Exercises Measure Theoretic Probability 2003-2004

Week 1

1. Prove the following statements.
   (a) The intersection of an arbitrary family of d-systems is again a d-system.
   (b) The intersection of an arbitrary family of σ-algebras is again a σ-algebra.
   (c) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are collections of subsets of Ω with $\mathcal{C}_1 \subset \mathcal{C}_2$, then $d(\mathcal{C}_1) \subset d(\mathcal{C}_2)$.

2. Let $\mathcal{G}$ and $\mathcal{H}$ be two σ-algebras on Ω. Let $\mathcal{C} = \{G \cap H : G \in \mathcal{G},\, H \in \mathcal{H}\}$. Show that $\mathcal{C}$ is a π-system and that $\sigma(\mathcal{C}) = \sigma(\mathcal{G}, \mathcal{H})$.

3. Show that $\mathcal{D}_2$ (Williams, page 194) is a π-system.

4. If h_1 and h_2 are measurable functions, then $h_1 h_2$ is measurable too.

5. Let Ω be a countable set. Let $\mathcal{F} = 2^\Omega$ and let $p : \Omega \to [0,1]$ satisfy $\sum_{\omega\in\Omega} p(\omega) = 1$. Put $\mathbb{P}(A) = \sum_{\omega \in A} p(\omega)$ for $A \in \mathcal{F}$. Show that $\mathbb{P}$ is a probability measure.

6. Let Ω be a countable set. Let $\mathcal{A}$ be the collection of $A \subset \Omega$ such that A or its complement has finite cardinality. Show that A is an algebra. What is $d(\mathcal{A})$?

Week 2

1. Let X be a random variable. Show that $\Pi(X) := \{X^{-1}(-\infty, x] : x \in \mathbb{R}\}$ is a π-system and that it generates $\sigma(X)$.

2. Let $\{Y_\gamma : \gamma \in C\}$ be an arbitrary collection of random variables and $\{X_n : n \in \mathbb{N}\}$ be a countable collection of random variables, all defined on the same probability space.
   (a) Show that $\sigma\{Y_\gamma : \gamma \in C\} = \sigma\{Y_\gamma^{-1}(B) : \gamma \in C, B \in \mathcal{B}\}$.
   (b) Let $\mathcal{X}_n = \sigma\{X_1, \ldots, X_n\}$ $(n \in \mathbb{N})$ and $\mathcal{A} = \bigcup_{n=1}^\infty \mathcal{X}_n$. Show that $\mathcal{A}$ is an algebra and that $\sigma(\mathcal{A}) = \sigma\{X_n : n \in \mathbb{N}\}$.

3. Show that the X^+ and X^- are measurable functions and that X^+ is right-continuous and X^- is left-continuous (notation as in section 3.12).

4. Let $\mathcal{F}$ be a σ-algebra on Ω with the property that for all $F \in \mathcal{F}$ it holds that $\mathbb{P}(F) \in \{0, 1\}$. Let $X : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Show that for some $c \in \mathbb{R}$ one has $\mathbb{P}(X = c) = 1$. (*Hint:* $\mathbb{P}(X \leq x) \in \{0, 1\}$ for all x.)

5. Let Ω be a countable set. Let $\mathcal{A}$ be the collection of $A \subset \Omega$ such that A or its complement has finite cardinality. Show that $\mathcal{A}$ is an algebra. What is $d(\mathcal{A})$?

6. Show that a finitely additive map $\mu : \Sigma_0 \to [0, \infty]$ is countably additive if $\mu(H_n) \to 0$ for every decreasing sequence of sets $H_n \in \Sigma_0$ with $\bigcap_n H_n = \emptyset$. If μ is countably additive, do we necessarily have $\mu(H_n) \to 0$ for every decreasing sequence of sets $H_n \in \Sigma_0$ with $\bigcap_n H_n = \emptyset$?

Week 3

1. Let (S, Σ, μ) be a measure space and f a nonnegative simple function. Show that $\mu(f)$ doesn't depend on the chosen representation of f.

