Exercises Measure Theoretic Probability 2002-2003

Week 1

1. Prove the following statements.
   (a) The intersection of an arbitrary family of d-systems is again a d-system.
   (b) The intersection of an arbitrary family of σ-algebras is again a σ-algebra. Characterize $\sigma(\mathcal{C})$ for a given collection $\mathcal{C} \subset 2^\Omega$.
   (c) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are collections of subsets of Ω with $\mathcal{C}_1 \subset \mathcal{C}_2$, then $d(\mathcal{C}_1) \subset d(\mathcal{C}_2)$.
2. Let $\mathcal{G}$ and $\mathcal{H}$ be two σ-algebras on Ω. Let $\mathcal{C} = \{G \cap H : G \in \mathcal{G}, H \in \mathcal{H}\}$. Show that $\mathcal{C}$ is a π-system and that $\sigma(\mathcal{C}) = \sigma(\mathcal{G}, \mathcal{H})$.
3. Show that $\mathcal{D}_2$ (Williams, page 194) is a π-system.
4. If h_1 and h_2 are measurable functions, then $h_1 h_2$ is measurable too.
5. Let Ω be a countable set. Let $\mathcal{F} = 2^\Omega$ and let $p : \Omega \to [0, 1]$ satisfy $\sum_{\omega \in \Omega} p(\omega) = 1$. Put $\mathbb{P}(A) = \sum_{\omega \in A} p(\omega)$ for $A \in \mathcal{F}$. Show that $\mathbb{P}$ is a probability measure.
6. Let Ω be a countable set. Let $\mathcal{A}$ be the collection of $A \subset \Omega$ such that A or its complement has finite cardinality. Show that A is an algebra. What is $d(\mathcal{A})$?

Week 2

1. Let X be a random variable. Show that $\Pi(X) := \{X^{-1}(-\infty, x] : x \in \mathbb{R}\}$ is a π-system and that it generates $\sigma(X)$.
2. Let $\{Y_\gamma : \gamma \in C\}$ be an arbitrary collection of random variables and $\{X_n : n \in \mathbb{N}\}$ be a countable collection of random variables, all defined on the same probability space.
   (a) Show that $\sigma\{Y_\gamma : \gamma \in C\} = \sigma\{Y_\gamma^{-1}(B) : \gamma \in C, B \in \mathcal{B}\}$.
   (b) Let $\mathcal{X}_n = \sigma\{X_1, \ldots, X_n\}$ $(n \in \mathbb{N})$ and $\mathcal{A} = \bigcup_{n=1}^\infty \mathcal{X}_n$. Show that $\mathcal{A}$ is an algebra and that $\sigma(\mathcal{A}) = \sigma\{X_n : n \in \mathbb{N}\}$.
3. Show that the X^+ and X^- are measurable functions and that X^+ is right-continuous and X^- is left-continuous (notation as in section 3.12).
4. Williams, exercise E4.1.

5. Williams, exercise E4.6.

6. Let $\mathcal{F}$ be a σ-algebra on Ω with the property that for all $F \in \mathcal{F}$ it holds that $\mathbb{P}(F) \in \{0, 1\}$. Let $X : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Show that for some $c \in \mathbb{R}$ one has $\mathbb{P}(X = c) = 1$. (*Hint:* $\mathbb{P}(X \leq x) \in \{0, 1\}$ for all x.)

Week 3

1. Let X and Y be simple random variables. Show that $\mathbb{E}\,X$ doesn't depend on the chosen representation of X. Show also that $\mathbb{E}\,(X+Y) = \mathbb{E}\,X + \mathbb{E}\,Y$.

2. Show that the expectation is a linear operator on $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

3. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}, \text{Leb})$ and let $f, f_1, f_2, \ldots$ be densities (nonnegative measurable functions that integrate to 1) on $[0,1]$. Assume that $f_n \to f$ a.s. Show that for all $x \in [0,1]$ it holds that $F_n(x) \to F(x)$, where $F_n(x) = \int_{[0,x]} f_n(t)\,dt$ and $F(x) = \int_{[0,t]} f(t)\,dx$.

