Portfolio theory

Additional exercises

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1 The proof of the first part of Theorem A.1 says "one easily shows that $\lambda \geq 1$". You may have noticed that in the text before that, convexity of C is not used, and this is what is needed to establish $\lambda \geq 1$. Define $x_t = tx + (1-t)x_0$, $t \in [0,1]$. For which t is $\|x_t\|$ minimal? Exploit this to prove the assertion.

2 Assume that a markt admits an arbitrage opportunity. Show there exists a $\bar{\xi} \in \mathbb{R}^{d+1}$ such that $W_0 = 0$, $W_1 \geq 0$ a.s., and $\mathbb{P}(W_1 > 0) > 0$. [Conversely, such a $\bar{\xi}$ is an arbitrage opportunity in the sense of Definition 1.1. Hence Definition 1.1 has an equivalent version with $W_0 = 0$ instead of $W_0 \leq 0$.]

3 Suppose that the initial price vector $\bar{\pi}$ is different from $\mathbb{E}^* \frac{\bar{S}}{1+r}$. Construct an arbitrage opportunity.

4 Consider the sets M_0 and M_1 of Theorem 1.15. Show by a direct argument that $\inf M_1 \geq \sup M_0$.

5 Complete the proof of Theorem 1.15, i.e. show that $\inf \Pi(C) = \max M_0$.

6 Complete the proof of Proposition 1.18, i.e. show that $\sup \Pi(C) \notin \Pi(C)$.

7 Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume Ω has three elements, $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Suppose there is, next to the riskless S_0, one risky asset S_1 that is not constant. Construct a non-attainable claim C and show that the market extended with C is complete.

8 Let $\mathcal{X} = [0,1] \times [0,1]$ endowed with the lexicographical order $\succ$. Show by a direct argument, not referring to Theorem 2.6, that $\mathcal{X}$ has no countable order dense subset.

9 I started to doubt whether the proof of Theorem 2.12 is entirely correct (under the assumption that u_0 and u_1 are not attained). Perhaps there is a hidden assumption that the set of equivalence classes for $\sim$ is uncountable, perhaps this is solved by connectedness, perhaps there are hidden assumptions on the topology.

(a) Assume $\mathcal{X} = \mathcal{Z}$. Is the statement of the Theorem still valid?

(b) Suppose that the countable dense subset is in fact finite. Is the statement of the Theorem still valid?

10 Let $f : S \to \mathbb{R}$ be concave, where S is an interval. Show that f is continuous on the interior of S and give an example where f is not continuous in a boundary point of S (which is assumed to belong to S).

11 Let u be an exponential utility function, $u(x) = -\exp(-\alpha x)$, $x \in \mathbb{R}$, $\alpha > 0$. Find the maximizing λ for the problem in Proposition 4.6 in each of the cases (a) X assumes two values only, (b) X has an exponential distribution, (c) X has a log-normal distribution. [I have not checked whether explicit solutions exist.]

12 A random variable X has a log-normal distribution with parameters α and σ, if $X = \exp(\alpha + \sigma Z)$, where $\sigma \geq 0$ and Z has a standard normal distribution.

(a) Compute $\mathbb{E}\, X^p$ for $p > 0$. In particular, one has $\mathbb{E}\, X = \exp(\alpha + \frac{1}{2}\sigma^2)$.

(b) Let μ_i be log-normal distributions ($i = 1, 2$) with parameters α_i, σ_i. Show that $\mu_1 \succeq_{\rm uni} \mu_2$ implies $m(\mu_1) \geq m(\mu_2)$ and $\sigma_1 \leq \sigma_2$.

