Tinbergen Institute
Measure Theory Exam Questions

1. If X and Y are *independent* random variables with $\mathbb{E}\,|X| < \infty$ and $\mathbb{E}\,|Y| < \infty$ (assumed to hold throughout this exercise), then the product formula $\mathbb{E}\,(XY) = \mathbb{E}\,X \cdot \mathbb{E}\,Y$ holds. To show this you have to apply (parts of) the standard machine[1] a couple of times.

   (a) First a special case. Let X be positive but arbitrary otherwise, and $Y = \mathbf{1}_A$ for some set $A \in \mathcal{F}$. Use the standard machine to show that $\mathbb{E}\,(X\mathbf{1}_A) = \mathbb{E}\,X \cdot \mathbb{P}(A)$.

   (b) Prove now, using the previous item and the standard machine again, the product formula for $X \geq 0$ and $Y \geq 0$.

   (c) Why are X^+ and Y^- also independent random variables?

   (d) Complete the proof for arbitrary X and Y.

2. Let X and Y be random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} = \sigma(Y)$.

   (a) Show that the collection of events $\{Y \in B\}$, where B runs through the Borel sets $\mathcal{B}(\mathbb{R})$, forms a σ-algebra (so you show that it has all the defining properties of a σ-algebra). This σ-algebra will be denoted $\mathcal{H}$.

   (b) Show the two inclusions $\mathcal{H} \subset \mathcal{G}$ and $\mathcal{G} \subset \mathcal{H}$. For the latter you need the 'minimality property' of $\sigma(Y)$.

   (c) Let $X = \mathbf{1}_G$ for some $G \in \mathcal{G}$. Find a function $f : \mathbb{R} \to [0, 1]$ that is Borel-measurable (and check this property!) such that $X = f(Y)$.

   (d) Use the standard machine to prove the following result. If X is $\mathcal{G}$-measurable, then there exists a Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $X = f(Y)$.

3. Let $X_1, X_2, \ldots$ be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that the X_i are nonnegative and let $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$. It is known that the S_n are random variables (measurable functions) as well. We define $S(\omega) = \lim_{n\to\infty} S_n(\omega)$, which exists for every $\omega \in \Omega$ but may be infinite.

   (a) Show that S is a random variable (*Hint:* show first that $\{S > a\} = \bigcup_{n=1}^\infty \{S_n > a\}$ for $a > 0$).

[1] Recall that the standard machine is a method of proving along steps: (1) for indicator functions; (2) for nonnegative simple functions; (3) for nonnegative functions by approximation with simple functions (the approximating sequence always exists); (4) general case.

(b) Note that $\mathbb{E}\, S \leq \infty$ is well defined. Show that $\mathbb{E}\, S = \sum_{i=1}^{\infty} \mathbb{E}\, X_i$.

(c) Assume that $\sum_{i=1}^{\infty} \mathbb{E}\, X_i < \infty$. Show that $\mathbb{P}(S < \infty) = 1$.

4. Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A well known property is that $\mathbb{E}\, X = 0$ if $X = 0$ a.s. In this exercise you will show this.

   (a) Suppose that X assumes finitely many values $y_0, y_1, \ldots, y_n$ and also that $X = 0$ a.s. Show that $\mathbb{E}\, X = 0$.

   (b) Suppose that $X \geq 0$, but also $X = 0$ a.s. Argue by using lower Lebesgue sums and the previous item that $\mathbb{E}\, X = 0$.

   (c) Let X be arbitrary but still $X = 0$ a.s. Show again that $\mathbb{E}\, X = 0$.

5. Recall the definition of *infimum*, written as inf. If $x_1, x_2, \ldots$ is a finite or infinite sequence of real numbers, then $x = \inf\{x_1, x_2, \ldots\}$ iff (1) $x \leq x_k$ for all k and (2) if $y > x$, there exists x_k such that $x_k < y$. It may happen that $x = -\infty$. For finite sequences $x_1, \ldots, x_n$, $\inf\{x_1, \ldots, x_n\}$ is the *minimum* of the x_k. An example with an infinite sequence is $\inf\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} = 0$, another example is $\inf\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots\} = 0$.

   If we have an infinite sequence of random variables $X_1, X_2, \ldots$, we say that the random variable X is $\inf\{X_1, X_2, \ldots\}$ if for every $\omega \in \Omega$ one has $X(\omega) = \inf\{X_1(\omega), X_2(\omega), \ldots\}$. From now on we assume to have a sequence of *nonnegative* random variables $X_1, X_2, \ldots$. For each n we define the random variable $Y_n := \inf\{X_n, X_{n+1}, X_{n+2}, \ldots\}$, also written as $Y_n = \inf_{m \geq n} X_m$.

   (a) Show that (each) Y_n is a random variable by considering events like $\{Y_n \geq a\}$.

   (b) Show that the Y_n form an *increasing* sequence of random variables. They then have a limit $Y_\infty \leq \infty$.

   (c) Show that $Y_n \leq X_m$ for all $m \geq n$, and conclude that $\mathbb{E}\, Y_n \leq y_n := \inf\{\mathbb{E}\, X_n, \mathbb{E}\, X_{n+1}, \ldots\}$. Note that the y_n form an increasing sequence too.

