Central Limit Theorem

(telegram style notes)

P.J.C. Spreij

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1 Introduction

In these notes the main topics are two proofs of the *Central Limit Theorem*. One is based on *smoothing by convolution* and *small disturbances*. The other one is based on manipulations with characteristic functions and Lévy's continuity theorem.

2 The Central Limit Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We will assume that all random variables that we encounter below are defined on this space and real valued. Let $X, X_1, X_2, \ldots$ be random variables and recall that $X_n \xrightarrow{w} X$ was defined as

$$\mathbb{E}\, f(X_n) \to \mathbb{E}\, f(X) \tag{2.1}$$

for all bounded continuous functions f. As a matter of fact one can show that weak convergence takes place, if (2.1) holds for all bounded *uniformly continuous* functions (exercise 4.1). We take this as our characterization of weak convergence. In the sequel $||f||$ denotes the sup norm of a function f.

Lemma 2.1 *Let X and Y be random variables and f a bounded uniformly continuous function. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|\mathbb{E}\, f(X) - \mathbb{E}\, f(X+Y)| \leq \varepsilon + 2||f||\,\mathbb{P}(|Y| \geq \delta). \tag{2.2}$$

Proof Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Then

$$\begin{aligned}|\mathbb{E}\, f(X) - \mathbb{E}\, f(X+Y)| &\leq \mathbb{E}\,(1_{\{|Y|<\delta\}}(f(X) - f(X+Y))) \\ &\quad + \mathbb{E}\,(1_{\{|Y|\geq\delta\}}(f(X) + f(X+Y))) \\ &\leq \varepsilon + 2||f||\,\mathbb{P}(|Y| \geq \delta).\end{aligned}$$

□

Lemma 2.2 *Let $Y, X, X_1, X_2, \ldots$ be random variables such that for all $\sigma > 0$ it holds that $X_n + \sigma Y \xrightarrow{w} X + \sigma Y$. Then also $X_n \xrightarrow{w} X$ (the $\sigma = 0$ case).*

Proof Let a uniformly continuous function f and $\varepsilon > 0$ be given and choose $\delta > 0$ as in the previous lemma. From (2.2) it follows that

$$|\mathbb{E}\, f(X) - \mathbb{E}\, f(X+\sigma Y)| \leq \varepsilon + 2||f||\,\mathbb{P}(|Y| \geq \frac{\delta}{\sigma})$$

and

$$|\mathbb{E}\, f(X_n) - \mathbb{E}\, f(X_n+\sigma Y)| \leq \varepsilon + 2||f||\,\mathbb{P}(|Y| \geq \frac{\delta}{\sigma}).$$

Now we consider

$$\begin{aligned}
|\mathbb{E}\, f(X_n) - \mathbb{E}\, f(X)| &\leq |\mathbb{E}\, f(X_n) - \mathbb{E}\, f(X_n + \sigma Y)| \\
&\quad + |\mathbb{E}\, f(X_n + \sigma Y) - \mathbb{E}\, f(X + \sigma Y)| \\
&\quad + |\mathbb{E}\, f(X) - \mathbb{E}\, f(X + \sigma Y)| \\
&\leq 2\varepsilon + 4||f||\, \mathbb{P}(|Y| \geq \frac{\delta}{\sigma}) \\
&\quad + |\mathbb{E}\, f(X_n + \sigma Y) - \mathbb{E}\, f(X + \sigma Y)|.
\end{aligned}$$

By assumption, the last term tends to zero for $n \to \infty$. Letting then $\sigma \downarrow 0$, we obtain $\limsup_n |\mathbb{E}\, f(X_n) - \mathbb{E}\, f(X)| \leq 2\varepsilon$, which finishes the proof, since ε is arbitrary. □