(c) Conversely, if $m(\mu_1) \geq m(\mu_2)$ and $\sigma_1 \leq \sigma_2$, then $\mu_1 \succeq_{\rm uni} \mu_2$. To prove this, proceed as follows. Let $X_1 = \exp(\alpha_1 + \sigma Z_1)$ and $X_2 = \exp(\alpha_2 + \sigma Z_2)$ (in obvious notation). Let further $X_3 = \exp(\alpha_2 - \alpha_1 + \sqrt{\sigma_2^2 - \sigma_1^2} Z_3$, where Z_3 is standard normal, independent of Z_1. Verify that $X_1 X_3$ has the same distribution as X_2 and that $\mathbb{E}\, X_3 = \frac{m(\mu_2)}{m(\mu_1)}$. Use then Jensen's inequality for conditional expectations to show that $\mathbb{E}\, u(X_2) \leq \mathbb{E}\, u(X_1)$.

13 Consider a market with one risky good, its value at $t = 1$ is S and price π (at $t = 0$). Assume that S has under $\mathbb{P}$ a Poisson distribution with parameter $\alpha > 0$. Consider the exponential family of Definition 6.10.

(a) Show that $Z(\lambda) < \infty$ for all $\lambda \in \mathbb{R}$

(b) Show that S has a Poisson distribution with parameter αe^λ under $\mathbb{P}_\lambda$.

(c) Compute the minimizer of $\lambda \mapsto Z(\lambda)$ directly.

(d) Verify that the minimizer is in agreement with Proposition 6.13.

14 Let F be a distribution function and q any of its quantile functions. Let q^- and q^+ be the extremal quantile functions and note that $q^- \leq q^+$.

(a) Show that $\{q^- = q = q^+\}$ has Lebesgue measure one. You may use Theorem 3.10 of the MTP lecture notes.

(b) If U is a random variable with the uniform distribution on $(0, 1)$, show that $q(U)$ has distribution function F.

15 Give a concrete example where the X^* in Theorem 7.12 is different from X_0.

16 The proof of Proposition 8.21 is a lot simpler if $\mathcal{F}_0$ is the trivial σ-algebra $\{\emptyset, \Omega\}$. In this case all ξ_n is the proof are just vectors in $\mathbb{R}^d$ and the spaces N and $N^\perp$ are closed linear subspaces of $\mathbb{R}^d$. Rewrite (and shorten) the proof under this additional assumption and make clear that Lemma 8.19 and Lemma 8.20 can be circumvented by using standard analysis arguments instead.

17 Show that it follows from the proof of Proposition 8.21 that under the same assumption also $\mathcal{K}$ is closed in L^0.

18 If one drops the no arbitrage assumption in Proposition 8.21, the assertion is no longer true in general. Here is an example. Assume that market contains only one risky asset ($d = 1$). Let in $(\Omega, \mathcal{F}, \mathbb{P})$, $\Omega = [0, 1]$, $\mathcal{F}$ the Borel σ-algebra, and $\mathbb{P}$ the Lebesgue measure. Assume that $Y : \Omega \to \mathbb{R}$ is given by $Y(\omega) = \omega$.

(a) Show that the no arbitrage condition is violated.

(b) Let $Z \geq 1$ be a constant. Show that Z cannot belong to $\mathcal{C}$, and conclude that $\mathcal{C}$ is not all of L^1.

(c) Let $Z \in L^1$ and define $Z_n = (Z^+ \wedge n)\mathbf{1}_{[\frac{1}{n}, 1]} - Z^-$. Show that $Z_n \in \mathcal{C}$ (establish first that $(Z^+ \wedge n)\mathbf{1}_{[\frac{1}{n}, 1]} \leq c_n Y$ for some constant c_n) and that $Z^n \to Z$ in L^1 for $n \to \infty$.

(d) Conclude that $\mathcal{C}$ is not closed.

19 Prove the independence lemma, $\mathbb{E}\left[f(X,Y)|\mathcal{G}\right] = \hat{f}(X)$ with $\hat{f}(x) = \mathbb{E}\, f(x,Y)$ if X is $\mathcal{G}$-measurable and Y independent of $\mathcal{G}$, under the assumption that X and Y are discrete.