**9.8** Suppose that at each time $t \in \{0, \dots, T\}$ an investor has a certain capital $W_t$ at her disposal. She consumes part of this, $C_t \geq 0$ say, and invests the remaining $W_t - C_t \geq 0$. The latter she does partly, a deterministic fraction $\pi_t$, in a riskless asset with fixed return $1 + r =: R$ and the remaining money in a risky asset with random yield $S_t$. This leads to the evolution

$$W_{t+1} = (W_t - C_t)\left(\pi_t R + (1 - \pi_t)S_t\right),\, t = 0, \dots T - 1.$$

It is assumed that the $S_t$ are *i.i.d.* random variables, all having the same distribution as a random variable $S$, and that all relevant expectations exist and are finite. Consider the utility function $u(x) = \frac{1}{\gamma}x^\gamma$, $x \geq 0$ and $\gamma < 1$. Let $\rho \in (0, 1)$ be a discount factor. The aim is to maximize $\sum_{t=0}^{T} \rho^t\, \mathbb{E}\, u(C_t)$ by appropriately selecting the $\pi_t$, $t = 0, \dots, T-1$, and the consumption $C_t$, $t = 0, \dots, T$. Assume $C_T = W_T$. The purpose is to characterize the optimal consumption pattern and to derive it by dynamic programming. We need more notation. We denote by $\pi^*$ the solution, assumed to exist, of the equation $\mathbb{E}\,(\pi R + (1-\pi)S)^{\gamma-1}(R - S) = 0$, and $\xi = \rho\mathbb{E}\left(\pi^* R + (1 - \pi^*)S\right)^\gamma$.

(a) Look at the theory of dynamic programming and rewrite Algorithm 9.7 in terms of the variables of this exercise.

(b) Compute the optimal consumption $C^*_{T-1}$ at time $T-1$, show that $C^*_{T-1} = \alpha_{T-1}W_{T-1}$, with $\alpha_{T-1} = \frac{(\beta_T\xi)^{1/(\gamma-1)}}{1+(\beta_T\xi)^{1/(\gamma-1)}}$ for $\beta_T = 1$.

(c) Show that the optimal value function at time $T-1$ is given by $v_{T-1}(w) = \rho^{T-1}\beta_{T-1}\frac{w^\gamma}{\gamma}$, where $\beta_{T-1} = \alpha^\gamma_{T-1} + \beta_T(1 - \alpha_{T-1})^\gamma\xi$.

(d) Show that the optimal $\pi^*_t$ are the same for all $t \leq T-1$ and that the optimal consumption is given by $C^*_t = \alpha_t W_t$, where the (nonrandom) constants $\alpha_t \in (0, 1)$ are given by $\alpha_{t-1} = \frac{(\beta_t\xi)^{1/(\gamma-1)}}{1+(\beta_t\xi)^{1/(\gamma-1)}}$, and (recursively) $\beta_{t-1} = \alpha^\gamma_{t-1} + \beta_t(1 - \alpha_{t-1})^\gamma\xi$. In passing you can show that the value functions are $v_t(x) = \rho^t\beta_t\frac{x^\gamma}{\gamma}$, $t = 0, \dots, T$.

(e) Note that we can write $\alpha_{t-1} = \frac{p_t}{1+p_t}$, with $p_t = (\beta_t\xi)^{1/(\gamma-1)}$. Show by a simple computation the formula $\beta_{t-1} = (\frac{p_t}{1+p_t})^{\gamma-1}$, which equals $\alpha^{\gamma-1}_{t-1}$. Show also the backward recursion $\frac{p_{t-1}}{a} = \frac{p_t}{1+p_t}$, where $a = \xi^{1/(\gamma-1)}$.

(f) Here we do some time reversion, we put $q_k = \frac{1+p_{T-k}}{p_{T-k}} = 1 + \frac{1}{p_{T-k}}$. Show that $q_k = 1 + \frac{q_{k-1}}{a}$, leading with $q_0 = 0$ to $q_k = \sum_{j=0}^{k} a^{-j} = \frac{a^{k+1}-1}{a^k(a-1)}$, if $a \neq 1$ (and $k + 1$ otherwise).

(g) Finally, show that the optimal consumption is given by $C^*_t = \frac{1}{q_{T-t}}W_t$, $t \leq T-1$. What is $v_0(x)$?

(h) Sketch the solution to the optimal consumption problem for the situation of logarithmic utility, $u(x) = \log x$. [For this utility function one has $u'(x) = x^{-1}$, which corresponds to $\gamma = 0$ above.]

[This exercise can be seen as a dynamic version of the situation in Proposition 4.7 and has been derived from the paper Paul A. Samuelson: Portfolio Selection By Dynamic Stochastic Programming, *The Review of Economics and Statistics* **51(3)**, 239–246, 1969.]