Opgaven kansrekening 2003-2004

1. Let $\{Y_\gamma : \gamma \in C\}$ be an arbitrary collection of random variables and $\{X_n : n \in \mathbb{N}\}$ be a countable collection of random variables, all defined on the same probability space.

   (a) Show that $\sigma\{Y_\gamma : \gamma \in C\} = \sigma\{Y_\gamma^{-1}(B) : \gamma \in C, B \in \mathcal{B}\}$.

   (b) Let $\mathcal{X}_n = \sigma\{X_1, \dots, X_n\}$ $(n \in \mathbb{N})$ and $\mathcal{A} = \bigcup_{n=1}^\infty \mathcal{X}_n$. Show that $\mathcal{A}$ is an algebra and that $\sigma(\mathcal{A}) = \sigma\{X_n : n \in \mathbb{N}\}$.

2. Let $\mathcal{F}$ be a σ-algebra on Ω with the property that for all $F \in \mathcal{F}$ it holds that $\mathbb{P}(F) \in \{0, 1\}$. Let $X : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Show that for some $c \in \mathbb{R}$ one has $\mathbb{P}(X = c) = 1$. (*Hint:* $\mathbb{P}(X \leq x) \in \{0, 1\}$ for all x.)

3. Let X be a (real) random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $\Lambda(B) = \mathbb{P}(X^{-1}[B])$ for every $B \in \mathcal{B}(\mathbb{R})$ and $F(x) = \Lambda((-\infty, x])$, $x \in \mathbb{R}$. Prove the following.

   (a) Λ is a probability measure on $\mathcal{B}(\mathbb{R})$.

   (b) F is increasing with $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$ and F is rightcontinuous.

   (c) For every $d \in \mathbb{R}$ we have $\mathbb{P}(X = d) = F(d) - F(d-)$ (where $F(d-) = \lim_{x\uparrow d} F(x)$). Show that the set $D = \{d \in \mathbb{R} : \mathbb{P}(X = d) > 0\}$ is at most countable.

4. If c is convex on a convex set $G \subset \mathbb{R}$, then for all $u < v < w$ in G one has

$$\frac{c(v) - c(u)}{v - u} \leq \frac{c(w) - c(v)}{w - v}.$$

   Show this inequality. Give an example of a set G and a convex function on it that is not continuous.

5. Let $p \geq 1$ and show that for all $x, y \in \mathbb{R}$ one has $|x+y|^p \leq 2^{p-1}(|x|^p + |y|^q)$. (*Hint:* $x \mapsto x^p$ is convex on $[0, \infty)$.)

6. Let $X, Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Show that $XY \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and that the (Cauchy-)Schwartz inequality

$$|\mathbb{E}XY| \leq \left(\mathbb{E}X^2 \mathbb{E}Y^2\right)^{1/2}$$

   holds. (*Hint:* Use that $\mathbb{E}(X + aY)^2 \geq 0$, for all $a \in \mathbb{R}$.)

7. If $Z_1, Z_2, \dots$ is a sequence of nonnegative random variables, then $\mathbb{E}\sum_{k=1}^\infty Z_k = \sum_{k=1}^\infty \mathbb{E}\, Z_k$. Show that this follows from Fubini's theorem.

8. Show that $\mathbb{E}\, X^2 1_{\{|X|>\varepsilon\}} \leq \mathbb{E}\, |X|^{2+\delta} \varepsilon^{-\delta}$ for all $\delta, \varepsilon > 0$.

9. Let $F, G : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right-continuous. Use Fubini's theorem to show the integration by parts formula, valid for all $a < b$,

$$F(b)G(b) - F(a)G(a) = \int_{(a,b]} F(s-)\, dG(s) + \int_{(a,b]} G(s)\, dF(s).$$

Hint: integrate $1_{(a,b]^2}$ and split the square into a lower and an upper triangle.

10. Let F be the distribution function of a nonnegative random variable X and assume that $\mathbb{E}\, X^\alpha < \infty$ for some $\alpha > 0$. Use exercise 9 to show that
$$\mathbb{E}\, X^\alpha = \alpha \int_0^\infty x^{\alpha-1}(1 - F(x))\, dx.$$

11. Let X be a random variable and let $\Pi(X) = \{X^{-1}(-\infty, x] : x \in \mathbb{R}\}$. Show that $\Pi(X)$ is a π-system that generates $\sigma(X)$.

12. Let the vector of random variables (X, Y) have a joint probability density function f. Let f_X and f_Y be the (marginal) probability density functions of X and Y respectively. Show that X and Y are independent iff $f(x, y) = f_X(x)f_Y(y)$ for all x, y except in a set of Leb$\times$Leb-measure zero.