For small σ, we view $X + \sigma Y$ as a perturbation of X. Let us take a standard normally distributed random variable Y, independent of X and the X_n. Notice that $Z := X + \sigma Y$ given $X = x$ has a $N(x, \sigma^2)$ distribution. Let f be bounded and uniformly continuous. Then $\mathbb{E}\, f(X + \sigma Y) = \mathbb{E}\,\mathbb{E}\,[f(Z)|X]$ and

$$\mathbb{E}\,[f(Z)|X = x] = \int_{-\infty}^{\infty} f(z) \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2\sigma^2}(z-x)^2)\, \mathrm{d}z =: f_\sigma(x).$$

Hence

$$\mathbb{E}\, f(X + \sigma Y) = \mathbb{E}\, f_\sigma(X). \tag{2.3}$$

Let $p_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}x^2)$, the density of a $N(0, \sigma^2)$ distributed random variable. The function f_σ is obtained by convolution of f with the normal density p_σ. By the Dominated Convergence Theorem, one can show (exercise 4.2) that f has derivatives of all orders given by

$$f_\sigma^{(k)}(x) = \int_{-\infty}^{\infty} f(z) p_\sigma^{(k)}(z - x)\, \mathrm{d}z. \tag{2.4}$$

Hence f_σ is a smooth function. Write $\mathcal{C}^\infty$ for the class of bounded functions that have *bounded* derivatives of all orders. Examples of such function are p_σ and f_σ. We have already weakened the requirement for weak convergence that convergence is assumed to hold for expectations involving uniformly continuous functions. The next step is to drastically reduce this class of functions.

Theorem 2.3 *Let $X, X_1, X_2, \ldots$ be random variables. The weak convergence $X_n \xrightarrow{w} X$ takes place iff $\mathbb{E}\, f(X_n) \to \mathbb{E}\, f(X)$, for all $f \in \mathcal{C}^\infty$.*

Proof Suppose that $\mathbb{E}\, f(X_n) \to \mathbb{E}\, f(X)$, for all $f \in \mathcal{C}^\infty$, then it holds in particular for any f_σ. In view of (2.3), this means that $X_n + \sigma Y \xrightarrow{w} X + \sigma Y$ for all $\sigma > 0$. Now lemma 2.2 applies. □

As a preparation for the proof of the Central Limit Theorem we proceed with some analytic technicalities that eventually lead to the crucial inequality (2.8). Let $f \in \mathcal{C}^\infty$ and put

$$R(x,y) = f(x+y) - f(x) - yf'(x) - \frac{1}{2}y^2 f''(x).$$

Replacing x and y above by independent random variables X and Y and taking expectations, then yields

$$\mathbb{E}\, f(X+Y) - \mathbb{E}\, f(X) - \mathbb{E}\, Y\, \mathbb{E}\, f'(X) - \frac{1}{2}\mathbb{E}\, Y^2\, \mathbb{E}\, f''(X)| = \mathbb{E}\, R(X,Y).$$

Let W be another random variable, independent of X, and assume that $\mathbb{E}\, W = \mathbb{E}\, Y$ and $\mathbb{E}\, W^2 = \mathbb{E}\, Y^2$. Then a similar equality is valid and we then obtain by taking the difference the inequality

$$|\mathbb{E}\, f(X+Y) - \mathbb{E}\, f(X+W)| \leq \mathbb{E}\,|R(X,Y)| + \mathbb{E}\,|R(X,W)|. \tag{2.5}$$

We are now going to find bounds on the remainder terms in this equation. The mean value theorem yields for any x and y that $R(x,y) = \frac{1}{6}y^3 f'''(\theta_1(x,y))$ for some $\theta_1(x,y)$ between x and $x+y$. Alternatively, we can express $R(x,y)$ by another application of the mean value theorem as

$$R(x,y) = f(x+y) - f(x) - yf'(x)) - \frac{1}{2}y^2 f''(x) = \frac{1}{2}y^2(f''(\theta_2(x,y)) - f''(x)),$$

for some $\theta_2(x,y)$ between x and $x+y$. Let $C = \max\{||f''||, \frac{1}{6}||f'''||\}$. Then we have the estimate $|R(x,y)| \leq C|y|^3$, as well as for every $\varepsilon > 0$ the estimate

$$|R(x,y)| \leq C(|y|^3 1_{\{|y|<\varepsilon\}} + y^2 1_{\{|y|\geq\varepsilon\}}) \leq Cy^2(\varepsilon + 1_{\{|y|\geq\varepsilon\}}).$$