2. Show that the integral is a linear operator on $\mathcal{L}^1(S, \Sigma, \mu)$.

3. Prove the second part of Scheffé's lemma (see page 55).

4. Let $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Show that $\lim_{n\to\infty} n\mathbb{P}(|X| > n) = 0$.

5. Prove the assertions (a)-(c) of section 6.5.

6. Prove lemma 6.12 (you find the 'standard machine' in section 5.12).

Week 4

1. Williams, exercise E4.1.

2. Williams, exercise E4.6.

3. Prove part (b) of Fubini's theorem in section 8.2 for $f \in \mathcal{L}^1(S, \Sigma, \mu)$ (you already know it for $f \in \mathrm{m}\Sigma^+$). Explain why $s_1 \mapsto f(s_1, s_2)$ is in $\mathcal{L}^1(S_1, \Sigma_1, \mu_1)$ for all s_2 outside a set N of μ_2-measure zero and that I_2^f is well defined on N^c.

4. If $Z_1, Z_2, \ldots$ is a sequence of nonnegative random variables, then

$$\mathbb{E} \sum_{k=1}^{\infty} Z_k = \sum_{k=1}^{\infty} \mathbb{E}\, Z_k. \tag{1}$$

   Show that this follows from Fubini's theorem. If $\sum_{k=1}^{\infty} \mathbb{E}\, Z_k < \infty$, what is $\mathbb{P}(\sum_{k=1}^{\infty} Z_k = \infty)$. Formulate a result similar to (1) for random variables Z_i that may assume negative values as well.

5. Let the vector of random variables (X, Y) have a joint probability density function f. Let f_X and f_Y be the (marginal) probability density functions of X and Y respectively. Show that X and Y are independent iff $f(x, y) = f_X(x)f_Y(y)$ for all x, y except in a set of Leb×Leb-measure zero.

6. Let f be defined on $\mathbb{R}^2$ such that for all $a \in \mathbb{R}$ the function $y \mapsto f(a, y)$ is Borel and such that for all $b \in \mathbb{R}^2$ the function $x \mapsto f(x, b)$ is continuous. Show that for all $a, b, c \in \mathbb{R}$ the function $(x, y) \mapsto bx + cf(a, y)$ is Borel-measurable on $\mathbb{R}^2$. Let $a_i^n = i/n$, $i \in \mathbb{Z}, n \in \mathbb{N}$. Define

$$f^n(x, y) = \sum_i 1_{(a_{i-1}^n, a_i^n]}(x)(\frac{a_i^n - x}{a_i^n - a_{i-1}^n} f(a_{i-1}^n, y) + \frac{x - a_{i-1}^n}{a_i^n - a_{i-1}^n} f(a_i^n, y)).$$

Show that the f^n are Borel-measurable on $\mathbb{R}^2$ and conclude that f is Borel-measurable on $\mathbb{R}^2$.

Week 5

1. Prove the conditional version of Fatou's lemma, property (f) on page 88 (Williams).

2. Prove the conditional Dominated Convergence theorem, property (g) on page 88 (Williams).

3. Let (X, Y) have a bivariate normal distribution with $\mathbb{E}\, X = \mu_X$, $\mathbb{E}\, Y = \mu_Y$, $\mathrm{Var}\, X = \sigma_X^2$, $\mathrm{Var}\, Y = \sigma_Y^2$ and $\mathrm{Cov}\,(X, Y) = c$. Let

$$\hat{X} = \mu_x + \frac{c}{\sigma_Y^2}(Y - \mu_Y).$$

Show that $\mathbb{E}\,(X - \hat{X})Y = 0$. Show also (use a special property of the bivariate normal distribution) that $\mathbb{E}\,(X - \hat{X})g(Y) = 0$ if g is a Borel-measurable function such that $\mathbb{E}\, g(Y)^2 < \infty$. Conclude that $\hat{X}$ is a version of $\mathbb{E}\,[X|Y]$.

Week 6

1. Exercise E9.1 (Williams).

2. Exercise E9.2 (Williams)

3. Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be uniformly integrable collections of random variables on a common probability space. Show that $\bigcup_{k=1}^n \mathcal{C}_k$ is uniformly integrable. (In particular is a finite collection in $\mathcal{L}^1$ uniformly integrable).

4. Williams, exercise E13.1.

5. Let $\mathcal{C}$ be a uniformly integrable collection of random variables.

(a) Consider $\bar{\mathcal{C}}$, the closure of $\mathcal{C}$ in $\mathcal{L}^1$. Use E13.1 to show that also $\bar{\mathcal{C}}$ is uniformly integrable.

