Exercises Portfolio Theory

1. Give the details of the proof on the bottom of page 11 of the lecture notes. Adjust the proof for the possible case where A_1 or A_2 is an empty set.

2. Let $\preceq$ be an ordering on the commodity space $\mathcal{B} = \mathbb{R}^n_+$. Assume that $\preceq$ is transitive, complete (= linear), monotone and continuous. Let $\mathbf{1} = (1, \ldots, 1)^\top$ and consider for given $x \in \mathcal{B}$ the sets $U = \{\alpha \geq 0 : \alpha x \preceq x\}$ and $L = \{\alpha \geq 0 : \alpha x \succeq x\}$. Show that (i) $L \cap U \neq \emptyset$ and that (ii) actually this intersection consists of one point only.

3. Assume that an ordering $\preceq$ on the space of lotteries satisfies the *independence* property. Show that for all lotteries L, L', L'' and $\alpha \in (0,1)$ the following properties hold.

   (a) $L \prec L' \Leftrightarrow \alpha L + (1-\alpha)L'' \prec \alpha L' + (1-\alpha)L''$

   (b) $L \approx L' \Leftrightarrow \alpha L + (1-\alpha)L'' \approx \alpha L' + (1-\alpha)L''$

   (c) If $L \prec L'$ and $\alpha \in (0,1)$, then $L \prec \alpha L + (1-\alpha)L' \prec L'$.

   (d) Let $L \prec L'$ and $\alpha, \beta \in [0,1]$. Then $\beta L + (1-\beta)L' \prec \alpha L + (1-\alpha)L'$ iff $\beta < \alpha$.

4. Let $\preceq$ be an ordering on the space of lotteries that satisfies the *transitivity*, *completeness*, *continuity* and *independence* properties. Assume $L \prec L' \prec L''$. Show that there exists a unique $\alpha \in [0,1]$ such that $\alpha L + (1-\alpha)L'' \approx L'$.

5. Let V be a utility function on the space of lotteries with respect to some preference ordering. Show that V is of expected utility form if and only if it is linear (for convex combinations we have $V(\sum \alpha_i L_i) = \sum \alpha_i V(L_i)$).

6. Let $f : I \to \mathbb{R}$ be a concave function (in short: $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$, for $t \in [0,1]$. Show the following property: for all $v \in I$ there exists a constant c_v such that for all $x \in I$ one has $f(x) \leq f(v) + c_v(x - v)$. (*Hint:* For all $u < v < w$ it holds that

$$\frac{f(v) - f(u)}{v - u} \geq \frac{f(w) - f(v)}{w - v},$$

   and both ratios are monotone functions of u and v respectively.) Show also that concavity of f follows if the afore mentioned property holds.

7. Assume that an ordering $\preceq$ on the space of lotteries can be represented by an expected utility form. Show that $\preceq$ satisfies the *independence* property.

8. Let U be an increasing and concave utility function $U : I \to \mathbb{R}$. Consider the fair game represented by a random variable ξ with values $x \pm \varepsilon$ in I that are attained with equal probabilities $\frac{1}{2}$. Given x, ε, the *probability premium* $\pi = \pi(x, \varepsilon)$ is by definition such that the lottery with the same

outcomes but with probability $\mathbb{P}(\xi = x - \varepsilon) = \frac{1}{2} - \pi$ has expected utility $U(x)$. Show that an individual with utility function U is risk averse iff $\pi(x, \varepsilon) \geq 0$ for all x, ε. Sketch a picture of the graph of U and construct $\pi(x, \varepsilon)$.

9. Consider a strictly risk averse decision maker with utility function U, who has an initial wealth of w euro. He is faced with a potential loss of ℓ euro (on some specified future date). The probability that the loss occurs is p. To protect against loss, he can by an insurance paying out 1 euro per purchased unit of insurance, if the loss occurs (otherwise nothing). The cost of the insurance is p euro per unit (check that this make the insurance *fair*). Suppose that he buys α units of insurance. Let $X(\alpha)$ be the (binary) random variable representing his resulting wealth on the future date. Let α^* be the optimal number of purchased insurance units in the sense that $\alpha \mapsto \mathbb{E}\, U(X(\alpha))$ is maximized at α^*. Show that $\alpha^* = \ell$. What is the (nonrandom!) resulting value of X?

10. Let U be a twice differentiable utility function, such that the Arrow-Pratt coefficient of risk $R(x) = -U''(x)/U'(x)$ is a constant $\alpha > 0$ not depending on x.

    (a) Show that there exist constants $a > 0$ and b such that $U(x) = ae^{-\alpha x} + b$.

    (b) If the column vector Z has a d-dimensional multivariate normal $N(\mu, \Sigma)$ distribution, then $\mathbb{E}\, \exp(u^\top Z) = \exp(u^\top mu + \frac{1}{2} u^\top \Sigma u)$ for every $v \in \mathbb{R}^d$. Show this.

