Introduction to stochastic finance in continuous time

Additional exercises

this version: October 4, 2017

1 Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}\, X_i = 1$ for all i. Define $P_n = \prod_{i=1}^n X_i$ and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ $(n \geq 1)$. Show that the P_n form a martingale sequence.

2 Suppose a continuous model for the stock price is such that $\log(S(T)/S(t))$ has a normal $N((r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))$ distribution (under $\mathbb{Q}$). Assume that at time t the price $S(t)$ is known to be equal to s. Then the price of the usual European call option at time t is known to be

$$e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(s\frac{S(T)}{S(t)} - K)^+.$$

Show by computation of an integral that the explicit expression of this price is given by the Black-Scholes formula of Equation (1.27).

3 Give a direct proof of Proposition 1.5.

4 Suppose one doesn't use the risk-neutral probabilities $q_u(N)$ and $q_d(N)$ in Theorem 1.4, but instead $p_u(N) = \frac{1}{2} + (\mu - \frac{1}{2}\sigma^2)\frac{\sqrt{\Delta}}{2\sigma}$ and the corresponding $p_d(N)$ for some $\mu \in \mathbb{R}$ and sufficiently small positive $\Delta = \frac{T}{N}$. What would then be the limit laws of $\log S_N(t)$ and $\log S_N(t) - \log S_N(s)$ (for $t > s$)?

5 Consider the interval $[0,1]$. A partition Π of $[0,1]$ is a set $\Pi = \{t_0, \ldots, t_n\}$ with $0 = t_0 < t_1 < \ldots < t_n = 1$. The *mesh* $\mu(\Pi)$ of Π is defined as the biggest 'gap': $\mu(\Pi) = \max\{t_j - t_{j-1} : j = 1, \ldots, n\}$.

For $p > 0$ the p-th order variation of a function $f : [0,1] \to \mathbb{R}$ over a partition $\Pi = \{t_0, \ldots, t_n\}$ of $[0,1]$ with $0 = t_0 < t_1 < \ldots < t_n = 1$ is defined as $V^p(f;\Pi) = \sum_{i=1}^n |f(t_i) - f(t_{i-1})|^p$. Put $V^p(f) = \lim V^p(f;\Pi)$, where the limit is taken over (sequences of) partitions with mesh tending to zero.

(a) Let $p = 1$ and $f \in C^1[0,1]$, so with bounded (left/right in the endpoints) derivative. Argue (use Riemann sums) that $V^1(f) = \int_0^1 |f'(t)|\, \mathrm{d}t$.

(b) Let $p = 2$ and $f \in C^1[0,1]$. Show first that there exists $C > 0$ such that $|f(t) - f(s)| \leq C|t-s|$ for all $s, t \in [0,1]$, then $\sum_j (t_j - t_{j-1})^2 \leq \mu(\Pi)$ for $t_j \in \Pi$ and finally that $V^2(f) = 0$.

6 Let X have the standard normal distribution and $\phi(\lambda) = \mathbb{E}\exp(\mathrm{i}\lambda X)$, for $\lambda \in \mathbb{R}$.

(a) Argue that $\phi'(\lambda) = \mathrm{i}\mathbb{E}\,(X \exp(\mathrm{i}\lambda X))$.

(b) Show (use integration by parts) that $\phi'(\lambda) = -\lambda\phi(\lambda)$.

(c) Conclude that $\phi(\lambda) = \exp(-\frac{1}{2}\lambda^2)$.

7 Let X be a random variable with $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X > 0) > 0$.

(a) Show that there exists $n \in \mathbb{N}$ such that $\mathbb{P}(X > 1/n) > 0$. (*Reason by contradiction, assume that* $\mathbb{P}(X > 1/n) = 0$ *for all* $n \in \mathbb{N}$.)

(b) Show that $\mathbb{E}\, X > 0$.

(c) Suppose X is such that $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{E}\, X = 0$. Show that it follows that $\mathbb{P}(X > 0) = 0$, equivalently $\mathbb{P}(X = 0) = 1$.

8 Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and that X is random variable having the $N(0,1)$ distribution. Let ϕ denote the density of the $N(0,1)$ distribution and let ϕ_a denote the density of the $N(a,1)$ distribution. Let $z(x) = \frac{\phi_a(x)}{\phi(x)}$. Define $\mathbb{Q}(F) = \mathbb{E}\,(\mathbf{1}_F Z)$ for $F \in \mathcal{F}$, where $Z = z(X)$.

(a) Compute $\mathbb{E}\,Z = 1$ (use an integral).

(b) Show that $\mathbb{Q}(X \leq x) = \Phi(x - a)$, where Φ is the distribution function of $N(0,1)$.

(c) Conclude that $X^{\mathbb{Q}} := X - a$ has the standard normal distribution under $\mathbb{Q}$.

(d) Show that $z(x) = \exp(ax - \frac{1}{2}a^2)$ and that Z has the same distribution as $\exp(aW(1) - \frac{1}{2}a^2)$, where W is a Brownian motion.

We see that the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ is 'neutralized' by replacing X by $X - a$ in the sense that $X - a$ has, under $\mathbb{Q}$, the same distribution as X under $\mathbb{P}$.

9 Show that (the value of) the Itô integral of (6.3) doesn't depend on the specific representation of the simple process a. *Hint: If* $a(t) = \sum_{i=1}^m b_i \mathbf{1}_{(s_{i-1},s_i]}(t)$ *is another representation, consider* $a(t) = \sum_{i=1}^m \sum_{j=1}^n b_i \mathbf{1}_{(s_{i-1},s_i]\cap(t_{j-1},t_j]}(t)$.

10 Here are some consequences of Example 6.2.

(a) Show Equality (6.4).

(b) What is the limit of its right hand side when the t_i come from a sequence of partitions of $[0,T]$ with mesh tending to zero? What is in the same situation the limit of $W^n(t)$?

(c) What is the 'reasonable' value of $\int_0^T W(s)\,\mathrm{d}W(s)$?

11 Prove Proposition 7.2.

12 Here you prove the converse of Theorem 7.6. Suppose (the context of the theorem applies) that a claim $F(S(T), Z(T))$ can be hedged by a self-financing portfolio with value process V if the type $V(t) = v(t, S(t), Z(t))$, where v is sufficiently differentiable. Show that v satisfies Equation (7.7).