Notes on the Black-Scholes model in mathematical finance

Dr. Andreas E. Kyprianou
University of Utrecht, The Netherlands.

These notes accompany Tuesday's lectures given in the week 5th–9th March at the Stieltjes Institute. They are not self contained. Some prior experience of measure theory, probability theory, stochastic processes is assumed. Preferable but not essential is also some knowledge of Brownian motion and stochastic calculus. Some good texts for further reading are the following:

1. D. Lamberton and B. Lapeyre. *Introduction of Stochastic Calculus Applied to Finance.* Chapman and Hall publishers 1996.
2. N. Bingham and R. Keisel. *Risk Neutral Valuation.* Springer Finance series 1998.
3. B. Øksendal. *Stochastic Differential Equations (5th ed.).* Springer 1999.
4. D. Williams. *Probability with Martingales.* Cambridge University Press 1991.
5. M. Baxter and A. Rennie. *Financial Calculus.* Cambridge University Press 1997.
6. J. Hull. Options. *Futures and other derivatives.* Prentice Hall.
7. I. Karatzas and S.Shreve. *Brownian Motion and Stochastic Calculus.* Springer. 1988.

1 Introduction

An important event in the history of options trading, and a key event in the development of financial mathematics, occurred in 1973 when Myron Scholes and Fischer Black published a paper which showed how to price the European call option (we shall define this object formally in the next section). Their paper has been the basis for the paradigm used by practitioners in financial institutions the world over. The market model used by Black and Scholes for pricing is not a perfect fit with reality but it offers a mathematically robust framework which can be applied to the study of financial derivatives other than the European call.

In this short series of lectures our aim is to show how to recover the Black-Scholes option pricing formula from well known probability theory. Let us thus state formally the problem of European option pricing.

2 European Call Options

Suppose for simplicity we assume that a financial market consists of two elements. The first is a bank account which offers an interest rate $r > 0$ such that if at time 0 one has a credit/debit of R_0 units of currency, then after time t this credit/debit will have grown to $R_0 \exp\{rt\}$. The second element is a 'risky asset', typically a stock or share. This asset is risky in the sense that its value $\{S_t : t \geq 0\}$ changes in time as some kind of random process.

A financial institution will offer to sell an European Call option with parameters K and T. This means that at any instance of time this institution will write a contract that gives the holder of the contract at a time T later the option to buy one unit of the stock for price K. Since the value of the stock will fluctuate randomly, there is a chance that whoever holds the contract will be lucky at time T and will be able to buy the stock for K when it is worth $S_T > K$, thus making a profit of $S_T - K$. On the other hand, the holder may be unlucky and the price of the stock may fall below the value K at time T. In which case there is no point to exercise the right to buy at price K as then one would be paying more than the value of the stock. In summary, the holder stands to earn $(S_T - K)^+$ where $x^+ = \max\{x, 0\}$. Either way, the contract must be worth something. That is to say, anyone who wants to hold this contract should pay for it as there is the possibility that they can make money from it. There are two fundamental questions we should ask.

Q1: *What is the value of this contract?*

Q2: *Is it possible that financial institution can use the money paid for this contract to honour the claim $(S_T - K)^+$ without losing or making money?*

Obviously the answer to these question depends on the initial value of the stock S_0, the selling or *strike* price K, how far away in time the contract will expire T, the random evolution of the stock in the mean time and the fact that one can borrow and save money at the bank which gives a fixed interest rate r. We should also point out that in the second is it equally important that the financial institution does not lose or again money. If this were permitted in the solution to the problem then we could allow silly things like charging ridiculous amounts of money. Even if the financial institution were to charge slightly more than enough to honour the contract then their customers would soon realize it is more profitable to be selling options than buying options. In that case the market would become unstable.

The first step to answering the above questions is to use some kind of sensible mathematical model for the evolution of the asset value. In the original work of Black and Scholes, they chose to use exponential Brownian motion because, amongst other things, this process has exponential growth on average - a feature that is commonly shared by stocks.

We now pause with our discussion about finance and jump into the world of Brownian motion and stochastic analysis in order to review the probabilistic tools we shall use to solve the problem of pricing the European option just as Black and Scholes did.

3 A brief review of some aspects of Brownian motion and stochastic analysis

3.1 Brownian Motion

To really define Brownian motion mathematically rigorously requires quite a lot of work. The definition we shall offer here is somewhat primitive but sufficient to work with.

Definition 1 *The process $B = \{B_t : t \in [0,T]\}$ is a standard Brownian motion with respect to a probability measure $\mathbb{P}$ if*

(i) $B_0 = 0$ $\mathbb{P}$*-almost surely,*

(ii) *for $s \le t$, $B_t - B_s$ is independent of the evolution of the process up to time s, which we shall typically refer to as $\{B_r : r \le s\}$* [1]*,*

(iii) *for $0 \le s \le t$, $B_t - B_s$ and $B_{t-s} - B_0$ have the same distribution which is Normal with mean zero and variance $t - s$.*

Definition 2 *The process $W = \{W_t : t \in [0,T]\}$ is a Brownian motion with volatility $\sigma > 0$ and drift $\mu \in \mathbb{R}$ if for all $t \ge 0$, $W_t = \mu t + \sigma B_t$.*

Denoting $\mathbb{E}$ expectation with respect to $\mathbb{P}$ we see that $\mathbb{E}(W_t) = \mu t$, so that on average the process W is *drifting* with velocity μ.