Hence we have the following bounds on $\mathbb{E}\,|R(X,Y)|$:

$$\mathbb{E}\,|R(X,Y)| \leq C\mathbb{E}\,|Y|^3 \tag{2.6}$$

and

$$\mathbb{E}\,|R(X,Y)| \leq C(\varepsilon\mathbb{E}\, Y^2 + \mathbb{E}\, Y^2 1_{\{|Y|\geq\varepsilon\}}). \tag{2.7}$$

The proof of the Central Limit Theorem is based on the following idea. Consider a sum of independent random variables $S = \sum_{j=1}^n \xi_j$, where n is 'big'. If we replace one of the ξ_j by another random variable, then we can think of a small perturbation of S and the expectation of $f(S)$ will hardly change. This idea will be repeatedly used, all the ξ_j that sum up to S will be step by step replaced with other, *normally distributed*, random variables. We assume that the ξ_j are have finite second moments and expectation zero. Let $\eta_1, \ldots, \eta_n$ be independent normal random variables, also independent of the ξ_j, with expectation zero and $\mathbb{E}\,\eta_j^2 = \mathbb{E}\,\xi_j^2$. Put $Z = \sum_{j=1}^n \eta_j$ and notice that also Z has a normal distribution with variance equal to $\sum_{j=1}^n \mathbb{E}\,\xi_j^2$. We are interested in $\mathbb{E}\, f(S) - \mathbb{E}\, f(Z)$. The following notation is convenient. Put $X_j = \sum_{i=1}^{j-1} \xi_i + \sum_{i=j+1}^{n} \eta_i$. Notice that $S = X_n + \xi_n$ and $Z = X_1 + \eta_1$. Repetitive use of the triangle inequality and application of (2.5) gives

$$
\begin{aligned}
|\mathbb{E}\, f(S) - \mathbb{E}\, f(Z)| &\leq \sum_{j=1}^{n} |\mathbb{E}\, f(X_j + \xi_j) - \mathbb{E}\, f(X_j + \eta_j)| \\
&\leq \sum_{j=1}^{n} |\mathbb{E}\, R(X_j, \xi_j) + \mathbb{E}\, R(X_j, \eta_j)|. \qquad (2.8)
\end{aligned}
$$

Theorem 2.4 (Central Limit Theorem) *Let for each $n \in \mathbb{N}$ be given a sequence $\xi_{n1}, \ldots, \xi_{nk_n}$ of independent random variables with $\mathbb{E}\, \xi_{nj} = 0$ and $\sum_{j=1}^{k_n} \mathrm{Var}\, \xi_{nj} = 1$. Let for every $\varepsilon > 0$*

$$L_n(\varepsilon) = \sum_{j=1}^{k_n} \mathbb{E}|\xi_{nj}|^2 1_{\{|\xi_{nj}| > \varepsilon\}}.$$

Suppose that the Lindeberg condition *holds: $L_n(\varepsilon) \to 0$ as $n \to \infty$ for every $\varepsilon > 0$. Then $S_n := \sum_{j=1}^{k_n} \xi_{nj} \xrightarrow{w} Z$, where Z has a N(0,1) distribution.*

Proof Let $S_n = \sum_{j=1}^{k_n} \xi_{nj}$ and let η_{nj} $(j = 1, \ldots, k_n,\ n \in \mathbb{N})$ be a double array of zero mean normal random variables, independent of all the ξ_{nj}, such that also for every n the η_{nj} $(j = 1, \ldots, k_n)$ are independent and such that $\mathbb{E}\, \eta_{nj}^2 = \mathbb{E}\, \xi_{nj}^2$. Let $Z_n = \sum_{j=1}^{k_n} \eta_{nj}$. Notice that the distributions of the Z_n are all standard normal and thus $\mathbb{E}\, f(Z_n) = \mathbb{E}\, f(Z)$ for every f in $\mathcal{C}^\infty$. Recall theorem 2.3. Take such $f \in \mathcal{C}^\infty$ and apply (2.8) to get