13. Let $X, X_1, X_2, \ldots$ be random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that the set $\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}$ is measurable.

14. Suppose that the real random variables $X_1, X_2, \ldots$ are defined on a common probability space and that they are *iid* with a uniform distribution on $[0, 1]$. Let $M_n = \max\{X_1, \ldots, X_n\}$.
    (a) Show that $M_n \xrightarrow{P} 1$.
    (b) Show that $M_n \xrightarrow{a.s.} 1$.

15. Suppose that there are two real random variables X and X' such that a sequence $X_1, X_2, \ldots$ converges in probability to both X and X'. Show that $\mathbb{P}(X = X') = 1$. Same question for almost sure convergence.

16. Let real random variables $X_1, X_2, \ldots$ be defined on a common probability space. Prove that '$X_n \xrightarrow{P} X$' is equivalent with 'every subsequence of (X_n) has a further subsequence that converges a.s. to X'.

17. Give an example of a sequence of random variables that converge in probability, but not almost surely.

18. Let $X_1, X_2, \ldots$ be an a.s. bounded sequence of random variables $\mathbb{P}(|X_n| \leq M) = 1$, for some real number M. Assume that for some random variable X one has $X_n \xrightarrow{P} X$. Show that also $\mathbb{P}(|X| \leq M) = 1$ and that for all $p \geq 1$ one has $X_n \xrightarrow{\mathcal{L}^p} X$.

19. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $\mathbb{E}\, X_1^2 < \infty$. The aim is to show is that both $\overline{X}_n \xrightarrow{\mathcal{L}^2} \mu$ where $\mu = \mathbb{E}\, X_1$ and $\overline{X}_n \xrightarrow{a.s.} \mu$.
    (a) Show the $\mathcal{L}^2$ convergence.

(b) Use Chebychev's inequality to show that $\sum_n \mathbb{P}(|\overline{X}_{n^2} - \mu| > \varepsilon) < \infty$ and deduce from a wellknown lemma that $\overline{X}_{n^2} \stackrel{a.s.}{\to} \mu$.

(c) Show the almost sure convergence of $\overline{X}_n$ by "filling the gaps".

20. Let $\alpha > 1$ and $\beta_k = [\alpha^k]$. Show that

(a) $\beta_k \geq \alpha^k(1 - \frac{1}{\alpha})$

(b) $\sum_{k=m}^{\infty} \frac{1}{\beta_k^2} \leq (\frac{\alpha}{\alpha-1})^4 \frac{1}{\beta_m^2}$.

(c) $\frac{\beta_{k+1}}{\beta_k} \to \alpha$.

21. Let $X_1, X_2, \ldots$ be real random variables and $g : \mathbb{R} \to \mathbb{R}$ a uniformly continuous function. Show that $g(X_n) \stackrel{P}{\to} g(X)$ if $X_n \stackrel{P}{\to} X$. What can be said of the $g(X_n)$ if $X_n \stackrel{a.s.}{\to} X$?

22. Let x_n be real numbers with $x_n \to x$. Let $y_n = \frac{1}{n}\sum_{i=1}^n x_i$. Show that $y_n \to x$.

23. Let $X_1, Y_1, X_2, Y_2, \ldots$ be an i.i.d. sequence whose members have a uniform distribution on $[0,1]$ and let $f : [0,1] \to [0,1]$ be continuous. Define $Z_i = 1_{\{f(X_i) > Y_i\}}$.

(a) Show that $\frac{1}{n}\sum_{i=1}^n Z_i \to \int_0^1 f(x)\,dx$ a.s.

(b) Show that $\mathbb{E}\,(\frac{1}{n}\sum_{i=1}^n Z_i - \int_0^1 f(x)\,dx)^2 \leq \frac{1}{4n}$.

(c) Explain why these two results are useful.

24. If $X_n \stackrel{P}{\to} X$ and g is a continuous function, then also $g(X_n) \stackrel{P}{\to} g(X)$. Show this.

25. Let X be a random variable with $\mathbb{E}\,X^2 < \infty$ and let $\phi(\theta) = \mathbb{E}\,e^{i\theta X}$. Show that $\phi''(0) = -\mathbb{E}\,X^2$.

26. Let X be a random variable with values in $\mathbb{Z}$ and ϕ its characteristic function. Show that $\mathbb{P}(X = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(\theta)e^{-ki\theta}\,d\theta$ for $k \in \mathbb{Z}$. Is $\int_{\mathbb{R}} |\phi(\theta)|\,d\theta < \infty$?