Definition 3 *With W as in the previous definition, the process $\{\exp W_t : t \in [0,T]\}$ is an exponential Brownian motion with volatility $\sigma \in \mathbb{R}$.*

Note that $\mathbb{E}(\exp W_t) = \exp\{(\mu + \sigma^2/2)\, t\}$ so that exponential Brownian motion grows or decays (depending on the sign of $(\mu + \sigma^2/2)$) on average as an exponential function of time. [Recall that B_t is Normally $B_t \sim N(0,t)$ so it has moment generating function $\mathbb{E}(\exp\{\sigma B_t\}) = \exp\{\sigma^2 t/2\}$]

The paths of Brownian motion are very rough. In fact it can be proved that, even though the paths are continuous, they are nowhere differentiable. This raises questions concerning the existence of a process that satisfies the conditions in Definition 1. *Brownian motion does exist* as a well defined mathematical

[1] This kind of statement will appear through this text. Strictly speaking, for each $t \in [0,T]$ we should define a sigma-algebra $\mathcal{F}_t$ generated by the paths of the Brownian motion up to time t. This means take a countable set of times $\{t_i : i \ge 1\}$ in $[0,T]$ and define correspondingly a set of open intervals $A_i \subseteq \mathbb{R}$. Then $\mathcal{F}_t$, or alternitively $\sigma\{B_r : r \le t\}$, is the smallest sigma algebra containing all events of the form $\{B_{t_i} \in A_i : i \ge 1\}$.

In this text we use $\{B_r : r \le t\}$ in place of 'the evolution (or path) of the Brownian motion up to time t'. In fact what we really mean is $\mathcal{F}_t$.

object and indeed there are several proofs one can consult. Conceptually one of the most pleasing, which also would convince the reader that the paths are nowhere differentiable, concerns demonstrating that paths of Brownian motion can be seen as (infinitely) rescaled paths of simple random walks. Despite the rather erratic behaviour of Brownian motion, it is an extremely flexible object mathematically speaking. One can use Brownian motion for example to construct many examples of continuous martingales. Here is a quick reminder of what a martingale is.

3.2 Martingales

Definition 4 *A continuous stochastic process $M = \{M_t : t \in [0,T]\}$ is a martingale if*

(ii) $\mathbb{E}|M_t| < \infty$ *of each* $t \in [0,T]$,

(iii) *for* $0 \le s \le t \le T$, $\mathbb{E}(M_t |\ M_r : r \le s) = M_s$.

Note that it follows that $\mathbb{E}(M_t) = \mathbb{E}(M_0)$ for all $t \in [0,T]$. Here are three examples of martingales which are simple functional of Brownian motion:

1. B_t,
2. $B_t^2 - t$
3. and $\exp\{\sigma B_t - \sigma^2 t/2\}$

for $t \in [0,T]$.

Exercise 5 *Prove that the first two are martingales.*

For the third we shall provide a short proof that is really is a martingale. First let $M_t = \exp\{\sigma B_t - \sigma^2 t/2\}$. Using the properties of Brownian motion in Definition 1 it follows that for $0 \le s \le t \le T$,

$$\begin{aligned}\mathbb{E}(M_t |\ M_r : r \le s) &= \mathbb{E}\left(M_s \times \exp\left\{\sigma\left(B_t - B_s\right) - \sigma^2\left(t-s\right)/2\right\} \middle|\ M_r : r \le s\right) \\ &= M_s \times \exp\left\{-\sigma^2\left(t-s\right)/2\right\} \times \mathbb{E}(\exp\{\sigma B_{t-s}\}) \\ &= M_s.\end{aligned}$$

Note that going from the first to the second equality, we have used the fact that knowing the path $\{M_r : r \le s\}$ is equivalent to knowing $\{B_r : r \le s\}$. Note also that using the law of total probability

$$\mathbb{E}[\mathbb{E}(M_t |\ M_r : r \le s)] = \mathbb{E}[M_t]$$

and hence $\mathbb{E}[M_t] = \mathbb{E}[M_s]$ for all s, t. In particular this means that $\mathbb{E}[M_t] = \mathbb{E}[M_0] = 1$ for all t.

This martingale is special for many reasons. One of its special properties is that it can be used to make a new probability measure with respect to which the Brownian motion B is no longer standard but has drift. Formally said this is the Cameron-Martin-Girsanov Theorem as follows.

Theorem 6 *For each $\lambda \in \mathbb{R}$, there exists a probability measure $\mathbb{P}^{\lambda}$ such that for each $t \in [0,T]$, and event A which can be described in terms of $\{B_r : r \leq t\}$*

$$\mathbb{P}^{\lambda}(A) = \mathbb{E}\left(\mathbf{1}_A \exp\{-\lambda B_T - T\lambda^2/2\}\right).$$

Further the process B^{λ} defined by $B_t^{\lambda} = B_t + \lambda t$ for $t \in [0,T]$ is a standard Brownian motion with respect to $\mathbb{P}^{\lambda}$. That is to say, under the new measure $\mathbb{P}^{\lambda}$, a drift of rate $-\lambda$ has been introduced to the Brownian motion B.

This is a very deep result which really deserves a lot more attention. It will be of great use later in pricing the European option. Note that a standard alternative to writing a probability $\mathbb{P}(A)$ is $\mathbb{E}(\mathbf{1}_A)$. And hence we see that this change of measure is really re-distributing the probability mass on different events. It is clear to see that $\mathbb{P}^{\lambda}$ is indeed a probability measure because by taking the even $A = \{B_t^{\lambda} \in \mathbb{R}\}$ for any $t \in [0,T]$ we see that we are really taking the expectation of the exponential martingale which is 1. One notices that when B moves to a high position at time T the re-weighting in the definition of $\mathbb{P}^{\lambda}$ is small. On the other hand, if B moves to a very low, negative position at time T the re-weighting is large. This is consistent with the idea that under $\mathbb{P}^{\lambda}$ the Brownian motion acquires drift $-\lambda$, that is to say, $\mathbb{P}^{\lambda}$ is favorable to paths which move downwards.

The following exercise is designed to give a little 'hands-on' experience with this result to see how it works.