$$
\begin{aligned}
|\mathbb{E}\, f(S_n) - \mathbb{E}\, f(Z)| &= |\mathbb{E}\, f(S_n) - \mathbb{E}\, f(Z_n)| \\
&\leq \sum_{j=1}^{k_n} \mathbb{E}\, |R(X_{nj}, \xi_{nj})| + \mathbb{E}\, |R(X_{nj}, \eta_{nj})|, \qquad (2.9)
\end{aligned}
$$

with an obvious meaning of the X_{nj}. For the first error terms in (2.9) we use the estimate of (2.7) which yields $\mathbb{E} \sum_{j=1}^{k_n} \mathbb{E}\, |R(X_{nj}, \xi_{nj})| \leq C(\varepsilon + L_n(\varepsilon))$. In view of the Lindeberg condition, this term can be made arbitrarily small. We now focus on the second error term in (2.9). Let $\sigma_{nj}^2 = \mathbb{E}\, \xi_{nj}^2 = \mathbb{E}\, \eta_{nj}^2$ and use (2.6) to obtain

$$\mathbb{E} \sum_{j=1}^{k_n} \mathbb{E}\, |R(X_{nj}, \eta_{nj})| \leq C \sum_{j=1}^{k_n} \mathbb{E}\, |\eta_{nj}|^3 = C \sum_{j=1}^{k_n} \sigma_{nj}^3 \mathbb{E}\, |N(0,1)|^3.$$

To finish the proof, we first observe that

$$\max_j \sigma_{nj}^2 = \max_j \mathbb{E}\, \xi_{nj}^2 = \max_j \mathbb{E}\, \xi_{nj}^2 (1_{\{|\xi_{nj}| \leq \varepsilon\}} + 1_{\{|\xi_{nj}| > \varepsilon\}}) \leq \varepsilon^2 + L_n(\varepsilon).$$

Hence (use $\sum_{j=1}^{k_n} \sigma_{nj}^2 = 1$)

$$\sum_{j=1}^{k_n} \sigma_{nj}^3 \leq \max_j \sigma_{nj} \sum_{j=1}^{k_n} \sigma_{nj}^2 \leq (\varepsilon^2 + L_n(\varepsilon))^{1/2}.$$

And, again, this term can be made arbitrarily small, because of the Lindeberg condition. □

3 Another proof of the Central Limit Theorem

The proof of the Central Limit Theorem that we present in this section is a classical one, just as the proof of the previous section. It is based on properties of characteristic functions, the first ones are contained in the following lemma.

Lemma 3.1 *Let X be a random variable with $\mathbb{E}\, X^2 < \infty$ and with characteristic function ϕ. Then*

$$|\phi(u) - 1| \leq \mathbb{E}\, \min\{2, |uX|\},$$

$$|\phi(u) - 1 - \mathrm{i}u\mathbb{E}\, X| \leq \mathbb{E}\, \min\{|u||X|, \frac{1}{2}u^2\mathbb{E}\, X^2\}$$

and

$$|\phi(u) - 1 - \mathrm{i}u\mathbb{E}\, X + \frac{1}{2}u^2\mathbb{E}\, X^2| \leq \mathbb{E}\, \min\{u^2X^2, \frac{1}{6}|u|^3|X|^3\}.$$

Proof Let $x \in \mathbb{R}$. Then $|e^{\mathrm{i}x} - 1| \leq 2$ and $|e^{\mathrm{i}x} - 1| = |\int_0^x \mathrm{i}e^{\mathrm{i}y}\, \mathrm{d}y| \leq |x|$. Hence $|e^{\mathrm{i}x} - 1| \leq \min\{2, |x|\}$. Since

$$e^{\mathrm{i}x} - 1 - \mathrm{i}x = \int_0^x (e^{\mathrm{i}y} - 1)\, \mathrm{d}y,$$

and

$$e^{\mathrm{i}x} - 1 - \mathrm{i}x + \frac{1}{2}x^2 = -\int_0^x \int_0^y (e^{\mathrm{i}t} - 1)\, \mathrm{d}t\, \mathrm{d}y,$$

we can for instance use the last inequality to arrive at $|e^{\mathrm{i}x} - 1 - \mathrm{i}x + \frac{1}{2}x^2| \leq \min\{x^2, |x|^3/6\}$. Replacing x with uX and taking expectations yields the assertions. □

We are now ready to give the announced proof of theorem 2.4.

