Additional exercises

1. Consider an ARMA(p,q) process (Y_t), $\phi(B)Y = \theta(Z)$, and let $r = p \vee q$.

   (a) Show that (Y_t) can be put in state space form as follows.

   $$\begin{aligned} X_{t+1} &= AX_t + BZ_{t+1} && (1) \\ Y_t &= CX_t + Z_t, && (2) \end{aligned}$$

   where

   $$A = \begin{pmatrix} \phi_1 & 1 & 0 & 0 & \cdots & 0 \\ \phi_2 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ \phi_{r-1} & 0 & & & 0 & 1 \\ \phi_r & 0 & & & \cdots & 0 \end{pmatrix},$$

   $$B = \begin{pmatrix} \phi_1 + \theta_1 \\ \vdots \\ \phi_r + \theta_r \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.$$

   Hint: Choose the elements of X_t as $X_{k,t} = \sum_{j \geq k} \phi_j Y_{t+k-j-1} + \sum_{j \geq k} \theta_j Z_{t+k-j-1}$.

   (b) Give a condition on the ϕ-polynomial such that the matrix A is stable (all its eigenvalues are inside the unit disk).

   (c) Show that the $r \times r$ matrix

   $$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{pmatrix} \qquad (3)$$

   is invertible.

   (d) Suppose that the equations (1) and (2) are given for certain A, B and C and that the matrix $\mathcal{O}$ in (3) is invertible. Show that we can represent (Y_t) in ARMA form.

2. Consider a partitioned matrix

   $$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

   and assume that A, D are square and M and D invertible.

   (a) Show the decomposition

   $$M = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}.$$

(b) Compute M^{-1} in partitioned form, in terms of the inverse of the matrix $\Delta = A - BD^{-1}C$.

(c) Apply the preceding to quantify the loss of efficiency when estimating the AR-parameters with a too large order p as in Section 11.1.1. (Make the expression $(\Gamma_p)^{-1}_{s,t=1,\ldots,p_0} - \Gamma_{p_0} \geq 0$ precise.)

3. Prove the Kolmogorov-Szegö formula $\int_{-\pi}^{\pi} \log f_\theta(\lambda)\,\mathrm{d}\lambda = 2\pi \log(\sigma^2/2\pi)$, page 224, for f_θ as given. *Hint:* take first the case $p = 1$ and use the Taylor expansion $\log(1 - z) = \sum_{k=1}(-z)^k/k$, valid for $|z| < 1$.
(*Remark:* In fact, the Kolmogorov-Szegö formula is valid in more generality for $\sigma^2 := \mathrm{E}(X_t - \Pi_{t-1}X_t)^2$ and f_θ replaced with f, the spectral density of a stationary time series. For the causal AR case, we know that $X_t - \Pi_{t-1}X_t = Z_t$.)

4. Consider a stationary, causal and invertible ARMA(p, q) process (X_t) given by $\phi(B)X_t = \theta(B)Z_t$. Write $\theta = (\phi, \theta) = (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q)$ and $I(\theta)$ for the Gaussian Fisher information matrix. Let J_p be the p-dimensional shift matrix with ij-element equal to $\delta_{i,j+1}$ and J_q its q-dimensional sister. Write e_p for the first standard column basis vector of $\mathbb{R}^p$ and likewise define e_q. Finally we have the $(p + q) \times (p + q)$ matrix

$$A(\theta) = \begin{pmatrix} J_p + e_p\phi & 0 \\ 0 & J_q - e_q\theta \end{pmatrix},$$

and $B^\top = \left(e_p^\top, e_q^\top\right)$. Show that $I(\theta)$ satisfies the linear equation

$$I(\theta) = A(\theta)I(\theta)A(\theta)^\top + BB^\top.$$