'simpler' terms in this case.

27. Let f be a simple function on some measure space (S, Σ, μ), with representation $f = \sum_{k=1}^n a_k 1_{A_k}$ say, where the A_k are measurable sets. Describe a procedure to turn this representation of f into $f = \sum_{j=1}^m d_j 1_{D_j}$, where the D_j are certain disjoint measurable sets. Show also that for this procedure one has $\sum_{k=1}^n a_k \mu(A_k) = \sum_{j=1}^m d_j \mu(D_j)$. If the distincts numbers $f_1, \ldots, f_r$ are all possible values of f, then $\mu(f) = \sum_{k=1}^r f_k \mu(\{f = f_k\})$.

28. Consider the measure space (S, Σ, μ), where $\Sigma = \sigma(\mathcal{P})$ for a finite partition $\mathcal{P}$ of S and μ a finite measure. Let $f : S \to \mathbb{R}$ be measurable. Show that f is constant on the elements P_k of $\mathcal{P}$. Let f_k be the common value of f on such a P_k. Show that $\mu(f)$ is well defined and express it in terms of the f_k. How would you interpret the result if μ is a probability measure?

29. Show that the Borel-Cantelli lemma 2.7 follows from (c) in section 6.5.

30. If c is convex on a convex set $G \subset \mathbb{R}$, then for all $u < v < w$ in G one has

$$\frac{c(v) - c(u)}{v - u} \leq \frac{c(w) - c(v)}{w - v}.$$

Show this inequality. Give an example of a set G and a convex function on it that is not continuous.

31. Let $p \geq 1$ and show that for all $x, y \in \mathbb{R}$ one has $|x+y|^p \leq 2^{p-1}(|x|^p + |y|^q)$. (*Hint:* $x \mapsto x^p$ is convex on $[0, \infty)$.)

32. Let $X, Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Show that $XY \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and that the (Cauchy-)Schwartz inequality

$$|\mathbb{E}XY| \leq \left(\mathbb{E}X^2 \mathbb{E}Y^2\right)^{1/2}$$

holds. (*Hint:* Use that $\mathbb{E}(X + aY)^2 \geq 0$, for all $a \in \mathbb{R}$.)

33. Consider a measure space (S, Σ, μ) and $f : S \to \mathbb{R}$ that is nonnegative and measurable. Define $\nu : \Sigma \to [0, \infty]$ by $\nu(A) = \mu(1_A f)$. Show that ν is a measure on (S, Σ) and that for $h \in \mathcal{L}^1(S, \Sigma, \nu)$ one has $\nu(h) = \mu(hf)$.

34. Consider the setting of the previous exercise. Let $E \in \Sigma$ be such that $\mu(E) = 0$. Show that $\nu(E) = 0$. Assume now that S is a countable set with Σ the power set of S and let μ be such that $\mu(\{s\}) = m(s) \in [0, \infty)$. Let ν be a measure of (S, Σ) such that $\nu(E) = 0$ as soon as $\mu(E) = 0$. Show that there is measurable function f on S such that $\nu(E) = \mu(1_E f)$. Can we do the same if some of the $m(s)$ are infinite.

35. Williams, exercise E4.1.

36. Williams, exercise E4.6.

37. If $Z_1, Z_2, \ldots$ is a sequence of nonnegative random variables, then $\mathbb{E} \sum_{k=1}^\infty Z_k = \sum_{k=1}^\infty \mathbb{E}\, Z_k$. Show that this follows from Fubini's theorem.

38. Show that $\mathbb{E}\, X^2 1_{\{|X|>\varepsilon\}} \leq \mathbb{E}\, |X|^{2+\delta} \varepsilon^{-\delta}$ for all $\delta, \varepsilon > 0$.

39. Call a measurable function on some (S, Σ, μ) essentially bounded if there is $M > 0$ such that $\mu(\{|f| > M\}) = 0$ and define $||f||_\infty = \inf\{M > 0 : \mu(\{|f| > M\}) = 0\}$. Show that $|| \cdot ||$ has all properties of a norm, except one (which one?). Show that Hölders inequality holds for $p = \infty$, $q = 1$.

40. Consider measure space (S_1, Σ_1, μ_1), (S_2, Σ_2, μ_2) and the product space $S_1 \times S_2$ with the product σ-algebra. Show that the set of measurable rectangles $A_1 \times A_2$ (with $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$) is a π-system that generates the product σ-algebra.

41. Use polar coordinates to show that $\int_{\mathbb{R}^2} \exp(-\frac{1}{2}(x^2 + y^2))\, dxdy = 2\pi$ and Fubini-'s theorem to show that $\int_{\mathbb{R}} \exp(-\frac{1}{2}x^2)\, dx = \sqrt{2\pi}$.