Proof of theorem 2.4 Let $\phi_{nj}(u) = \mathbb{E}\, \exp(\mathrm{i}u\xi_{nj})$ and $\phi_n(u) = \mathbb{E}\, \exp(\mathrm{i}uS_n)$. Because of independence we have

$$\phi_n(u) = \prod_{j=1}^{k_n} \phi_{nj}(u).$$

First we show that

$$\sum_{j=1}^{k_n}(\phi_{nj}(u)-1)\to -\frac{1}{2}u^2. \tag{3.10}$$

We write

$$\sum_{j=1}^{k_n}(\phi_{nj}(u)-1)=\sum_{j=1}^{k_n}(\phi_{nj}(u)-1+\frac{1}{2}u^2\mathbb{E}\,\xi_{nj}^2)-\sum_{j=1}^{k_n}\frac{1}{2}u^2\mathbb{E}\,\xi_{nj}^2.$$

The last term gives the desired limit, so it suffices to show that the first term converges to zero. By virtue of lemma 3.1, we can bound its absolute value by

$$\sum_{j=1}^{k_n}\mathbb{E}\,\min\{u^2\xi_{nj}^2,\frac{1}{6}|u|^3|\xi_{nj}|^3\}. \tag{3.11}$$

But

$$\mathbb{E}\,\min\{u^2\xi_{nj}^2,\frac{1}{6}|u|^3|\xi_{nj}|^3\}\le\frac{1}{6}|u|^3\varepsilon\mathbb{E}\,\xi_{nj}^2 1_{\{|\xi_{nj}|\le\varepsilon\}}+u^2\mathbb{E}\,\xi_{nj}^2 1_{\{|\xi_{nj}|>\varepsilon\}}.$$

Hence we get that the expression in (3.11) is majorized by

$$\frac{1}{6}|u|^3\varepsilon\sum_{j=1}^{k_n}\mathbb{E}\,\xi_{nj}^2+u^2L_n(\varepsilon)=\frac{1}{6}|u|^3\varepsilon+u^2L_n(\varepsilon),$$

which tends to $\frac{1}{6}|u|^3\varepsilon$. Since ε is arbitrary, we have proved (3.10). It then also follows that

$$\exp(\sum_{j=1}^{k_n}(\phi_{nj}(u)-1))\to\exp(-\frac{1}{2}u^2). \tag{3.12}$$

Recall that $u\mapsto\exp(-\frac{1}{2}u^2)$ is the characteristic function of $N(0,1)$. Hence, by application of Lévy's continuity theorem and (3.12), we are finished as soon as we have shown that

$$\prod_{j=1}^{k_n}\phi_{nj}(u)-\exp(\sum_{j=1}^{k_n}(\phi_{nj}(u)-1))\to 0. \tag{3.13}$$

The displayed difference is less than

$$\sum_{j=1}^{k_n}|\phi_{nj}(u)-\exp((\phi_{nj}(u)-1))|, \tag{3.14}$$

because of the following elementary result: if a_i and b_i are complex numbers with norm less than or equal to one, then

$$|\prod_{i=1}^{n}a_i-\prod_{i=1}^{n}b_i|\le\sum_{i=1}^{n}|a_i-b_i|.$$

To apply this result we have to understand that the complex numbers involved indeed have norm less than or equal to one. For the $\phi_{nj}(u)$ this is one of the basic properties of characteristic functions. But it turns out that $\exp(\phi_{nj}(\cdot)-1)$ is a characteristic function as well (see exercise 4.8).
Let $M_n(u) = \max_j |\phi_{nj}(u) - 1|$. Now we use the inequality $|e^z - 1 - z| \leq |z|^2 e^{|z|}$ (which easily follows from a Taylor expansion) with $z = \phi_{nj}(u) - 1$ to bound (3.14) by

$$\sum_{j=1}^{k_n} |\phi_{nj}(u) - 1|^2 \exp(|\phi_{nj}(u) - 1|) \leq M_n(u) e^{M_n(u)} \sum_{j=1}^{k_n} |\phi_{nj}(u) - 1|.$$