Exercise 7 *(i) Using the same notation as in the previous theorem, use the facts about the exponential Brownian motion to show that in fact*

$$\mathbb{P}^{\lambda}(A) = \mathbb{E}\left(\mathbf{1}_A \exp\{-\lambda B_t - t\lambda^2/2\}\right) \text{ for } t \in [0,T].$$

Note that it is important here that A is an event determined by the path up to time t. This exercise really shows that the reweighting 'localizes' for events 'up to time t' to the value of the exponential martingale at time t.

(ii) Now choose A to be the event $\{B_t + \lambda t \in I\}$ where I is some interval of the real line. Use the fact that Brownian motion has a Normal distribution at each fixed time t to deduce that

$$\mathbb{P}^{\lambda}(B_t + \lambda t \in I) = \mathbb{P}(B_t \in I)$$

and reason intuitively that $\mathbb{P}^{\lambda}$ has the effect of introducing drift $-\lambda$ to B.

3.3 Stochastic integration

Without defining any of the following symbols, we immediately recognize and understand the object

$$\text{“}\int f(s)\, d\mu(s)\text{”}$$

to be some kind of Lebesgue or perhaps Riemann integral. Suppose now that the integrand f was a random function of s and the measure μ is replaced by Brownian motion B. Could such an integral as

$$\text{“}\int f(s,\omega)\, dB_s(\omega)\text{”}$$

be well defined for almost every path ω of a Brownian motion? Further, if so, then what are its properties? These are elementary and natural questions which leads in to the theory of stochastic integration. The first problem we must address is: what exactly does dB_s mean? As we have already mentioned, the paths of Brownian motion, whilst continuous, are differentiable nowhere. So the naive interpretation

$$dB_s = \frac{dB_s}{ds} ds$$

is senseless. Just like the theory of Lebesgue integration we must start from the case when f is a 'simple function' and work our way up.

We shall define $\{\Phi_t : 0 \leq t \leq T\}$ a *simple process* if it can be written as

$$\Phi_t(\omega) = \sum_{i=1}^{p} \phi_{i-1}(\omega)\, \mathbf{1}_{(t_{i-1},t_i]}(t) \tag{1}$$

where $0 = t_0 < \ldots < t_p = T$ and each $\phi_i \in \mathbb{R}$ is bounded and its value can be determined by the path $\{B_r : r \leq t_{i-1}\}$; that is to say it is a process *adapted* to the increasing flow of information supplied by the Brownian motion. A reasonable definition for our stochastic integral *for simple processes* is thus when $t_k < t \leq t_{k+1} \leq T$

$$\begin{aligned}\int_0^t \Phi_s dB_s &= \sum_{1\leq i\leq k} \phi_{i-1}\left(B_{t_i} - B_{t_{i-1}}\right) + \phi_k\left(B_t - B_{t_k}\right) \\ &= \sum_{1\leq i\leq p} \phi_{i-1}\left(B_{t_i\wedge t} - B_{t_{i-1}\wedge t}\right). \end{aligned} \tag{2}$$

This stochastic integral for simple processes have the following important features.

- The integral as a function of time is a continuous process. This is obvious from (2)
- Given $\{B_r : r \leq t\}$ the value of $\int_0^t \Phi_s dB_s$ can be determined (so again it is an adapted process).
- Stochastic integration on simple functions is a linear operator. That is for two simple functions $\Phi^{(1)}$ and $\Phi^{(2)}$ and constants α_1 and α_2,

$$\int_0^t \left(\alpha_1\Phi_s^{(1)} + \alpha_2\Phi_s^{(2)}\right) dB_s = \alpha_1 \int_0^t \Phi_s^{(1)} dB_s + \alpha_2 \int_0^t \Phi_s^{(2)} dB_s$$

- Note that

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t \Phi_s dB_s\right)^2\right] &= \mathbb{E}\left[\sum_{i,j} \phi_{i-1}\phi_{j-1}\left(B_{t_i} - B_{t_{i-1}}\right)\left(B_{t_j} - B_{t_{j-1}}\right)\right] \\ &= \sum_{i,j} \mathbb{E}\left[\phi_{i-1}\phi_{j-1}\left(B_{t_i} - B_{t_{i-1}}\right)\left(B_{t_j} - B_{t_{j-1}}\right)\right].\end{aligned}$$

For $i < j$ in the above expression we can use the fact that B is a martingale and show that

$$\begin{aligned}&\mathbb{E}\left[\phi_{i-1}\phi_{j-1}\left(B_{t_i} - B_{t_{i-1}}\right)\left(B_{t_j} - B_{t_{j-1}}\right)\right] \\ =\ &\mathbb{E}\left[\phi_{i-1}\phi_{j-1}\left(B_{t_i} - B_{t_{i-1}}\right)\mathbb{E}\left[\left(B_{t_j} - B_{t_{j-1}}\right)|B_r : r \leq t_{j-1}\right]\right] \\ =\ &0\end{aligned}$$

and likewise for $j < i$. For those terms where $i = j$ we also have from the fact that $B_t^2 - t$ is a martingale that

$$\begin{aligned}\mathbb{E}\left[\phi_{i-1}^2\left(B_{t_i} - B_{t_{i-1}}\right)^2\right] &= \mathbb{E}\left[\phi_{i-1}^2\mathbb{E}\left[\left(B_{t_i} - B_{t_{i-1}}\right)^2|B_r : r \leq t_{i-1}\right]\right] \\ &= \mathbb{E}\left[\phi_{i-1}^2(t_i - t_{i-1})\right].\end{aligned}$$

Thus it follows that

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t \Phi_s dB_s\right)^2\right] &= \sum_i \mathbb{E}\left[\phi_{i-1}^2(t_i - t_{i-1})\right] \\ &= \mathbb{E}\left[\sum_i \phi_{i-1}^2(t_i - t_{i-1})\right] \\ &= \mathbb{E}\left[\int_0^t \Phi_s^2 ds\right]. \qquad (3)\end{aligned}$$

This property (3) is known as the Itô isometry.