4. (a) Show that for $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ it holds that $|\mathbb{E}\,X| \leq \mathbb{E}\,|X|$.
   (b) Prove the second part of Scheffé's lemma for random variables X and X_n (see page 55).

5. Let $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Show that $\lim_{n\to\infty} n\mathbb{P}(|X| > n) = 0$.

Week 4

1. Prove the assertions (a)-(d) of section 5.14.

2. Prove the assertions (a)-(c) of section 6.5.

3. Prove lemma 6.12 (you find the 'standard machine' in section 5.12).

4. Prove part (b) of Fubini's theorem in section 8.2.

5. Complete the proof of theorem 6.11: show the a.s. uniqueness of Y and show that if $X - Y \perp Z$ for all Z in $\mathcal{K}$, then $||X - Y||_2 = \inf\{||X - Y'||_2 : Y' \in \mathcal{K}\}$.

Week 5

1. Finish the proof of theorem 9.2: Take arbitrary $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and show that the existence of the conditional expectation of X follows from the existence of the conditional expectations of X^+ and X^-.

2. Prove the conditional version of Fatou's lemma, property (f) on page 88.

3. Prove the conditional Dominated Convergence theorem, property (g) on page 88.

4. Exercise E9.1.
5. Exercise E9.2
6. Let (X,Y) have a bivariate normal distribution with $\mathbb{E}\,X = \mu_X$, $\mathbb{E}\,Y = \mu_Y$, $\operatorname{Var} X = \sigma_X^2$, $\operatorname{Var} Y = \sigma_Y^2$ and $\operatorname{Cov}(X,Y) = c$. Let

$$\hat{X} = \mu_x + \frac{c}{\sigma_Y^2}(Y - \mu_Y).$$

   Show that $\mathbb{E}\,(X - \hat{X})Y = 0$. Show also (use a special property of the bivariate normal distribution) that $\mathbb{E}\,(X - \hat{X})g(Y) = 0$ if g is a Borel-measurable function such that $\mathbb{E}\,g(Y)^2 < \infty$. Conclude that $\hat{X}$ is a version of $\mathbb{E}\,[X|Y]$.

Week 6

In all exercises below we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F}$.

1. Let X be an adapted process and T a stopping time that is finite. Show that X_T is $\mathcal{F}$-measurable. Show also that for arbitrary stopping times T (so the value infinity is also allowed) the stopped process X^T is adapted.
2. Show that an adapted process X is a martingale iff $\mathbb{E}[X_{n+m}|\mathcal{F}_n] = X_n$ for all $n, m \geq 0$.
3. Read first the definition of a submartingale. Let f be a convex function on $\mathbb{R}$. Suppose that X is a martingale and that $\mathbb{E}|f(X_n)| < \infty$ for all n. Show that the process $(f(X_n))_{n\geq 0}$ is a submartingale.
4. For every n we have a measurable function f_n on $\mathbb{R}^n$. Let $Z_1, Z_2, \ldots$ be independent random variables and $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. Show that (you may assume sufficient integrability) that $X_n = f_n(Z_1, \ldots, Z_n)$ defines a martingale under the condition that $\mathbb{E} f_n(z_1, \ldots, z_{n-1}, Z_n) = f_{n-1}(z_1, \ldots, z_{n-1})$ for every n.
5. If S and T are stopping times, then also $S + T$, $S \vee T$ and $S \wedge T$ are stopping times. Show this.

Week 7

In all exercises below we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F}$.

1. (a) If X is a martingale is and f a convex function such that $\mathbb{E}|f(X_n)| < \infty$, then Y defined by $Y_n = f(X_n)$ is a submartingale. Show this (keyword: Jensen).

(b) Show that Y is a submartingale, if X is one and f is an increasing function.

2. Prove Corollaries (c) and (d) on page 101.

3. Show that the process C of section 11.1 is previsible.

4. Let X be a submartingale with $\sup_{n\geq 0} \mathbb{E}|X_n| < \infty$. Show that there exists a random variable X_∞ such that $X_n \to X_\infty$ a.s.