(b) Let $\mathcal{D}$ be the convex hull of $\mathcal{C}$. Then both $\mathcal{D}$ and its closure in $\mathcal{L}^1$ are uniformly integrable

6. In this exercise you prove (fill in the details) the following characterization: a collection $\mathcal{C}$ is uniformly integrable iff there exists a function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$ and $M := \sup\{\mathbb{E}G(|X|) : X \in \mathcal{C}\} < \infty$.
The necessity you prove as follows. Let $\varepsilon > 0$ choose $a = M/\varepsilon$ and c such that $\frac{G(t)}{t} \geq a$ for all $t > c$. To prove uniform integrability of $\mathcal{C}$ you use that $|X| \leq \frac{G(|X|)}{a}$ on the set $\{|X| \geq c\}$.
It is less easy to prove sufficieny. Proceed as follows. Suppose that we have a sequence (g_n) with $g_0 = 0$ and $\lim_{n\to\infty} g_n = \infty$. Define $g(t) = \sum_{n=0}^{\infty} 1_{[n,n+1)}(t) g_n$ and $G(t) = \int_0^t g(s)ds$. Check that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$.
With $a_n(X) = \mathbb{P}(|X| > n)$, it holds that $\mathbb{E}\, G(|X|) \leq \sum_{n=1}^{\infty} g_n a_n(|X|)$. Furthermore, for every $k \in \mathbb{N}$ we have $\int_{|X|\geq k} |X|\, d\mathbb{P} \geq \sum_{m=k}^{\infty} a_m(X)$. Pick for every n a constant $c_n \in \mathbb{N}$ such that $\int_{|X|\geq c_n} |X|\, d\mathbb{P} \leq 2^{-n}$. Then $\sum_{m=c_n}^{\infty} a_m(X) \leq 2^{-n}$ and hence $\sum_{n=1}^{\infty}\sum_{m=c_n}^{\infty} a_m(X) \leq 1$. Choose then the sequence (g_n) as the 'inverse' of (c_n): $g_n = \#\{k : c_k \leq n\}$.

7. Prove that a collection $\mathcal{C}$ is uniformly integrable iff there exists an *increasing and convex* function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$ and $M := \sup\{G(|X|) : X \in \mathcal{C}\} < \infty$.
Let $\mathcal{D}$ be the closure of the convex hull of a uniformly integrable collection $\mathcal{C}$ in $\mathcal{L}^1$. With the function G as above we have $\sup\{\mathbb{E}G(|X|) : X \in \mathcal{D}\} = M$, whence also $\mathcal{D}$ is uniformly integrable.

8. Let $p \geq 1$ and let $X, X_1, X_2, \ldots$ be random variables. Then X_n converges to X in $\mathcal{L}^p$ iff the following two conditions are satisfied.

   (a) $X_n \to X$ in probability,

   (b) The collection $\{|X_n|^p : n \in \mathbb{N}\}$ is uniformly integrable.

Week 7

In all exercises below we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F}$.

1. Let X be an adapted process and T a stopping time that is finite. Show that X_T is $\mathcal{F}$-measurable. Show also that for arbitrary stopping times T (so the value infinity is also allowed) the stopped process X^T is adapted.

2. For every n we have a measurable function f_n on $\mathbb{R}^n$. Let $Z_1, Z_2, \ldots$ be independent random variables and $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. Show that (you may

assume sufficient integrability) that $X_n = f_n(Z_1, \ldots, Z_n)$ defines a martingale under the condition that $\mathbb{E} f_n(z_1, \ldots, z_{n-1}, Z_n) = f_{n-1}(z_1, \ldots, z_{n-1})$ for every n.

3. If S and T are stopping times, then also $S + T$, $S \vee T$ and $S \wedge T$ are stopping times. Show this.

4. (a) If X is a martingale is and f a convex function such that $\mathbb{E}|f(X_n)| < \infty$, then Y defined by $Y_n = f(X_n)$ is a submartingale. Show this.

   (b) Show that Y is a submartingale, if X is a submartingale and f is an increasing function.

5. Prove Corollaries (c) and (d) on page 101.

6. Let X be an adapted process and $a < b$ real numbers. Let $S_1 = \inf\{n : X_n < a\}$, $T_1 = \inf\{n > S_1 : X_n > b\}$, etc. Show that the S_k and T_k are stopping times. Show also that the process C of section 11.1 is previsible (synonymous for predictable).

7. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0, 1)$, $\mathcal{F}$ the Borelsets of $[0, 1)$ and $\mathbb{P}$ the Lebesgue measure. Let $I^n_k = [k2^{-n}, (k+1)2^{-n})$ for $k = 0, \ldots, 2^n - 1$ and $\mathcal{F}_n$ be the σ-algebra by the I^n_k for $k = 0, \ldots, 2^n - 1$. Define $X_n = 1_{I^n_0} 2^n$. Show that X_n is a martingale is and that the conditions of theorem 11.5 are satisfied. What is X_∞ in this case? Do we have $X_n \xrightarrow{\mathcal{L}^1} X_\infty$? (This has something to do with 11.6).

Week 8

In all exercises below we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F}$.

1. Let $Y \in \mathcal{L}^1$, $(\mathcal{F}_n)$ and define for all $n \in \mathbb{N}$ the random variable $X_n = \mathbb{E}[Y|\mathcal{F}_n]$. We know that there is X_∞ such that $X_n \to X_\infty$ a.s. Show that for $Y \in \mathcal{L}^2$, we have $X_n \xrightarrow{\mathcal{L}^2} X_\infty$. Find a condition such that $X_\infty = Y$. Give also an example in which $P(X_\infty = Y) = 0$.

2. Let $X = (X_n)_{n \le 0}$ a (backward) supermartingale.

   (a) Show equivalence of the next two properties:
   (i) $\sup_n \mathbb{E}|X_n| < \infty$ and (ii) $\lim_{n \to -\infty} \mathbb{E} X_n < \infty$.
   (Use that $x \mapsto x^+$ is convex and increasing.)

   (b) Under the condition $\sup_n \mathbb{E}|X_n| =: A < \infty$ the supermartingale X is uniformly integrable. To show this, you may proceed as follows (*but other solutions are equally welcome*). Let $\varepsilon > 0$ and choose $K \in \mathbb{Z}$ such that for all $n < K$ one has $0 \le \mathbb{E} X_n - \mathbb{E} X_K < \varepsilon$. It is then sufficient to show that $(X_n)_{n \le K}$ is uniformly integrable. Let

$c > 0$ be arbitrary and $F_n = \{|X_n| > c\}$. Using the supermartingale inequality you show that

$$\int_{F_n} |X_n| \, d\mathbb{P} \leq \int_{F_n} |X_K| \, d\mathbb{P} + \varepsilon.$$

Because $\mathbb{P}(F_n) \leq \frac{A}{c}$ you conclude the proof.

3. Finish the proof of theorem 14.11: Show that $||Z_n||_p$ is increasing in n and that $||Z_\infty||_p = \sup\{||Z_n||_p : n \geq 1\}$.

4. Exercise E13.3.

5. Exercise E14.1.

6. Exercise E14.2.

Week 9

In all exercises below we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F}$.

1. Exercise 12.1.

2. Exercise 12.2

3. Let (H_n) be a predictable sequence of random variables with $\mathbb{E}\, H_n^2 < \infty$ for all n. Let (ε_n) be a sequence with $\mathbb{E}\, \varepsilon_n^2 = 1$, $\mathbb{E}\, \varepsilon_n = 0$ and ε_n independent of $\mathcal{F}_{n-1}$ for all n. Let $M_n = \sum_{k \leq n} H_k \varepsilon_k$, $n \geq 0$. Compute the conditional variance process A of (M_n). Take $p > 1/2$ and consider $N_n = \sum_{k \leq n} \frac{1}{(1+A_k)^p} H_k \varepsilon_k$. Show that there exists a random variable N_∞ such that $N_n \to N_\infty$ a.s. Show (use Kroneckers's lemma) that $\frac{M_n}{(1+A_n)^p}$ has an a.s. finite limit.

4. Suppose that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{Q} \ll \mathbb{P}$ with $d\mathbb{Q}/d\mathbb{P} = M_\infty$. Denote by $\mathbb{P}_n$ and $\mathbb{Q}_n$ the restrictions of $\mathbb{P}$ and $\mathbb{Q}$ to $\mathcal{F}_n$ $(n \geq 1)$. Show that $\mathbb{Q}_n \ll \mathbb{P}_n$ and that

$$\frac{d\mathbb{Q}_n}{d\mathbb{P}_n} = M_n,$$

where $M_n = \mathbb{E}_P[M_\infty | \mathcal{F}_n]$.

