Exercises and Problems
Brownse Beweging en Stochastische Differentiaalvergelijkingen II

1. KS, page 134, problem 2.5

2. KS, page 141, problem 2.12

3. Show the validity of (2.26) for simple processes X and Y by direct computation.

4. Let M be a continuous square integrable martingale and T a stopping time. Show that $\langle M^T \rangle = \langle M \rangle^T$. (For any process X and any stopping time we write $X_t^T = X_{T \wedge t}$.)

5. Let M be a continuous local martingale. Show that there exists a continuous increasing process such that the difference of M^2 and this process is a local martingale. Show that with zero initial condition this process is unique.

6. KS, page 147, problem 2.28. Do this first without Itô's formula, then also with.

7. KS, page 148, problem 2.29

8. Let M be a continuous local martingale and T a stopping time. Prove Lenglart's inequality: for all $\varepsilon > 0$ and $\delta > 0$ we have

$$\mathbb{P}(\sup_{t \le T} |M_t| > \varepsilon) \le \frac{\mathbb{E}(\delta \wedge \langle M \rangle_T)}{\varepsilon^2} + \mathbb{P}(\langle M \rangle_T \ge \delta).$$

9. Let M be a continuous square integrable martingale and T a finite stopping time. Prove that $\mathbb{P}(M_t = 0, \forall t \le T) = 1$ if $\langle M \rangle_T = 0$ a.s.

10. Let M, N be continuous square integrable martingales and T a stopping time. Show the a.s. inequality

$$\langle M, N \rangle_T^2 \le \langle M \rangle_T \langle N \rangle_T$$

11. Let W be standard Brownian motion. Find a sequence of piecewise constant processes W^n such that $\mathbb{E} \int_0^T |W_t^n - W_t|^2 dt \to 0$. Compute $\int_0^T W_t^n \, dW_t$ and show that it 'converges' (in what sense?) to $\frac{1}{2}(W_T^2 - T)$, if we consider smaller and smaller intervals of constancy. Deduce that $\int_0^T W_t \, dW_t = \frac{1}{2}(W_T^2 - T)$.

12. Prove Gronwall's inequality (KS problem 5.2.7)

13. KS problem 2.12. Argue as follows. Replace ξ with $\xi_k = \xi 1_{\{|\xi|\leq k\}}$. Define solution X^k of the SDE with ξ replaced by ξ_k. Compare X^l with X^k for $l \geq k$ and show that $\max_{s\leq T} |X^l_s - X^k_s| 1_{\{|\xi|\leq k\}} = 0$ a.s. (use the techniques of the proofs of theorem 2.5 and 2.9.). Then we can define a process X by $X_t(\omega) = X^k_t$, where we choose $k = k(\omega)$ such that $|\xi(\omega)| \leq k$. Show that $X_t 1_{\{|\xi|\leq k\}}$ is a solution of the SDE, and let $k \to \infty$.

14. KS problems 5.6.1 and 5.6.2 Find explicit solutions if $A(t) \equiv A$, $\sigma(t) = \sigma$ and $a(t) \equiv 0$. Find the parameters of the normal distribution of X_0 such that all X_t have the same distribution, if A has all its eigenvalues in the complex left half plane.

15. Show that $Z(X)$ in (5.2) of page 191 is a local martingale.

16. KS Problem 3.5.6

17. KS Problem 3.5.18

18. KS Problem 3.5.20

19. Show that the process $\hat{X}_t(\omega) = X_t(\omega) 1_{\{s\geq t\}} 1_A(\omega)$ with $A \in \mathcal{F}_s$ is progressive (this is the process that I needed at the end of the proof of proposition 3.4.14 in KS, but notice that the proof that I gave differs from the one in the book).

20. Show the validity of (4.46), KS page 184, by a Monotone Class argument.

21. Let $L(\cdot, x)$ be the local time of a Brownian motion B at x. If $\{\mathcal{F}_t\}$ is the *augmented* filtration generated by $|B|$, show then that $L(\cdot, x)$ is adapted to this filtration. (This was needed at the end of example 5.3.5 in KS).

22. Determine for all of the following processes M the representation as a (constant plus a) stochastic integral (in terms of the Martingale Representation Theorem).

    (a) $M_t = W_t^3 - c\int_0^t W_s\, ds$ for a suitable constant c (which one?).

    (b) For some fixed time T we have $M_t = \mathbb{E}[e^{W_T} | \mathcal{F}_t]$.

    (c) For some fixed time T we take $M_t = \mathbb{E}[\int_0^T W_s\, ds | \mathcal{F}_t]$.

    (d) If v is the solution of the partial differential equation

    $$\frac{\partial v}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x) = 0,$$

    then $M_t = v(t, W_t)$ is a martingale (show this by the Itô formula, you may assume enough integrability).