Stieltjes Institute Lectures on ARCH Modelling and Financial Time Series

March 8, 2001

1 Introductory Remarks

The computational tasks and exercises together will probably take up more time than is available. It is advised to spend some time on both. To get financial data, which will be required in order to do the Computational Tasks below, one can consult a great number of web-sites (apart from our own web-site!). Here we mention a few links:
http://www.economagic.com
http://www.lib.stat.cmu.edu/
http://www.econ.vu.nl/koopman/sv/svpdx.dat
http://www.econ.vu.nl/econometriclinks/#data
in the last web-site a large number of links are brought together. The first link might be the most useful for our purposes. In this site it is possible to click on the option "Display series in COPY/PASTE format" which allows to bring in the data into an S-PLUS data-sheet by copy-and-paste (in S-PLUS click on DATA, then FILL to get an empty data-sheet, then you can copy the series in there; it may be necessary to clean up the data-sheet afterwards by deleting some rows and columns, which contain no information or information that you do not need).

In the tasks you will work with the GARCH module of S-PLUS. There is a very good USER's GUIDE for the GARCH module, which tells you step-by-step what to do if you want to estimate a GARCH model. The USERS GUIDE can be found in .pdf format in the following location

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E:\splus\module\garch\garch.pdf
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which can be opened by the ACROBAT reader. You can either read from the screen or try to print a number of pages. Chapter 2 and Chapter 3 till page 34, say, will be probably be the parts you will use most this afternoon.

2 Computational Tasks

- **Task 1.** Estimate an ARCH model with an exchange rate return, a stock return and a stock index return (e.g. S&P500). Try various specifications like ARCH(1), ARCH(2), GARCH(1,1), GARCH(2,2). It is asserted that ARCH models are especially good for exchange rates. Is this in accordance with your findings? If time permits try different starting values for the parameters in the optimization algorithm. What are your findings?

- **Task 2.** Estimate an ARCH model for a stock return series and use the square of a stock index return series (like S&P500) as an explanatory variable in the conditional variance equation. Compare the result with the estimation of the same model, but then without this explanatory variable. This can be repeated for several stock return series and the results can be compared.

- **Task 3.** Choose a return series for which you have found a good description with an ARCH model and for which a long data series is available. Fit an ARCH model to the same series but now with a lower frequency. (So use e.g Z_{3t} instead of Z_t if Z_t is the prices series; the return should then be calculated as $\frac{Z_{3t}-Z_{3t-3}}{Z_{3t-3}}$ of course.) It is asserted that the ARCH effects diminish at lower frequency. Is this in accordance with your findings? Note that for a fair comparison one should use the same number of data points for both models; this implies that a shorter period needs to be used for the high-frequency model.

3 Exercises

- **Exercise 1.** In the lectures it was stated that if (ϵ_t) is a stationary sequence with finite fourth moment then for fixed t the sequence $\alpha^k \epsilon_{t-k}^2$ converges to zero for $k \longrightarrow \infty$ (with probability one) if $|\alpha| < 1$. Proof this.
- **Exercise 2.** Consider the GARCH model with notation as in the lecture notes. Assume $c > 0$. Show that if $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j \geq 1$ then it is not possible that the model is stationary with finite second moments.
- **Exercise 3.** Show that if X has a Gaussian distribution with zero mean and variance σ^2, then the fourth moment is $E(X^4) = 3\sigma^4$. (So the kurtosis is 3).
- **Exercise 4.** Consider the model $Y_t = W_t^3 W_{t-1}$, where W_t is standard Gaussian white noise. Show that $E(Y_t^2 | \underline{Y_{t-1}}) = W_{t-1}^2 E(W_t^6) = 15 W_{t-1}^2$.
- **Exercise 5.** (Gourieroux-Exercise 3.1 with r=2) Consider an ARCH(1) model. It can be written as

  $$Y_t = \sqrt{a_0 + a_1 Y_{t-1}^2}.\epsilon_t,$$

  where ϵ_t has a standard normal conditional distribution:

  $$\epsilon_t | \underline{\epsilon_{t-1}} \sim N(0, 1).$$

  Assume $3a_1^2 < 1$ and calculate $E(Y_t^2)$ and $E(Y_t^4)$.

4 Hints

- *Ad Exercise 1.* This can be shown by combining the Chebyshev inequality:

  $$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2},$$

where X is a random variable with mean μ and variance σ^2 and k an arbitrary positive number, with the Lemma of Borel-Cantelli, which says that if (A_t) is a sequence of (measurable) events and $A = \limsup_{t\to\infty} A_t = \cap_{s=t}^{\infty} \cup_{r=s}^{\infty} A_r$ then

$$\sum t = 1^{\infty} P(A_t) < \infty \Longrightarrow P(A) = 0.$$

Fix $\epsilon > 0$ and consider the event A that the sequence takes a value outside the ball centered at the origin, with radius ϵ infinitely often. Then it can be shown that $P(A) = 0$. Deduce from this that $\alpha^k \epsilon_{t-k}^2$ converges to zero for $k \longrightarrow \infty$.

- *Ad Exercise 2.*A proof by contradiction: Assume that the model is stationary with finite second moments and show that then the time-invariant second moment of ϵ_t is equal to

  $$E(\epsilon_t^2) = \frac{c}{1 - \sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i)}.$$

  Deduce from this that $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j \geq 1$ is impossible.

- *Ad Exercise 3.* First calculate the *characteristic function* $\phi(t) = E(\exp(itX))$ using the method of "splitting the square"(kwadraatafsplitsing). Then consider the fourth derivative of $\phi(t)$ at $t = 0$.

- *Ad Exercise 4.* We need to show that W_t is a function of $\underline{Y_t}$. To get the right intuition consider the logarithm of the absolute value on both sides of the model equation. Then one gets an MA(1) model. Show that this can be inverted to obtain an expression for $\log |W_t|$ in terms of $\log |Y_t|, \log |Y_{t-1}|, \ldots$.

- *Ad Exercise 5* Form the vector $W_t = (Y_t^4, Y_t^2)'$ and show that the conditional mean of this vector can be written as

  $$E(W_t | \underline{W_{t-1}}) = b + AW_{t-1}$$

  for some column vector b and some 2×2 upper triangular matrix A. Show that this matrix has its eigenvalues within the open unit disk (i.e. modulus of the eigenvalues less than one) if and only if

  $$3a_1^2 < 1.$$

  Calculate $E(W_t)$.