Exercise 8 *Prove that $\mathbb{E}\left(\int_0^t \Phi_s dB_s\right) = 0$ for all $t \in [0,T]$ and then deduce the final property below. [Hint: note that for simple functions the integral is simply a sum of weighted martingale differences].*

- The stochastic integral $\int_0^t \Phi_s dB_s$ is also a martingale.

Just as one constructs the Lebesgue integral, one can now define a more general set of integrands Φ for which the integral $\int_0^t \Phi_s dB_s$ exists by considering functions that can be approximated by simple functions.

In view of the fact that our simple functions are adapted processes and satisfy the relation (3), it is not a surprise to announce that we can define in general our stochastic integral $\int_0^t \Phi_s dB_s$ for integrands in the following class

$$\mathcal{H} = \left\{(\Phi_t)_{0\leq t\leq T}\text{, real adapted process, } \mathbb{E}\left[\int_0^T \Phi_s^2 ds\right] < \infty\right\}.$$

The fine details we shall leave out as they are quite demanding and is essentially based on functional analysis and the theory of Hilbert spaces. The main ideas however are the following.

Suppose we define $\mathcal{H}^s$ to be the space of simple functions in $\mathcal{H}$ and define a *distance metric* on $\mathcal{H}$ such that for $\Phi \in \mathcal{H}$

$$[\Phi] = \sqrt{\mathbb{E}\left[\int_0^T \Phi_s^2 ds\right]}.$$

An important first result is that $\mathcal{H}^s$ is dense in $\mathcal{H}$ with respect to $[\cdot]$. This means that all points in $\mathcal{H}$ can be written as limits of sequences of elements of $\mathcal{H}^s$. That is there exists a sequence $\left\{\Phi^{(n)}\right\}_{n\geq 0}$ such that

$$[\Phi - \Phi^{(n)}]^2 = \mathbb{E}\left[\int_0^T \left(\Phi_s - \Phi_s^{(n)}\right)^2 ds\right] \to 0 \tag{4}$$

as $n \to \infty$. The next step is to define another distance metric $||\cdot||$ on random variables with finite variance so that for any such X

$$||X|| = \sqrt{\mathbb{E}[X^2]}.$$

This distance metric is very closely related to the previous in that for simple functions $\Phi \in \mathcal{H}^s$, Itô's isometry implies that

$$\left\|\int_0^t \Phi_s dB_s\right\| \leq [\Phi].$$

for any $t \in [0,T]$. The convergence (4) implies that the sequence $\left\{\Phi^{(n)}\right\}_{n\geq 0}$ whose limit point is the $\Phi \in \mathcal{H}$ is a Cauchy sequence with respect to $[\cdot]$, in other words $[\Phi^{(n)} - \Phi^{(m)}] \to 0$ as n, m tend to infinity. The linearity of the stochastic integral as an operator on simple functions, and the previous facts imply that uniformly for $t \in [0,T]$

$$\left\|\int_0^t \Phi_s^{(n)} dB_s - \int_0^t \Phi_s^{(m)} dB_s\right\| \leq [\Phi^{(n)} - \Phi^{(m)}] \to 0 \tag{5}$$

as $n, m \to \infty$. Let $\mathcal{M}^2[0,T]$ be the space of continuous martingales indexed by $[0,T]$ with finite variance for each $t \in [0,T]$. The convergence in (5) suggests that the sequence of stochastic integral processes $\left\{\int_0^t \Phi_s^{(n)} dB_s : t \in [0,T]\right\}_{n\geq 0}$ is a Cauchy sequence of martingales in $\mathcal{M}^2$ with respect to another distance metric $||\cdot||^{\text{sup}}$ where given such a martingale $M \in \mathcal{M}^2[0,T]$

$$||M_\cdot||^{\text{sup}} = \sup_{t\in[0,T]} ||M_t|| = ||M_T||\,.$$

Note then that this metric measures distance between the entire path of martingales in $\mathcal{M}^2[0,T]$. The last equality, or equivalently the fact that these type of martingales have increasing variance, can be proved by Jensen's inequality. Note then that for stochastic integrals of simple functions

$$\left\| \int_0^\cdot \Phi_s dB_s \right\|^{\sup} = [\Phi].$$

With a little more work it can be shown that the existence of a Cauchy sequence with respect to $\|\cdot\|^{\sup}$ is enough to guarantee a limit (in the same sense of distance) exists and is a martingale in $\mathcal{M}^2[0,T]$.

Definition 9 *The stochastic integral can now be defined as the continuous, finite variance martingale which we denote "$\int_0^t \Phi_s dB_s$" for $t \in [0,T]$ such that*

$$\left\| \int_0^\cdot \Phi_s dB_s - \int_0^\cdot \Phi_s^{(n)} dB_s \right\|^{\sup} = \left\| \int_0^T \Phi_s dB_s - \int_0^T \Phi_s^{(n)} dB_s \right\| \to 0$$

where $[\Phi - \Phi^{(n)}] \to 0$. In addition to the martingale property being carried over in the limit, all the other properties of stochastic integrals of simple processes do too.

It seems from this definition that we have rather an abstract definition of a stochastic integral as some kind of limiting process. The real issue about the stochastic integral is that, given $\Phi \in \mathcal{H}$ we are able to find a martingale $M \in \mathcal{M}^2[0,T]$ which has all the same characteristics as the stochastic integral we defined for simple processes and in particular its relation to Φ is that its second moment at time $t \in [0,T]$ satisfies

$$\mathbb{E}\left(M_t^2\right) = \mathbb{E}\left(\int_0^t \Phi_s^2 ds\right).$$

The identified martingale M is what we mean by the 'stochastic integral' $\int_0^\cdot \Phi_s dB_s$. We need to find ways of manipulating these processes. That is to say, we need some calculus! This will come a little later. For the time being, we shall make some more technical remarks about stochastic integrals as processes.