From lemma 3.1, second assertion, we get $\sum_{j=1}^{k_n} |\phi_{nj}(u) - 1| \leq \frac{1}{2}u^2 \sum_{j=1}^{k_n} \mathbb{E}\,\xi_{nj}^2 = \frac{1}{2}u^2$. On the other hand, we have $\max_j \operatorname{Var}\xi_{nj} = \max_j \mathbb{E}\,\xi_{nj}^2 \leq \varepsilon^2 + L_n(\varepsilon)$. Hence

$$\max_j \mathbb{E}\,\xi_{nj}^2 \to 0 \tag{3.15}$$

and then by lemma 3.1 and Jensen's inequality

$$M_n(u) = \max_j |\phi_{nj}(u) - 1| \leq \max_j |u| \mathbb{E}\,|\xi_{nj}| \leq |u| (\max_j \mathbb{E}\,\xi_{nj}^2)^{1/2} \to 0.$$

This proves (3.13) and hence it completes the proof of the theorem. □

Remark 3.2 The Lindeberg condition in the theorem is almost necessary. One can show that if (3.15) holds and if the weak convergence as in the theorem takes place, then also the Lindeberg condition is satisfied.

4 Exercises

4.1 Show that $X_n \xrightarrow{w} X$ iff $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$ for all bounded uniformly continuous functions f.

4.2 Show, using the Dominated Convergence Theorem, that (2.4) holds. Show also that all the derivatives are bounded functions.

4.3 For each n we have a sequence $\xi_{n1}, \ldots, \xi_{nk_n}$ of independent random variables with $\mathbb{E}\xi_{nj} = 0$ and $\sum_{j=1}^{k_n} \operatorname{Var}\xi_{nj} = 1$. If $\sum_{j=1}^{k_n} \mathbb{E}|\xi_{nj}|^{2+\delta} \to 0$ as $n \to \infty$ for some $\delta > 0$, then $\sum_{j=1}^{k_n} \xi_{nj} \xrightarrow{w} N(0,1)$. Show that this follows from theorem 2.4.

4.4 The classical central limit theorem says that $\frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{n} (X_j - \mu) \xrightarrow{w} N(0,1)$, if the X_j are *iid* with $\mathbb{E}X_j = \mu$ and $0 < \operatorname{Var} X_j = \sigma^2 < \infty$. Show that this follows from theorem 2.4.

4.5 Let X and Y be independent, assume that Y has a $N(0,1)$ distribution. Let $\sigma > 0$. Let ϕ be the characteristic function of X: $\phi(u) = \mathbb{E}\exp(\mathrm{i}uX)$.
(a) Show that $Z = X + \sigma Y$ has density $p(z) = \frac{1}{\sigma\sqrt{2\pi}} \mathbb{E}\exp(-\frac{1}{2\sigma^2}(z - X)^2)$.
(b) Show that $p(z) = \frac{1}{2\pi\sigma} \int \phi(-y/\sigma) \exp(\mathrm{i}yz/\sigma - \frac{1}{2}y^2)\,\mathrm{d}y$.