5. Show that for a supermartingale X the condition $\sup\{\mathbb{E}\,|X_n| : n \in \mathbb{N}\} < \infty$ is equivalent to the condition $\sup\{\mathbb{E}\,X_n^- : n \in \mathbb{N}\} < \infty$.

6. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0,1)$, $\mathcal{F}$ the Borelsets of $[0,1)$ and $\mathbb{P}$ the Lebesgue measure. Let $I_k^n = [k2^{-n}, (k+1)2^{-n})$ for $k = 0, \ldots, 2^n - 1$ and $\mathcal{F}_n$ be the σ-algebra by the I_k^n for $k = 0, \ldots, 2^n - 1$. Define $X_n = 1_{I_0^n} 2^n$. Show that X_n is a martingale is and that the conditions of theorem 11.5 are satisfied. What is X_∞ in this case? Do we have $X_n \stackrel{\mathcal{L}^1}{\to} X_\infty$? (This has something to do with 11.6).

Week 8

1. Let $Y \in \mathcal{L}^1$, $(\mathcal{F}_n)$ and define for all $n \in \mathbb{N}$ the random variable $X_n = \mathbb{E}\,[Y|\mathcal{F}_n]$. We know that there is X_∞ such that $X_n \to X_\infty$ a.s. Show that for $Y \in \mathcal{L}^2$, we have $X_n \stackrel{\mathcal{L}^2}{\to} X_\infty$. Find a condition such that $X_\infty = Y$. Give also an example in which $P(X_\infty = Y) = 0$.

2. Show that every finite collection in $\mathcal{L}^1$ is uniformly integrable.

3. Williams, exercise E13.1.

4. Let $\mathcal{C}$ be a uniformly integrable collection of random variables.

   (a) Consider $\bar{\mathcal{C}}$, the closure of $\mathcal{C}$ in $\mathcal{L}^1$. Use E13.1 to show that also $\bar{\mathcal{C}}$ is uniformly integrable.

   (b) Let $\mathcal{D}$ be the convex hull of $\mathcal{C}$. Then both $\mathcal{D}$ and its closure in $\mathcal{L}^1$ are uniformly integrable

5. In this exercise you prove (fill in the details) the following characterization: a collection $\mathcal{C}$ is uniformly integrable iff there exists a function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$ and $M := \sup\{\mathbb{E}G(|X|) : X \in \mathcal{C}\} < \infty$.
   The necessity you prove as follows. Let $\varepsilon > 0$ choose $a = M/\varepsilon$ and c such that $\frac{G(t)}{t} \geq a$ for all $t > c$. To prove uniform integrability of $\mathcal{C}$ you use that $|X| \leq \frac{G(|X|)}{a}$ on the set $\{|X| \geq c\}$.
   It is less easy to prove sufficieny. Proceed as follows. Suppose that we have a sequence (g_n) with $g_0 = 0$ and $\lim_{n\to\infty} g_n = \infty$. Define $g(t) = \sum_{n=0}^\infty 1_{[n,n+1)}(t) g_n$ and $G(t) = \int_0^t g(s)ds$. Check that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$.

With $a_n(X) = \mathbb{P}(|X| > n)$, it holds that $\mathbb{E}\,G(|X|) \le \sum_{n=1}^\infty g_n a_n(|X|)$. Furthermore, for every $k \in \mathbb{N}$ we have $\int_{|X|\ge k} |X|\,d\mathbb{P} \ge \sum_{m=k}^\infty a_m(X)$. Pick for every n a constant $c_n \in \mathbb{N}$ such that $\int_{|X|\ge c_n} |X|\,d\mathbb{P} \le 2^{-n}$. Then $\sum_{m=c_n}^\infty a_m(X) \le 2^{-n}$ and hence $\sum_{n=1}^\infty \sum_{m=c_n}^\infty a_m(X) \le 1$. Choose then the sequence (g_n) as the 'inverse' of (c_n): $g_n = \#\{k : c_k \le n\}$.