42. Show (use a famous theorem) that $\lim_{T\to\infty} \int_0^\infty \int_0^T \sin x e^{-xt}\, dx\, dt = \frac{\pi}{2}$ and show (use another famous theorem) that also $\lim_{T\to\infty} \int_0^T \frac{\sin x}{x}\, dx = \frac{\pi}{2}$. Is the function $x \mapsto \frac{\sin x}{x}$ Lebesgue-integrable on $[0, \infty)$?

43. Let $F, G : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right-continuous. Use Fubini's theorem to show the integration by parts formula, valid for all $a < b$,

$$F(b)G(b) - F(a)G(a) = \int_{(a,b]} F(s-)\, dG(s) + \int_{(a,b]} G(s)\, dF(s).$$

Hint: integrate $1_{(a,b]^2}$ and split the square into a lower and an upper triangle.

44. Let F be the distribution function of a nonnegative random variable X and assume that $\mathbb{E}\, X^\alpha < \infty$ for some $\alpha > 0$. Use exercise 43 to show that

$$\mathbb{E}\, X^\alpha = \alpha \int_0^\infty x^{\alpha-1}(1 - F(x))\, dx.$$

45. Let X be a random variable and let $\Pi(X) = \{X^{-1}(-\infty, x] : x \in \mathbb{R}\}$. Show that $\Pi(X)$ is a π-system that generates $\sigma(X)$.

46. Let the vector of random variables (X, Y) have a joint probability density function f. Let f_X and f_Y be the (marginal) probability density functions of X and Y respectively. Show that X and Y are independent iff $f(x, y) = f_X(x) f_Y(y)$ for all x, y except in a set of Leb$\times$Leb-measure zero.

47. Let $X, X_1, X_2, \ldots$ be random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the set $\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}$ is measurable.

48. Let $X_1, X_2, \ldots$ be an a.s. bounded sequence of random variables $\mathbb{P}(|X_n| \leq M) = 1$, for some real number M. Assume that for some random variable X one has $X_n \xrightarrow{P} X$. Show that also $\mathbb{P}(|X| \leq M) = 1$ and that for all $p \geq 1$ one has $X_n \xrightarrow{\mathcal{L}^p} X$.

49. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $\mathbb{E}\, X_1^2 < \infty$. The aim is to show is that both $\overline{X}_n \xrightarrow{\mathcal{L}^2} \mu$ where $\mu = \mathbb{E}\, X_1$ and $\overline{X}_n \xrightarrow{a.s.} \mu$.

   (a) Show the $\mathcal{L}^2$ convergence.

   (b) Use Chebychev's inequality to show that $\sum_n \mathbb{P}(|\overline{X}_{n^2} - \mu| > \varepsilon) < \infty$ and deduce form a wellknown lemma that $\overline{X}_{n^2} \xrightarrow{a.s.} \mu$.

   (c) Show the almost sure convergence of $\overline{X}_n$ by "filling the gaps".

50. Let $\alpha > 1$ and $\beta_k = [\alpha^k]$. Show that

   (a) $\beta_k \geq \alpha^k(1 - \frac{1}{\alpha})$

   (b) $\sum_{k=m}^{\infty} \frac{1}{\beta_k^2} \leq (\frac{\alpha}{\alpha-1})^4 \frac{1}{\beta_m^2}$.

   (c) $\frac{\beta_{k+1}}{\beta_k} \to \alpha$.

51. Exercise E7.1 of Williams.

52. Let $X_1, X_2, \ldots$ be real random variables and $g : \mathbb{R} \to \mathbb{R}$ a uniformly continuous function. Show that $g(X_n) \xrightarrow{P} g(X)$ if $X_n \xrightarrow{P} X$. What can be said of the $g(X_n)$ if $X_n \xrightarrow{a.s.} X$?

53. Let x_n be real numbers with $x_n \to x$. Let $y_n = \frac{1}{n}\sum_{i=1}^n x_i$. Show that $y_n \to x$.

54. Let $X_1, Y_1, X_2, Y_2, \ldots$ be an i.i.d. sequence whose members have a uniform distribution on $[0,1]$ and let $f : [0,1] \to [0,1]$ be continuous. Define $Z_i = 1_{\{f(X_i) > Y_i\}}$.

   (a) Show that $\frac{1}{n}\sum_{i=1}^n Z_i \to \int_0^1 f(x)\,dx$ a.s.