3.4 Martingale representation

In the previous subsection is was pointed out that the defined *stochastic integrals are also continuous finite variance martingales.* We may ask ourselves is the converse of this previous statement true? That is to say, can any martingale (lets be reasonable and say continuous) be written as a stochastic integral?

Theorem 10 *Suppose that $M \in \mathcal{M}^2[0,T]$ is a continuous finite variance martingale such that M_t is adapted to $\{B_r : r \leq t\}$. Then there exists a process $\Phi \in \mathcal{H}$ such that for all $0 \leq t \leq T$*

$$M_t = M_0 + \int_0^t \Phi_s dB_s \quad a.s.$$

This will turn out to be very handy later!

3.5 Itô diffusions and exponential Brownian motion

It should be clear now that the stochastic integrals we have defined are more that just integrals. They are actually continuous stochastic processes on. We can use these stochastic integrals to define a more general class of continuous stochastic processes called Itô diffusions as follows.

Definition 11 *A stochastic process $\{X_t : 0 \leq t \leq T\}$ on is a real valued Itô diffusion if it can be written as*

$$X_t = X_0 + \int_0^t \Psi_s ds + \int_0^t \Phi_s dB_s \tag{6}$$

$\mathbb{P}$-almost surely where

(i) *X_0 is $\mathcal{F}_0$-measurable,*

(ii) *$\{\Psi_t : 0 \leq t \leq T\}$ and $\{\Phi_t : 0 \leq t \leq T\}$ are adapted to B*[2]

(iii) *$\int_0^t |\Psi_s| ds < \infty$ $\mathbb{P}$-a.s. and*

(iv) $\mathbb{E}\left(\int_0^t \Phi_s^2 ds\right) < \infty$

These conditions are merely sensible requirements so that the integrals in (6) exist. They are thus sufficient conditions.

Commonly the Itô diffusion in (6) is written in shorthand form

$$dX_t = \Psi_t dt + \Phi_s dB_t. \tag{7}$$

3.6 Basic stochastic calculus

There are two fundamental manipulations that any beginner of stochastic calculus should know. The Itô formula and Integration by parts formula.

Theorem 12 (Itô formula) *Let $\{X_t : 0 \leq t \leq T\}$ be an Itô diffusion as in (7) and suppose that $f(t,x)$ is once differentiable in t and twice differentiable in x with continuous partial derivatives. Then for $0 \leq t \leq T$*

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)\, dt + \frac{\partial f}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\, \Phi_s^2 dt. \tag{8}$$

[2] We understand this to mean that for each $t \in [0, T]$, Φ_t and Ψ_t are are adapted to $\{B_r : r \leq t\}$.

That is to say in integral form,

$$
\begin{aligned}
f(t,X_t) &= f(0,X_0)+\int_0^t \frac{\partial f}{\partial t}(s,X_s)\,ds \\
&\quad +\int_0^t \frac{\partial f}{\partial x}(s,X_s)\,dX_s+\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s)\,\Phi_s^2 ds \\
&= f(0,X_0)+\int_0^t \frac{\partial f}{\partial t}(s,X_s)\,ds \\
&\quad +\int_0^t \frac{\partial f}{\partial x}(s,X_s)\,\Psi_s ds+\int_0^t \frac{\partial f}{\partial x}(s,X_s)\,\Phi_s dB_s+\frac{1}{2}\int_0^t f''(X_s)\,\Phi_s^2 ds.
\end{aligned}
$$

The following exercise shows a classic example of how stochastic calculus can differ significantly from the calculus we know for Lebesgue integrals.

Exercise 13 *Let $X=B$, $f(x)=x^2$ and apply (8) to deduce that*

$$d\left(B_t^2\right)=2B_s dB_t+dt$$

and hence

$$\int_0^t B_s dB_s=\frac{1}{2}\left(B_t^2-t\right).$$

Compare this to the Lebesgue $\int_0^t s\ ds$.

Theorem 14 (Integration by parts) *Let $\{X_t : 0\le t\le T\}$ be an Itô diffusion, and suppose that $g(t)$ is a differentiable function with continuous derivatives. Then*

$$d\left(g(t)\,X_t\right)=g(t)\,dX_t+g'(t)\,X_t dt$$

Proof. This is a direct consequence of the Itô formula for the special choice $f(t,X_t)=g(t)\,X_t$. ■

Exercise 15 *It is possible to prove a more general result than the previous Theorem. Suppose that $X^{(1)}$ and $X^{(2)}$ are two Itô diffusions driven by the same Brownian motion satisfying $dX_t^{(i)}=\Psi_s^{(i)}ds+\Phi_s^{(i)}dB_s$, $i=1,2$. Use the Itô formula to write down expressions for $\left(X_t^{(1)}+X_t^{(2)}\right)^2$, $\left(X_t^{(1)}\right)^2$ and $\left(X_t^{(1)}\right)^2$. Then subtract $\left(X_t^{(1)}\right)^2$ and $\left(X_t^{(1)}\right)^2$ from $\left(X_t^{(1)}+X_t^{(2)}\right)^2$ and conclude that*

$$d\left(X_t^{(1)}X_t^{(2)}\right)=X_t^{(1)}dX_t^{(2)}+X_t^{(2)}dX_t^{(1)}+\Phi_t^{(1)}\Phi_t^{(2)}dt.$$

Exercise 16 *From the previous example consider the combinations of $\Psi_s^{(i)}=0,1$, $\Phi_s^{(i)}=0,1$ for $i=1,2$ to deduce the following* ***informal*** *'algebra' for differentials:*

$$d\left(X_t^{(1)}X_t^{(2)}\right)=X_t^{(1)}dX_t^{(2)}+X_t^{(2)}+dX_t^{(1)}\cdot dX_t^{(2)}$$

where $dX_t^{(1)} \cdot dX_t^{(2)}$ *is computed according to*

	dt	dB_t
dt	0	0
dB_t	0	dt

.