6. Prove that a collection $\mathcal{C}$ is uniformly integrable iff there exists an *increasing and convex* function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$ and $M := \sup\{G(|X|) : X \in \mathcal{C}\} < \infty$.
   Let $\mathcal{D}$ be the closure of the convex hull of a uniformly integrable collection $\mathcal{C}$ in $\mathcal{L}^1$. With the function G as above we have $\sup\{\mathbb{E}G(|X|) : X \in \mathcal{D}\} = M$, whence also $\mathcal{D}$ is unifromly integrable.

7. Let $p \ge 1$ and let $X, X_1, X_2, \ldots$ be random variables. Then X_n converges to X in $\mathcal{L}^p$ iff the following two conditions are satisfied.
   (a) $X_n \to X$ in probability,
   (b) The collection $\{|X_n|^p : n \in \mathbb{N}\}$ is uniformly integrable.

Week 9

1. Show that R, defined on page 142 of Williams, is equal to $q\mathbb{E}\,||X^{p-1}Y||$. Show also that the hypothesis of lemma 14.10 is true for $X \wedge n$ if it is true for X and complete the proof of this lemma.

2. Exercise E14.2 of Williams.

3. Let $X = (X_n)_{n\le 0}$ a (backward) supermartingale.
   (a) Show equivalence of the next two properties:
   (i) $\sup_n \mathbb{E}|X_n| < \infty$ and (ii) $\lim_{n\to-\infty} \mathbb{E}X_n < \infty$.
   (Use that $x \mapsto x^+$ is convex and increasing.)
   (b) Under the condition $\sup_n \mathbb{E}|X_n| =: A < \infty$ the supermartingale X is uniformly integrable. To show this, you may proceed as follows (*but other solutions are equally welcome*). Let $\varepsilon > 0$ and choose $K \in \mathbb{Z}$ such that for all $n < K$ one has $0 \le \mathbb{E}X_n - \mathbb{E}X_K < \varepsilon$. It is then sufficient to show that $(X_n)_{n\le K}$ is uniformly integrable. Let $c > 0$ be arbitrary and $F_n = \{|X_n| > c\}$. Using the supermartingale inequality you show that

$$\int_{F_n} |X_n|\,d\mathbb{P} \le \int_{F_n} |X_K|\,d\mathbb{P} + \varepsilon.$$

   Because $\mathbb{P}(F_n) \le \frac{A}{c}$ you conclude the proof.

Week 10

1. Let $\mu, \mu_1 \mu_2, \ldots$ be probability measures on $\mathbb{R}$. Show that $\mu_n \xrightarrow{w} \mu$ iff for all bounded Lipschitz continuous functions one has $\int f \, d\mu_n \to \int f \, d\mu$. (*Hint: for one implication the proof of lemma 17.2 is instructive.*)

2. Show the 'if part' of lemma 17.2.

3. If the random variables $X, X_1, X_2, \ldots$ are defined on the same probability space and if $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{w} X$. Prove this.

4. Suppose that $X_n \xrightarrow{w} X$ and that the collection $\{X_n, n \geq 1\}$ is uniformly integrable (you make a minor change in the definition of this notion if the X_n are defined on different probability spaces). Use the Skorohod representation to show that $X_n \xrightarrow{w} X$ implies $\mathbb{E}X_n \to \mathbb{E}X$.

5. Show the following variation on Fatou's lemma: if $X_n \xrightarrow{w} X$, then $\mathbb{E}|X| \leq \liminf_{n\to\infty} \mathbb{E}|X_n|$.

6. Show that the weak limit of a sequence of probability measures is unique.

Week 11

1. Consider the $N(\mu_n, \sigma_n^2)$ distributions, where the μ_n are real numbers and the σ_n^2 nonnegative. Show that this family is tight iff the sequences (μ_n) and (σ_n^2) are bounded. Under what condition do we have that the $N(\mu_n, \sigma_n^2)$ distributions converge to a (weak) limit? What is this limit?