4.6 Let $X, X_1, X_2, \ldots$ be a sequence of random variables and Y a $N(0,1)$-distributed random variable independent of that sequence. Let ϕ_n be the characteristic function of X_n and ϕ that of X. Let p_n be the density of $X_n + \sigma Y$ and p the density of $X + \sigma Y$.
(a) If $\phi_n \to \phi$ pointwise, then $p_n \to p$ pointwise. Invoke the previous exercise and the dominated convergence theorem to show this.
(b) Let $f \in C_b(\mathbb{R})$ be bounded by B. Show that

$$|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| \leq 2B \int (p(z) - p_n(z))^+ \, \mathrm{d}z.$$

(c) Show that $|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| \to 0$ if $\phi_n \to \phi$ pointwise.
(d) Prove the following theorem: $X_n \xrightarrow{w} X$ iff $\phi_n \to \phi$ pointwise.

4.7 Let $X_1, X_2, \ldots, X_n$ be an *iid* sequence having a distribution function F, a continuous density (w.r.t. Lebesgue measure) f. Let m be such that $F(m) = \frac{1}{2}$. Assume that $f(m) > 0$ and that n is odd, $n = 2k - 1$, say ($k = \frac{1}{2}(n+1)$).
(a) Show that m is the unique solution of the equation $F(x) = \frac{1}{2}$. We call m the *median* of the distribution of X_1.
(b) The *sample median* M_n of $X_1, \ldots, X_n$ is by definition X_k. Show that with $U_{nj} = 1_{\{X_j \leq m + n^{-1/2}x\}}$ we have

$$\mathbb{P}(n^{1/2}(M_n - m) \leq x) = \mathbb{P}(\sum_j U_{nj} \geq k).$$

(c) Let $p_n = \mathbb{E}\, U_{nj}$, $b_n = (np_n(1-p_n))^{1/2}$, $\xi_{nj} = (U_{nj} - p_n)/b_n$, $Z_n = \sum_{j=1}^n \xi_{nj}$, $t_n = (k - np_n)/b_n$. Rewrite the probabilities in the previous part as $\mathbb{P}(Z_n \geq t_n)$ and show that $t_n \to t := -2xf(m)$.
(d) Show that $\mathbb{P}(Z_n \geq t) \to 1 - \Phi(t)$, where Φ is the standard normal distribution.
(e) Show that $\mathbb{P}(Z_n \geq t_n) \to \Phi(2f(m)x)$ and conclude that the *Central Limit Theorem for the sample median* holds:

$$2f(m)n^{1/2}(M_n - m) \xrightarrow{w} N(0,1).$$

4.8 Let $X_1, X_2, \ldots$ be a sequence of *iid* random variables and N a Poisson(λ) distributed random variable, independent of the X_n. Put $Y = \sum_{n=1}^N X_n$. Let ϕ be the characteristic function of the X_n and ψ the characteristic function of Y. Show that $\psi = \exp(\lambda\phi - \lambda)$.

4.9 Let Y be a random variable with a Gamma$(t,1)$ distribution, so it has density $\frac{1}{\Gamma(t)} y^{t-1} e^{-y} 1_{\{y>0\}}$, where $\Gamma(t) = \int_0^\infty y^{t-1}e^{-y}\,\mathrm{d}y$ for $t > 0$. Put $X_t = \frac{Y-t}{\sqrt{t}}$.
(a) Show that X_t has a density on $(-\sqrt{t}, \infty)$ given by

$$f_t(x) = \frac{\sqrt{t}}{\Gamma(t)}(x\sqrt{t} + t)^{t-1} e^{-(x\sqrt{t}+t)}.$$

(b) Show that the characteristic function $\phi_t(u) = \mathbb{E}\, e^{iuX_t}$ of X_t is given by

$$\phi_t(u) = e^{-\mathrm{i}u\sqrt{t}} \frac{1}{(1 - \frac{\mathrm{i}u}{\sqrt{t}})^t}$$

and conclude that $\phi_t(u) \to e^{-\frac{1}{2}u^2}$ as $t \to \infty$.
(c) Show that

$$\frac{t^{t-\frac{1}{2}} e^{-t}}{\Gamma(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_t(u)\, \mathrm{d}u.$$

(d) Prove Stirling's formula

$$\lim_{t\to\infty} \frac{\Gamma(t)}{\sqrt{2\pi} e^{-t} t^{t-\frac{1}{2}}} = 1.$$