3.7 Exponential Brownian motion

We can use the results of the previous section to make a study of exponential Brownian motion, the diffusion we shall use to model the evolution of our stock. Consider a process $\{S_t : 0 \leq t \leq T\}$ which satisfies

$$dS_t = S_t\,(\mu dt + \sigma dB_t) \tag{9}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. We are looking for an adapted process that satisfies (9) such that the integrals $\int_0^t S_u du$ and $\int_0^t S_u dB_u$ exist for $0 \leq t \leq T$. The process S is defined in terms of itself and has the form of an Itô diffusion (providing the afore mentioned integrals exist). Processes that can be written as integrals of themselves as in the case of (9) are said to satisfy stochastic differential equations. Just like ordinary differential equations there are issues concerning existence and uniqueness of their solutions. This lies way beyond the scope of this text.

It is possible to prove using the Itô formula and the integration by parts formula to prove that indeed (9) has a solution and it is unique. The solution is what we defined earlier as exponential Brownian motion.

Exercise 17 (Existence) *If we think about ordinary differential equations, it is reasonable to guess that (9) has an exponential feel about it. So it would make sense to try and find a solution of the form* $c.\exp\{\alpha t + \beta B_t\}$ *where* c, α *and* β *are constants. In fact the correct choices of* α *and* β *are* $(\mu - \sigma^2/2)$ *and* σ *respectively so*

$$S_t = c.\exp\left\{(\mu - \sigma^2/2)t + \sigma B_t\right\}.$$

This can be confirmed by plugging $X_t = (\mu - \sigma^2/2)dt + \sigma B_t$ *into the Itô formula with the function* $f(x) = c.\exp\{x\}$.

Exercise 18 (*Uniqueness*) *Suppose now that* Y *is another solution to the stochastic differential equation (9). Consider the process* YZ *where* $Z = 1/X$. *Note that* $Z_t = \exp\left\{-(\mu - \sigma^2/2)t - \sigma B_t\right\}$ *and show (again from the Itô formula) that*

$$dZ_t = Z_t\left(-\left(\mu - \sigma^2\right)t - \sigma dB_t\right).$$

Applying the integration by parts formula in Exercise 15 deduce that

$$\begin{aligned} d\left(Y_t Z_t\right) &= Y_t Z_t\left(-\left(\mu - \sigma^2\right)t - \sigma dB_t\right) \\ &\quad + Y_t Z_t\left(\mu dt + \sigma dB_t\right) - \sigma^2 Z_t Y_t dt \\ &= 0, \end{aligned}$$

and hence uniqueness (up to a multiplicative constant).

4 Derivation of the Black-Scholes formula

4.1 The market

Lets get back to business! Recall that (following the original paper of Black and Scholes) we will assume our market to consist of two simple instruments. A bank account R process and a risky asset (stock or share) S.

The bank account R evolves as an exponential growth with rate $r > 0$, that is to say

$$\frac{dR_t}{R_t} = rdt.$$

This is rather a fancy way of writing $R_t = R_0 \exp\{rt\}$, however it is worth reminding ourselves that this is the simple ordinary differential equation satisfied by such an exponential growth as it will be of use later. Also from an economists perspective this is a more natural way of expressing the way in which a bank account works. It says in an instantaneous increment of time dt, the return on your investment R_t is proportional to the time period and at a fixed rate r. We will see shortly that stochastic differential equations also have similar interpretations.

The value of the risky asset S we will now assume to evolve as a stochastic process $\{S_t : t \geq 0\}$ driven by the following stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

the same as (9). Again let us consider this relation from the point of view of an economist. It says that in a infinitesimal increment of time, the proportional change in the asset value is like that of a deterministic return with rate μ together with some added random noise σdB. Of course we now know that this equation has a unique solution which is the exponential Brownian motion

$$S_t = S_0 \exp\left\{(\mu - \sigma^2/2)t + \sigma B_t\right\}. \tag{10}$$

Apart from the previous comments, this is a fitting model for the evolution of a risky asset because;

- the sample paths are continuous,
- the relative increments, $(S_t - S_u)/S_u$ where $u \leq t$ are independent of the history of the asset, namely $\{S_r : r \leq u\}$ (this follows directly as a consequence of the fact that B is a Brownian motion)
- the process is stationary, that is to say that $(S_t - S_u)/S_u$ where $u \leq t$ has the same distribution as $(S_{t-u} - S_0)/S_0$ (again this follows as a consequence of the underlying Brownian motion)

4.2 Self financing strategy

Let us put ourselves in the seat of the financial institution offering to sell the European option. Black and Scholes reasoned that the payment that you will receive for selling this option should be used to dynamically invest in the two available financial instruments, R and S, for the duration of the contract so that the end value of our investment is precisely equal to the claim of the contract holder, namely $(S_T - K)^+$. Technically speaking, as the seller of the European option we wish to *replicate* the claim $(S_T - K)^+$ by *hedging* with the market .

It seems a strange proposition; that we will begin with a deterministic amount (the money we have recovered by selling the European option) and by some how buying and selling on the market (R, S) over a period $[0, T]$ we end up with a random amount of money, but none the less for each outcome ω of the path of the path of the risky asset (equivalently the path of the Brownian motion) up to time T, this random amount is exactly equal to $(S_T(\omega) - K)^+$.