2. For each n we have a sequence $\xi_{n1}, \ldots, \xi_{nk_n}$ of independent random variables with $\mathbb{E}\xi_{nj} = 0$ and $\sum_{j=1}^{k_n} \operatorname{Var} \xi_{nj} = 1$. If $\sum_{j=1}^{k_n} \mathbb{E}|\xi_{nj}|^{2+\delta} \to 0$ as $n \to \infty$ for some $\delta > 0$, then $\sum_{j=1}^{k_n} \xi_{nj} \xrightarrow{w} N(0,1)$. Show that this follows from the Lindeberg Central Limit Theorem.

3. The classical central limit theorem says that $\frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{n} (X_j - \mu) \xrightarrow{w} N(0,1)$, if the X_j are *iid* with $\mathbb{E}X_j = \mu$ and $0 < \operatorname{Var} X_j = \sigma^2 < \infty$. Show that this follows from the Lindeberg Central Limit Theorem.

4. Show that $X_n \xrightarrow{w} X$ iff $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$ for all bounded uniformly continuous functions f.

5. Let X and Y be independent, assume that Y has a $N(0,1)$ distribution. Let $\sigma > 0$. Let ϕ be the characteristic function of X: $\phi(u) = \mathbb{E}\exp(\mathrm{i}uX)$.

   (a) Show that $Z = X + \sigma Y$ has density $p(z) = \frac{1}{\sigma\sqrt{2\pi}} \mathbb{E}\exp(-\frac{1}{2\sigma^2}(z-X)^2)$.

   (b) Show that $p(z) = \frac{1}{2\pi\sigma} \int \phi(-y/\sigma) \exp(\mathrm{i}yz/\sigma - \frac{1}{2}y^2) \, dy$.

6. Let $X, X_1, X_2, \ldots$ be a sequence of random variables and Y a $N(0,1)$-distributed random variable independent of that sequence. Let ϕ_n be the characteristic function of X_n and ϕ that of X. Let p_n be the density of $X_n + \sigma Y$ and p the density of $X + \sigma Y$.

   (a) If $\phi_n \to \phi$ pointwise, then $p_n \to p$ pointwise. Invoke the previous exercise and the dominated convergence theorem to show this.

   (b) Let $f \in C_b(\mathbb{R})$ be bounded by B. Show that $|\mathbb{E}f(X_n+\sigma Y) - \mathbb{E}f(X+\sigma Y)| \leq 2B \int (p(z) - p_n(z))^+ \, dz$.

   (c) Show that $|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| \to 0$ if $\phi_n \to \phi$ pointwise.

   (d) Prove the following theorem: $X_n \xrightarrow{w} X$ iff $\phi_n \to \phi$ pointwise.

Week 12

1. Consider the sequence of 'tents' (X^n), where $X^n_t = nt$ for $t \in [0, \frac{1}{2n}]$, $X^n_t = 1 - nt$ for $t \in [\frac{1}{2n}, \frac{1}{n}]$, and zero elsewhere (there is no randomness here). Show that all finite dimensional distributions of the X^n converge, but X^n does not converge in distribution.

2. Show that ρ as in (1.1) defines a metric.

3. Suppose that the ξ_i of section 4 are *iid* normally distributed random variables. Use Doob's inequality to obtain $\mathbb{P}(\max_{j \leq n} |S_j| > \gamma) \leq 3\gamma^{-4}n^2$.

4. Show that a finite dimensional projection on $C[0,\infty)$ (with the metric ρ) is continuous.

5. Consider $C[0,\infty)$ with the Borel σ-algebra $\mathcal{B}$ induced by ρ and some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $X : (\Omega, \mathcal{F}) \to (C[0,\infty), \mathcal{B})$ is measurable, then all maps $\omega \mapsto X_t(\omega)$ are random variables. Show this, as well as its converse. For the latter you need separability that allows you to say that the Borel σ-algebra $\mathbb{B}$ is a product σ-algebra (see also Williams, page 82).

6. Prove proposition 2.2 of the lecture notes.