   (d) Show that $\mathbb{E}\, Y_\infty \leq \lim_{n\to\infty} y_n$. *This property is often written as* $\mathbb{E} \lim_{n\to\infty} \inf_{m \geq n} X_m \leq \lim_{n\to\infty} \inf_{m \geq n} \mathbb{E}\, X_m$, *and is known as Fatou's lemma.*

   (e) In the previous item, a strict inequality may occur. Consider thereto the probability space with $\Omega = (0, 1)$, $\mathcal{F}$ the Borel sets in $(0, 1)$ and $\mathbb{P}$ the Lebesgue measure. We take $X_n(\omega) = n\, \mathbf{1}_{(0,1/n)}(\omega)$. Show that indeed strict inequality now takes place in Fatou's lemma (so you compute both sides of the inequality).

6. In this exercise we need limits of sequences of subsets of a given set Ω, which we define in two cases. Suppose that we have an *increasing* sequence of sets A_n ($n \geq 0$), i.e. $A_n \subset A_{n+1}$ for all $n \geq 0$. Then we define $\overline{A} = \lim_{n\to\infty} A_n := \bigcup_{n=0}^\infty A_n$. If the sequence is *decreasing*, $A_n \supset A_{n+1}$ for all n, we define $\underline{A} = \lim_{n\to\infty} A_n := \bigcap_{n=0}^\infty A_n$. We work with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we consider an increasing sequence of events A_n (so $A_n \in \mathcal{F}$ for all n). Let $D_0 = A_0$ and $D_n = A_n \setminus A_{n-1}$ for $n \geq 1$.

   (a) Show that $\mathbb{P}(A_n) = \sum_{k=0}^n \mathbb{P}(D_k)$.

   (b) Show that $\overline{A} = \bigcup_{k=0}^\infty D_k$.

   (c) Show that $\mathbb{P}(A_n) \to \mathbb{P}(\overline{A})$ for $n \to \infty$.

   (d) Suppose that events B_n ($n \geq 0$) form a decreasing sequence. Show that $\mathbb{P}(B_n) \to \mathbb{P}(\underline{B})$. (Hint: consider the B_n^c.)

7. Let X, Y be random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, so they are both $\mathcal{F}$-measurable.

   (a) Let $c \in \mathbb{R}$. Show (make a sketch!) that $\{(x,y) \in \mathbb{R}^2 : x + y > c\} = \bigcup_{q\in\mathbb{Q}} \{(x,y) \in \mathbb{R}^2 : x > q, y > c - q\}$.

   (b) Show that $X+Y$ is also $\mathcal{F}$-measurable. NB: For this it is sufficient to show that $\{X + Y > c\} \in \mathcal{F}$ for all $c \in \mathbb{R}$.

8. Let $x_1, x_2, \ldots$ be a sequence of real numbers. We put, for $n \geq 1$, $\overline{x}_n = \sup\{x_n, x_{n+1}, \ldots\}$ and $\underline{x}_n = \inf\{x_n, x_{n+1}, \ldots\}$. Note that the $\overline{x}_n$ form a decreasing sequence and the $\underline{x}_n$ an increasing one, and hence both sequences have a limit, denoted $\overline{x}$ and $\underline{x}$ respectively. One always has $\overline{x} \geq \underline{x}$ and $\overline{x} = \inf\{\overline{x}_1, \overline{x}_2, \ldots\}$. Moreover, the original sequence with the x_n has a limit x iff $x = \overline{x} = \underline{x}$.

   Consider now a sequence of random variables X_n defined on some $(\Omega, \mathcal{F}, \mathbb{P})$. As these are measurable functions, we can define $\overline{X}_n$ as the function s.t. $\overline{X}_n(\omega) = \sup\{X_n(\omega), X_{n+1}(\omega), \ldots\}$ and likewise $\underline{X}_n$, $\overline{X}$, $\underline{X}$.

   (a) Consider $E_a = \{\overline{X}_n \leq a\}$ for arbitrary $a \in \mathbb{R}$. Show that $E_a \in \mathcal{F}$ and conclude that $\overline{X}_n$ is a random variable (for every n).

   (b) Show that $\underline{X}_n$ is a random variable.

   (c) Show that $\overline{X}$ and $\underline{X}$ are random variables too.

   (d) Show that $\{\omega : \lim_{n\to\infty} X_n(\omega) \text{ exists}\} = \{\omega : \overline{X}(\omega) - \underline{X}(\omega) \leq 0\}$ and that this set belongs to $\mathcal{F}$.

(e) Assume that $X(\omega) = \lim_{n\to\infty} X_n(\omega)$ exists for every ω. Show that X is a random variable.

9. Consider a sequence of random variables X_n defined on some $(\Omega, \mathcal{F}, \mathbb{P})$ and put $S_n = \sum_{k=1}^n X_k$ for $n \geq 1$.

   (a) Assume all $X_n \geq 0$. Show that $\mathbb{E} \sum_{k=1}^\infty X_k = \sum_{k=1}^\infty \mathbb{E}\, X_k$. *Hint: apply the Monotone Convergence Theorem to the* S_n.

   From here on the assumption that the X_n are nonnegative is dropped.

   (b) Show that $\mathbb{E} \sum_{k=1}^\infty |X_k| = \sum_{k=1}^\infty \mathbb{E}\, |X_k|$.

   (c) Assume $\sum_{k=1}^\infty \mathbb{E}\, |X_k| < \infty$. Show that $\mathbb{E} \sum_{k=1}^\infty X_k = \sum_{k=1}^\infty \mathbb{E}\, X_k$.