Let us say that at time $0 \le t \le T$ we will hold Ψ_t units of R and Φ_t units of S. We call the process $\Pi := \{(\Psi_t, \Phi_t) : 0 \le t \le T\}$ our portfolio process, (Ψ_t, Φ_t) is the hedge at time t and the value of our portfolio at time t is thus

$$V_t(\Pi) = \Psi_t R_t + \Phi_t S_t.$$

Our job is therefore to show that we can find a hedging system such that the end value of the portfolio $V_T(\Pi) = (S_T - K)^+$, in which case $V_0(\Pi)$ should be the price of the option. Note that our choice of (Ψ_t, Φ_t) will end up to be a function of the asset value S_t and time t, a and therefore they will be random processes too.

The hedge should also be a self financing strategy. That is to say once we have made our initial investment (Ψ_0, Φ_0) (using the money paid by the holder of the option to buy it) at any instance of time our portfolio should only change in value as a consequence of the evolution in the value of our holdings R and S. That is to say the portfolio is not interfered with by the addition or removal of funds. Mathematically speaking we would like to express this as

$$dV_t(\Pi) = \Psi_t dR_t + \Phi_t dS_t. \tag{11}$$

This would imply that $V(\Pi)$ is an Itô diffusion providing certain conditions hold. Thus formally speaking we make the following mathematical definition of a self-financing strategy.

Definition 19 *The process (Ψ, Φ) is self financing if (11) holds, both Ψ and Φ are adapted to B and the following integrability conditions hold,*

$$\int_0^t |\Psi_s| ds < \infty \ \mathbb{P}\text{-}a.s. \text{ and } \mathbb{E}\left(\int_0^t \Phi_s^2 ds\right) < \infty \tag{12}$$

4.3 Discounted values

Since, according to the bank, any credit or debit, grows at a rate r, the whole problem of pricing over the time interval $[0, T]$ would be easier to handle if the value of money remained constant. To this end let us define $\widetilde{S}_t$ and $\widetilde{V}_t(\Pi)$ the discounted values of the asset and portfolio respectively. That is $\widetilde{S}_t = e^{-rt} S_t$ and $\widetilde{V}_t(\Pi) = e^{-rt} V_t(\Pi)$ so that the value of the asset and portfolio are expressed in the terms of the value of money at time 0. Using the integration by parts formula from earlier for the processes $X_t^{(1)} = e^{-rt}$ and $X_t^{(2)} = S_t$ we have

$$\begin{aligned} d\widetilde{S}_t &= -re^{-rt} S_t dt + e^{-rt} dS_t \\ &= \widetilde{S}_t \left((\mu - r) \, dt + \sigma dB_t \right). \end{aligned} \tag{13}$$

Likewise

$$\begin{aligned} d\widetilde{V}_t &= -re^{-rt} V_t dt + e^{-rt} dV_t \\ &= -re^{-rt} \left(\Psi_t e^{rt} + \Phi_t S_t \right) dt + e^{-rt} \left(\Psi_t r e^{rt} dt + \Phi_t dS_t \right) \\ &= -r\Phi_t \widetilde{S}_t dt + e^{-rt} \Phi_t S_t \left(\mu dt + \sigma dB_t \right) \\ &= \Phi_t \widetilde{S}_t \left((\mu - r) \, dt + \sigma dB_t \right) dt \\ &= \Phi_t d\widetilde{S}_t. \end{aligned}$$

Exercise 20 *Prove the converse, that when (12) holds $d\widetilde{V}_t = \Phi_t d\widetilde{S}_t$ implies we have a self financing portfolio and deduce that, again with condition (12)*

$$d\widetilde{V}_t = \Psi_t dR_t + \Phi_t dS_t \Longleftrightarrow d\widetilde{V}_t = \Phi_t d\widetilde{S}_t. \tag{14}$$

Mathematically we can read two things out of (13) and (14). The first is that when (Ψ, Φ) satisfies the conditions (12) then $\widetilde{V}_t(\Pi)$ has increments instantaneously proportional to the increments of $\widetilde{S}_t$ if and only if it is self financing. Secondly suppose that $\mu - r = 0$ so that $d\widetilde{S}_t$ is proportional to dB_t. From our previous discussion about stochastic integrals that would imply that $\widetilde{S}_t$ and hence $\widetilde{V}_t$ are martingales. However, in general, it will not be the case that $\mu - r = 0$. If we look back to our comments about Brownian motion, then we know the solution to (13) is an exponential Brownian motion of the form $\exp\{ \left(\mu - r - \sigma^2/2 \right) t + \sigma B_t \}$ which is not quite a martingale according to Definition 3. If we could introduce a drift of $(\mu - r)/\sigma$ to B then we would certainly have a martingale. But we know how to do this via the Cameron-Martin-Girsanov Theorem. Writing $B_t^{(\mu - r)/\sigma} = B_t + (\mu - r) \, t / \sigma$ we have

$$d\widetilde{S}_t = \sigma \widetilde{S}_t dB_t^{(\mu - r)/\sigma}$$

where $B^{(\mu - r)/\sigma}$ is a $\mathbb{P}^{(\mu - r)/\sigma}$-standard Brownian motion. It is now clear (modulo checking the integrability conditions) that under $\mathbb{P}^{(\mu - r)/\sigma}$ both $\widetilde{S}_t$ and $\widetilde{V}_t$ are martingales.

4.4 Black-Scholes formula

The essential idea behind Black and Scholes' pricing formula is that one can *replicate* the claim $(S_T - K)^+$ is the following way. Find a $\mathbb{P}^{(\mu - r)/\sigma}$-martingale whose terminal value at time T is exactly the claim $(S_T - K)^+$. Then since $\widetilde{V}_t(\Pi)$ is also $\mathbb{P}^{(\mu-r)/\sigma}$-martingale for any self financing strategy, the Martingale Representation Theorem gives us the instantaneous ratio of the increments of these two martingales. From this it is easy to show that a self financing strategy not only exists but it is unique. Here is the argument in more detail.