27. Verify the formulas for the characteristic functions in each of the following cases.

(a) $\phi_{N(0,1)}(\theta) = \exp(-\frac{1}{2}\theta^2)$

(b) $\phi_{N(\mu,\sigma^2)}(\theta) = \exp(i\theta\mu - \frac{1}{2}\sigma^2\theta^2)$

(c) If X has an exponential distribution with parameter λ, then $\phi_X(\theta) = \lambda/(\lambda - i\theta)$.

(d) If X has a Cauchy distribution, then $\phi_X(\theta) = \exp(-|\theta|)$.

28. Read the proof of the Helly-Bray lemma. Show that the function F defined on page 184 is (a) right-continuous and that (b) $\lim F_{n_i}(x) = F(x)$ for all x where F is continuous. *Hint:* Fix x and $\varepsilon > 0$. Then there is $c \in \mathbb{Q}$ such that $F(x) \leq H(c) < F(x) + \varepsilon$. If F is continuous at x, then there exists also $c' < c \in \mathbb{Q}$ and $y < x$ such that $F(x) - \varepsilon \leq F(y) \leq H(c') \leq H(c)$.

29. Let (F_n) be a sequence of distribution functions on $\mathbb{R}$ such that $\lim_{n\to\infty} F_n(x) = F(x)$ for all x where the distribution function F is continuous. Show that $\lim_{n\to\infty} \int_{\mathbb{R}} h\, dF_n = \int_{\mathbb{R}} h\, dF$ for all bounded and continuous $h : \mathbb{R} \to \mathbb{R}$.

30. Let $X, X_1, X_2, \ldots$ be real-valued random variables with $F_{X_n} \xrightarrow{w} F_X$. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous and put $Y = h(X)$ and $Y_n = h(X_n)$ for every $n \in \mathbb{N}$. Show that $F_{Y_n} \xrightarrow{w} F_Y$.

31. Suppose that $X, X_1, X_2, \ldots$ are real valued random variables, defined on one the same probability space, with $X_n \to X$ in probability. Show that $F_{X_n} \xrightarrow{w} F_X$. *Hint: use exercise 16.*

32. Let $\mu, \mu_1, \mu_2, \ldots$ be probability measures on $\mathbb{R}$ and suppose that for any open $G \subset \mathbb{R}$ that $\liminf \mu_n(G) \geq \mu(G)$. Then $\mu_n \to \mu$.Show this as follows. Let h be a bounded continuous function on $\mathbb{R}$. Assume w.l.og. that $0 \leq h < 1$. Let $k \in \mathbb{N}$ and define $F_i = \{x : \frac{i-1}{k} \leq h(x) < \frac{i}{k}\}$. Split $\mu(h)$ into integrals over the F_i. Then

$$\frac{1}{k}\sum_{i=1}^{k} \mu(h > \frac{i}{k}) \leq \mu(h) \leq \frac{1}{k}\sum_{i=1}^{k} \mu(h > \frac{i-1}{k})$$

and something similar for μ_n. Deduce that $\liminf \mu_n(h) \geq \mu(h)$ and complete the proof with the aid of an inequality for $\limsup \mu_n(h)$.

33. Suppose that the real random variables $X, X_1, X_2, \ldots$ are defined on a common probability space and that $F_{X_n} \xrightarrow{w} F_X$. Suppose that $X = x_0$ a.s. for some $x_0 \in \mathbb{R}$. Show that $X_n \to X$ in probability.

34. Let X_n have a $\text{Bin}(n, \lambda/n)$ distribution (for $n > \lambda$). Show that $X_n \xrightarrow{w} X$, where X has a $\text{Poisson}(\lambda)$ distribution.

35. Exercise 18.3

36. Show the following two statements. If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$, then $X_n + Y_n \xrightarrow{a.s.} X + Y$. If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then $X_n + Y_n \xrightarrow{P} X + Y$.

37. Show that $\sum_{n=1}^{\infty} \mathbb{P}(|X| \geq n) \leq \mathbb{E}\,|X| \leq 1 + \sum_{n=1}^{\infty} \mathbb{P}(|X| \geq n)$.

38. Let $X_1, X_2, \ldots$ be an *iid* sequence of random variables with a (common) Cauchy distribution. Find the distribution of the sample means $\overline{X}_n = \sum_{i=1}^{n} X_i/n$ (use characteristic functions). Do the $\overline{X}_n$ obey the strong or weak law of large numbers? If so, what is the limit?

39. Let N have a Poisson(λ) distribution, $Y_1, Y_2, \ldots$ be an *iid* sequence, independent of N. Show that $\phi_X = \exp(\lambda(\phi_Y - 1))$.