    (c) Consider an investor with utility function U who wants to invest an initial capital X_0. There is one riskless asset with return r_0 and n risky assets with random returns r_i. Suppose that the random vector $(r_1, \ldots, r_n)$ has a multivariate normal $N(\mu, \Sigma)$ distribution with nonsingular covariance matrix Σ. Find the optimal investment strategy π^* (notation of Lemma 4.10).

11. In exercise 10 the optimization problems turns out to be of the form: maximize $\mathbb{E}\, Z - c\mathrm{Var}\, Z$. This seems reasonable, if one thinks of Z as a random revenue. One wants to maximize the expected revenue and to keep the 'risk' in terms of variance low. In general such a maximization problems leads to odd results. Consider the following example. In a first lottery the random pay-off Z satisfies $\mathbb{P}(Z = 1) = 1$ and $\mathbb{P}(Z = 0) = 0$, where as in a second lottery $\mathbb{P}(Z = 1) = 1 - \mathbb{P}(Z = 0) = p > 0$. Find an example of values of p and c such that the second lottery is preferred to the first one.

12. Consider a twice differentiable utility function $U : I \to \mathbb{R}$. Define for fixed x such that $tx \in I$ the function $t \mapsto v_x(t) = U(tx)$. A way to establish the *relative* risk around x can obtained by inspection of $v_x(t)$ in a neighbourhood of $t = 1$. A measure of relative risk at x is given by

$r(x) = -v_x''(1)/v_x'(1)$. Show that $r(x) = xR(x)$ ($R(x)$ the Arrow-Pratt risk measure).

13. A utility function U is said to exhibit decreasing risk aversion if the function $x \mapsto R(x)$ is decreasing. Show that this property is equivalent to saying that for every $x_1 < x_2$ there exists a concave function g such that $U(x_2 + z) = g(U(x_1 + z))$ for all z (for which the given expressions make sense).

14. *In this exercise (and the next one) on Stochastic Domination you should reply the $1 + r_i$ and $1 + r_j$ in Definition 5.1 with other in this context relevant random variables.*
Consider a lottery with random outcome ξ. Given the outcome of this lottery the result of a second lottery with random outcome η (independent of ξ) is added to it. Assume that $\mathbb{P}(\eta \geq 0) = 1$. The final result is $\zeta = \xi + \eta$. Show that $\zeta \succeq_{FSD} \xi$. Conversely, show that there exists a random variable η that is independent of ξ with $\mathbb{P}(\eta \geq 0) = 1$, if $\zeta \succeq_{FSD} \xi$.[1]

15. Consider the two lotteries in the previous exercise, but assume that η has mean zero, instead of being a.s. nonnegative. Show that $\xi \succeq_{SSD} \zeta$. Show that also the converse statement holds true.

16. Consider Example 3, of *Korn*, page 10. Solve the dual optimization problem (i.e. of the type of equation (3)) under the constraint that $\pi_1 + \frac{3}{2}\pi_2 \geq 5/4 + \sqrt{6}/12$. Do this both analytically and graphically. Verify the assertion of Proposition 1.

17. Requirement (14) (*Korn*, page 29) together with the basic model assumptions of Chapter 2 imply that $\mathbb{E}\, H(T)B < \infty$. Show this. *Hint: Let* $Z = \exp(-\frac{1}{2}\int_0^T ||\theta(s)||^2\, ds - \int_0^T \theta(s)^\top dW(s))$. *Show that for any* $\nu > 0$ $\mathbb{E}\, Z^\nu < \infty$. *Apply Hölder's inequality to* $B(T)Z$.

18. Let X be a nonnegative local martingale (with $X(0) = 0$). Show that X is a supermartingale. (*You may want to use a famous lemma.*)

19. Look at Remark 8c of Chapter 2 (*Korn*). The claim is that the 'hedge' portfolio process is unique (in some sense, L is Lebesgue measure on $[0, T]$). Show this. (I don't understand immediately how Korn derives the equality at the bottom of page 28.)

20. Much of the theory of section 2.3 (*Korn*) goes through (given all the assumptions) for the case $m > n$, with some appropriate modifications. Give a quick review of this and show that things break down in the proof of Theorem 7b.

21. About Theorem 7b of Chapter 2 (*Korn*): Consider two consumption rate processes c_1 and c_2 satisfying the assumptions. How are the corresponding

[1] If necessary, you may assume that ξ and η are bounded.

portfolio processes π_1 and π_2 are related? (I have no idea what comes out of this, just give it a try).

22. Go to page 46 of *Korn*, step 2. There are here and there some errors in the computations. Give the correct version (of course, the final answer is correct).

23. Prove theorem C2a (*Korn*, page 324). *Apply the Itô rule to* $\tilde{v}(T, X_T)$*, then you take expectations. Convince yourself that either the (local)martingale term has expectation zero under the assumptions of Chapter 2, or use an appropriate sequence of stopping times* T_n *to justify this property by studying* $\tilde{v}(T_n, X_{T_n})$. By similar reasoning you can also prove part b of this theorem.