5. Let M be a nonnegative martingale with $\mathbb{E}\, M_n = 1$ for all n. Define $\mathbb{Q}_n(F) = \mathbb{E}\, 1_F M_n$ for $F \in \mathcal{F}_n$ $(n \geq 1)$. Show that for all n and k one has $\mathbb{Q}_{n+k}(F) = \mathbb{Q}_n(F)$ for $F \in \mathcal{F}_n$. Assume that M is uniformly integrable. Show that there exists a probability measure $\mathbb{Q}$ on $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$ that is absolutely continuous w.r.t. $\mathbb{P}$ and that is such that for all n the restriction of $\mathbb{Q}$ to $\mathcal{F}_n$ coincides with $\mathbb{Q}_n$. Characterize $d\mathbb{Q}/d\mathbb{P}$.

6. Consider the set up of section 14.17 (Williams). Assume that

$$\prod_{k=1}^{n} \mathbb{E}_{\mathbb{P}} \sqrt{\frac{g_k(X_k)}{f_k(X_k)}} \to 0.$$

Suppose one observes $X_1, \ldots, X_n$. Consider the testing problem H_0: the densities of the X_k are the f_k against H_1: the densities of the X_k are the g_k and the test that rejects H_0 if $M_n > c_n$, where $\mathbb{P}(M_n > c_n) = \alpha \in (0,1)$ (likelihood ratio test). Show that this test is *consistent*: $\mathbb{Q}(M_n \leq c_n) \to 0$. (Side remark: the content of the Neyman-Pearson lemma is that this test is most powerful among all test with significance level less than or equal to α.)

Week 10

1. Let $\mu, \mu_1 \mu_2, \ldots$ be probability measures on $\mathbb{R}$. Show that $\mu_n \stackrel{w}{\to} \mu$ iff for all bounded Lipschitz continuous functions one has $\int f \, d\mu_n \to \int f \, d\mu$. (*Hint: for one implication the proof of lemma 17.2 is instructive.*)

2. Show the 'if part' of lemma 17.2 without referring to the Skorohod representation. *First you take for given $\varepsilon > 0$ a $K > 0$ such that $F(K) - F(-K) > 1 - \varepsilon$ (why does such a K exist?). Approximate a continuous f on the interval $(-K, K]$ with a piecewise constant function and you compute the integrals of this approximating function and use the convergence of the $F_n(x)$ at continuity points x of F etc.*

3. If the random variables $X, X_1, X_2, \ldots$ are defined on the same probability space and if $X_n \stackrel{P}{\to} X$, then $X_n \stackrel{w}{\to} X$. Prove this.

4. Suppose that $X_n \stackrel{w}{\to} X$ and that the collection $\{X_n, n \geq 1\}$ is uniformly integrable (you make a minor change in the definition of this notion if the X_n are defined on different probability spaces). Use the Skorohod representation to show that $X_n \stackrel{w}{\to} X$ implies $\mathbb{E}X_n \to \mathbb{E}X$.

5. Show the following variation on Fatou's lemma: if $X_n \stackrel{w}{\to} X$, then $\mathbb{E}|X| \leq \liminf_{n\to\infty} \mathbb{E}|X_n|$.

6. Show that the weak limit of a sequence of probability measures is unique.

7. The proof of the Helly-Bray lemma that I gave was not complete. You show that the $F_{n_k}(c_i)$ converge for all c_i (the n_k were obtained by the Cantor-type diagonalization procedure) and that $F_{n_k}(x)$ converges to $F(x)$ at all continuity points x of F.

Week 11

1. Consider the $N(\mu_n, \sigma_n^2)$ distributions, where the μ_n are real numbers and the σ_n^2 nonnegative. Show that this family is tight iff the sequences (μ_n) and (σ_n^2) are bounded. Under what condition do we have that the $N(\mu_n, \sigma_n^2)$ distributions converge to a (weak) limit? What is this limit?

2. For each n we have a sequence $\xi_{n1}, \ldots, \xi_{nk_n}$ of independent random variables with $\mathbb{E}\xi_{nj} = 0$ and $\sum_{j=1}^{k_n} \operatorname{Var} \xi_{nj} = 1$. If $\sum_{j=1}^{k_n} \mathbb{E}|\xi_{nj}|^{2+\delta} \to 0$ as $n \to \infty$ for some $\delta > 0$, then $\sum_{j=1}^{k_n} \xi_{nj} \stackrel{w}{\to} N(0,1)$. Show that this follows from the Lindeberg Central Limit Theorem.

3. Show that $X_n \stackrel{w}{\to} X$ iff $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$ for all bounded uniformly continuous functions f.