   (b) Show that $\mathbb{E}\,(\frac{1}{n}\sum_{i=1}^n Z_i - \int_0^1 f(x)\,dx)^2 \leq \frac{1}{4n}$.

   (c) Explain why these two results are useful.

55. If $X_n \xrightarrow{P} X$ and g is a continuous function, then also $g(X_n) \xrightarrow{P} g(X)$. Show this.

56. Let X be a random variable with $\mathbb{E}\, X^2 < \infty$ and let $\phi(\theta) = \mathbb{E}\, e^{i\theta X}$. Show that $\phi''(0) = -\mathbb{E}\, X^2$.

57. Let X be a random variable with values in $\mathbb{Z}$ and ϕ its characteristic function. Show that $\mathbb{P}(X = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\theta) e^{-ki\theta}\,d\theta$ for $k \in \mathbb{Z}$. Is $\int_{\mathbb{R}} |\phi(\theta)|\,d\theta < \infty$?

58. Verify the formulas for the characteristic functions in each of the following cases.

   (a) $\phi_{N(0,1)}(\theta) = \exp(-\frac{1}{2}\theta^2)$

   (b) $\phi_{N(\mu,\sigma^2)}(\theta) = \exp(i\theta\mu - \frac{1}{2}\sigma^2\theta^2)$

(c) If X has an exponential distribution with parameter λ, then $\phi_X(\theta) = \lambda/(\lambda - i\theta)$.

(d) If X has a Cauchy distribution, then $\phi_X(\theta) = \exp(-|\theta|)$.

59. Read the proof of the Helly-Bray lemma. Show that the function F defined on page 184 is (a) right-continuous and that (b) $\lim F_{n_i}(x) = F(x)$ for all x where F is continuous. *Hint:* Fix x and $\varepsilon > 0$. Then there is $c \in \mathbb{Q}$ such that $F(x) \leq H(c) < F(x) + \varepsilon$. If F is continuous at x, then there exists also $c' < c \in \mathbb{Q}$ and $y < x$ such that $F(x) - \varepsilon \leq F(y) \leq H(c') \leq H(c)$.

60. Let (F_n) be a sequence of distribution functions on $\mathbb{R}$ such that $\lim_{n\to\infty} F_n(x) = F(x)$ for all x where the distribution function F is continuous. Show that $\lim_{n\to\infty} \int_{\mathbb{R}} h\, dF_n = \int_{\mathbb{R}} h\, dF$ for all bounded and continuous $h : \mathbb{R} \to \mathbb{R}$.

61. Let $X, X_1, X_2, \ldots$ be real-valued random variables with $F_{X_n} \xrightarrow{w} F_X$. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous and put $Y = h(X)$ and $Y_n = h(X_n)$ for every $n \in \mathbb{N}$. Show that $F_{Y_n} \xrightarrow{w} F_Y$.

62. Suppose that $X, X_1, X_2, \ldots$ are real valued random variables, defined on one the same probability space, with $X_n \to X$ in probability. Show that $F_{X_n} \xrightarrow{w} F_X$.

63. Let $\mu, \mu_1, \mu_2, \ldots$ be probability measures on $\mathbb{R}$ and suppose that for any open $G \subset \mathbb{R}$ that $\liminf \mu_n(G) \geq \mu(G)$. Then $\mu_n \to \mu$.Show this as follows. Let h be a bounded continuous function on $\mathbb{R}$. Assume w.l.og. that $0 \leq h < 1$. Let $k \in \mathbb{N}$ and define $F_i = \{x : \frac{i-1}{k} \leq h(x) < \frac{i}{k}\}$. Split $\mu(h)$ into integrals over the F_i. Then

$$\frac{1}{k}\sum_{i=1}^{k} \mu(h > \frac{i}{k}) \leq \mu(h) \leq \frac{1}{k}\sum_{i=1}^{k} \mu(h > \frac{i-1}{k})$$

and something similar for μ_n. Deduce that $\liminf \mu_n(h) \geq \mu(h)$ and complete the proof with the aid of an inequality for $\limsup \mu_n(h)$.

64. Suppose that the real random variables $X, X_1, X_2, \ldots$ are defined on a common probability space and that $F_{X_n} \xrightarrow{w} F_X$. Suppose that $X = x_0$ a.s. for some $x_0 \in \mathbb{R}$. Show that $X_n \to X$ in probability.

65. Let X_n have a $\text{Bin}(n, \lambda/n)$ distribution (for $n > \lambda$). Show that $X_n \xrightarrow{w} X$, where X has a $\text{Poisson}(\lambda)$ distribution.

66. Exercise 18.3