24. Simple dynamic programming exercise. Printed version will be handed over.

25. I don't believe that *in general* Korn's statement on page 325: "Theorem C2 stays correct if in the HJB-equation we replace $L(t, x, u)$ by $L(x, u) - \rho v(t, x)$" holds true. But, show that it is true for $t = 0$. If the SDE for the controlled diffusion has coefficients that are independent of time (like in equation (2), page 320), then you show that $v(t, x) = e^{-\rho t} v(0, x)$, if one considers the *infinite horizon* problem on page 325. Then Korn's statement applies and the HJB-equation (8) for $v(x) = v(0, x)$ holds true.

26. Show that the expectation in problem (P) (*Korn*, page 49) is finite for the optimal wealth process X^*. Try also to compute it explicitly (if this can be done within a reasonable time span).

27. Prove the assertion of Corollary 7* (page 53 in *Korn*).

28. The utility functions U_γ are for $\gamma \in (0, 1)$ defined as $U_\gamma(x) = \frac{1}{\gamma} x^\gamma$ for $x > 0$. Notice that $U_\gamma'(x) = x^{\gamma-1}$. It is common to define (as kind of a limit) U_0 as $U_0(x) = \log x$. Solve the infinite horizon optimization problem (P) on page 49 of *Korn* via the HJB approach .

29. Take U_0 as in exercise 28. Solve the finite horizon problem (P) on page 55 of *Korn* for this case via the HJB approach.

30. Same questions as in exercises 28 and 29 for exponential utility: $U(x) = \alpha - \frac{\beta}{\gamma} \exp(-\gamma x)$, $x > 0$ for positive parameters β, γ.

31. Solve the finite horizon problem (P) on page 55 for the case of power utility ($U_1(x) = U_2(x) = \frac{1}{\gamma} x^\gamma$) via the Martingale approach. What is the solution for the corresponding infinite horizon problem with $U_2 = 0$?

32. Same questions as in exercise 31 for exponential utility: $U_1(x) = U_2(x) = \alpha - \frac{\beta}{\gamma} \exp(-\gamma x)$, $x > 0$ for positive parameters β, γ.

33. Compute along the lines of Example 24 in *Korn* the optimal consumption process and optimal portfolio process when $U_1(t,x) = \exp(-\mu t)\log x$, $U_2(x) = \log x$. Check the result in the special case $\mu = 0$ with Example 19.

34. Consider a Cox-Ross-Rubinstein market with time set $\{0,\dots,T\}$. In this market the bond price evolves as $B_t = (1+r)^t$ and the stock price S satisfies $S_t = \prod_{s=1}^t Z_s$. The filtration is given by $\mathcal{F}_t = \sigma\{Z_1,\dots,Z_t\}$. Under the risk neutral measure $\mathbb{Q}$, the Z_s are *iid* with $\mathbb{Q}(Z_1 = u) = q = 1 - \mathbb{Q}(Z_s = d)$ (assume that $0 < d < 1+r < u$ and recall that $q = (1+r-d)/(u-d)$). We consider an American put option with payoff at time t given by $f_t = (K - S_t)^+$. Let P_t be the fair price of this option at time t.

    (a) Show that $P_t = p(t, S_t)$, where $p(T,x) = (K-x)^+$ and for $t \leq T-1$ it holds that

    $$p(t,x) = \max\{(K-x)^+, \frac{\mu(t+1,x)}{1+r}\},$$

    where $\mu(t+1,x) = (1-q)p(t+1,xd) + qp(t+1,xu)$.

    (b) Show that

    $$p(0,x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{Q}}[(1+r)^{-\tau}(K - xV_\tau)^+],$$

    where $\mathcal{T}$ is the set of stopping times bounded by T and where for fixed t the random variable V_t can be written as a product $V_t = \prod_{i=1}^t U_i$ of *iid under* $\mathbb{Q}$ random variables U_i. Give the distribution of $(U_1,\dots,U_T)$ under $\mathbb{Q}$.

    (c) Show that $x \mapsto p(0,x)$ is convex and non-increasing.

    (d) Assume that $d < 1$. Show that there exists $x^* \in [0,K]$ such that $p(0,x) = (K-x)^+$ if $x \in [0,x^*]$ and that $p(0,x) > (K-x)^+$ if $x \in (x^*, Kd^{-T}]$.

    (e) Suppose that you owe an American put at time $t = 0$. For which values of S_0 would you immediately exercise your option?

    (f) Compute a hedge strategy for the American put for $T = 2$ and the fair prices P_t for $t = 0,1,2$.

35. Show that the function P in Lemma 8.2.8 of Elliott & Kopp is convex and non-increasing in each of its arguments.

36. Show that the derivative $\frac{\partial}{\partial x}P$ (notation of section 8.2) exists in (t,x) such that $P(x,t) > (K-x)^+$.

37. Give the full proof (including details) of proposition 8.5.2.