4. Let X and Y be independent, assume that Y has a $N(0,1)$ distribution. Let $\sigma > 0$. Let ϕ be the characteristic function of X: $\phi(u) = \mathbb{E}\exp(\mathrm{i}uX)$.
   (a) Show that $Z = X + \sigma Y$ has density $p(z) = \frac{1}{\sigma\sqrt{2\pi}} \mathbb{E}\exp(-\frac{1}{2\sigma^2}(z-X)^2)$.
   (b) Show that $p(z) = \frac{1}{2\pi\sigma} \int \phi(-y/\sigma) \exp(\mathrm{i}yz/\sigma - \frac{1}{2}y^2)\, dy$.

5. Let $X, X_1, X_2, \ldots$ be a sequence of random variables and Y a $N(0,1)$-distributed random variable independent of that sequence. Let ϕ_n be the characteristic function of X_n and ϕ that of X. Let p_n be the density of $X_n + \sigma Y$ and p the density of $X + \sigma Y$.
   (a) If $\phi_n \to \phi$ pointwise, then $p_n \to p$ pointwise. Invoke the previous exercise and the dominated convergence theorem to show this.
   (b) Let $f \in C_b(\mathbb{R})$ be bounded by B. Show that $|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| \le 2B \int (p(z) - p_n(z))^+\, dz$.
   (c) Show that $|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| \to 0$ if $\phi_n \to \phi$ pointwise.
   (d) Prove the following theorem: $X_n \stackrel{w}{\to} X$ iff $\phi_n \to \phi$ pointwise.

6. Let $X_1, X_2, \ldots, X_n$ be an *iid* sequence having a distribution function F, a density (w.r.t. Lebesgue measure) f. Let m be such that $F(m) = \frac{1}{2}$. Assume that $f(m) > 0$ and that n is odd, $n = 2k - 1$, say ($k = \frac{1}{2}(n+1)$).
   (a) Show that m is the unique solution of the equation $F(x) = \frac{1}{2}$. We call m the *median* of the distribution of X_1.
   (b) The *sample median* M_n of $X_1, \ldots, X_n$ is by definition X_k. Show that with $U_{nj} = 1_{\{X_j \le m + n^{-1/2}x\}}$ we have
   $$\mathbb{P}(n^{1/2}(M_n - m) \le x) = \mathbb{P}(\sum_j U_{nj} \ge k).$$
   (c) Let $p_n = \mathbb{P}U_{nj}$, $b_n = (np_n(1-p_n))^{1/2}$, $\xi_{nj} = (U_{nj} - p_n)/b_n$, $Z_n = \sum_{j=1}^n \xi_{nj}$, $t_n = (k - np_n)/b_n$. Rewrite the probabilities in part 6b as $\mathbb{P}(Z_n \ge t_n)$ and show that $t_n \to t := -2xf(m)$.

(d) Show that $\mathbb{P}(Z_n \geq t) \to 1 - \Phi(t)$, where Φ is the standard normal distribution.

(e) Show that $\mathbb{P}(Z_n \geq t_n) \to \Phi(2f(m)x)$ and conclude that the *Central Limit Theorem for the sample median* holds:

$$2f(m)n^{1/2}(M_n - m) \xrightarrow{w} N(0,1).$$

Week 12

1. Consider the sequence of 'tents' (X^n), where $X_t^n = nt$ for $t \in [0, \frac{1}{2n}]$, $X_t^n = 1 - nt$ for $t \in [\frac{1}{2n}, \frac{1}{n}]$, and zero elsewhere (there is no randomness here). Show that all finite dimensional distributions of the X^n converge, but X^n does not converge in distribution.

2. Show that ρ as in (1.1) defines a metric.

3. Suppose that the ξ_i of section 4 are *iid* normally distributed random variables. Use Doob's inequality to obtain $\mathbb{P}(\max_{j \leq n} |S_j| > \gamma) \leq 3\gamma^{-4}n^2$.

4. Show that a finite dimensional projection on $C[0,\infty)$ (with the metric ρ) is continuous.

5. Consider $C[0,\infty)$ with the Borel σ-algebra $\mathcal{B}$ induced by ρ and some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $X : (\Omega, \mathcal{F}) \to (C[0,\infty), \mathcal{B})$ is measurable, then all maps $\omega \mapsto X_t(\omega)$ are random variables. Show this, as well as its converse. For the latter you need separability that allows you to say that the Borel σ-algebra $\mathbb{B}$ is a product σ-algebra (see also Williams, page 82).

6. Prove proposition 2.2 of the lecture notes.