First begin by noticing that

$$M_t := \mathbb{E}^{(\mu-r)/\sigma}\left(e^{-rT}(S_T - K)^+ | B_r : r \leq t\right) \text{ for } 0 \leq t \leq T$$

is martingale by the tower property of conditional expectation. Also the martingale has the terminal value $M_T = e^{-rT}(S_T - K)^+$ and since

$$\mathbb{E}^{(\mu-r)/\sigma}\left[\left((S_T - K)^+\right)^2\right] \leq \mathbb{E}^{(\mu-r)/\sigma}\left[S_T^2\right] < \infty$$

(this can be shown by manual computation using the moment generating function of Normal distributions) it follows that $\mathbb{E}^{(\mu-r)/\sigma}\left(M_t^2\right) < \infty$ for all $0 \leq t \leq T$ (recall that finite variance martingales have increasing variance by Jensen's inequality). Therefore according to Theorem 10 there exists a process $\Lambda = \{\Lambda_t : 0 \leq t \leq T\}$ such that $\mathbb{E}^{(\mu-r)/\sigma}\left(\int_0^T \Lambda_t^2 dt\right) < \infty$ and $dM_t = \Lambda_t dB_t^{(\mu-r)/\sigma}$

Now define

$$\Phi_t = \frac{\Lambda_t}{\sigma \widetilde{S}_t} \text{ and } \Psi_t = M_t - \widetilde{S}_t \Phi_t \text{ for } 0 \leq t \leq T$$

and note that the pair (Ψ, Φ) form a self financing strategy. Indeed it can be checked that $\int_0^t |\Psi_s| ds < \infty$ $\mathbb{P}$-a.s. and $\mathbb{E}^{(\mu-r)/\sigma}\left(\int_0^t \Phi_s^2 ds\right) < \infty$ Further

$$\widetilde{V}_t(\Pi) = \Psi_t + \Phi_t \widetilde{S}_t = M_t \tag{15}$$

so that

$$d\widetilde{V}_t = dM_t = \Lambda_t dB_t^{(\mu-r)/\sigma} = \Phi_t \times \sigma \widetilde{S}_t dB_t^{(\mu-r)/\sigma} = \Phi_t d\widetilde{S}_t$$

so that by (14) $dV_t = \Psi_t dR_t + \Phi_t dS_t$.

¿From (15) it would also appear that

$$V_t(\Pi) = e^{rt} M_t = \mathbb{E}^{(\mu-r)/\sigma}\left(e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t\right)$$

so that indeed $V_T(\Pi) \equiv (S_T - K)^+$.

We now have the answer to our first question. The value of a European Call option is

$$V_0(\Pi) = \mathbb{E}^{(\mu-r)/\sigma}\left(e^{-rT}(S_T - K)^+\right) \tag{16}$$

and since under $\mathbb{P}^{(\mu-r)/\sigma}$

$$\begin{aligned} S_T &= S_0 \exp\left\{(\mu - \sigma^2/2)T + \sigma B_T\right\} \\ &= S_0 \exp\left\{(r - \sigma^2/2)T + \sigma B_T^{(\mu-r)/\sigma}\right\} \end{aligned} \tag{17}$$

where $B^{(\mu-r)/\sigma}$ is a $\mathbb{P}^{(\mu-r)/\sigma}$-Brownian motion, we have an exact distribution for S_T, namely

$$\log S_T \sim \mathcal{N}\left(\log S_0 + r - \sigma^2/2, \sigma^2 T\right).$$

Thus the expectation in (16) can be exactly evaluated. It is a long and tedious calculation, but none the less uses nothing more than high school integration.

Exercise 21 *Show that the price of a European Call option with parameters K and T when the starting value of the underlying asset is S_0 is*

$$\begin{aligned} C(T, S_0, K) = S_0 \mathcal{N}&\left(\frac{\log(S_0/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}\right) \\ -Ke^{-rT}\mathcal{N}&\left(\frac{\log(S_0/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right) \end{aligned} \tag{18}$$

where $\mathcal{N}(x)$ is the cumulative distribution of a standard Normal variable.

4.5 Hedging the European Call option

We have in fact more than answered the first question, Q1, posed in Section 2. We have an expression for the *value* of the option at all times during its lifetime. From (17) we can decompose the exponential Brownian motion as follows

$$\begin{aligned} S_T &= S_t \exp\left\{\left(r - \sigma^2/2\right)(T-t) + \sigma\left(B_T^{(\mu-r)/\sigma} - B_t^{(\mu-r)/\sigma}\right)\right\} \\ &= S_t \times S'_{T-t} \end{aligned}$$

where S'_{T-t} is a copy of $\exp\{(r-\sigma^2/2)(T-t) + \sigma B_{T-t}^{(\mu-r)/\sigma}\}$ and independent of $\mathcal{F}_t$. Hence

$$\begin{aligned} V_t(\Pi) &= \mathbb{E}^{(\mu-r)/\sigma}\left(e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t\right) \\ &= \mathbb{E}^{(\mu-r)/\sigma}\left(e^{-r(T-t)}(S_t \times S'_{T-t} - K)^+ | \mathcal{F}_t\right) \\ &= C(\tau, S_t, K) \end{aligned}$$

where $\tau = T - t$.

Lets now address Q2. The function $C(\tau, s, K)$, being defined in terms of $\mathcal{N}(\cdot)$ is differentiable with respect to τ and twice differentiable with respect to s with continuous derivatives. Therefore applying Itô's Lemma we have that

$$\begin{aligned} dV_t &= -\frac{\partial C}{\partial \tau}(T-t, S_t, K)\,dt \\ &\quad + \frac{\partial C}{\partial s}(T-t, S_t, K)\,dS_t + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}(T-t, S_t, K)\,\sigma^2 S_t^2\,dt \\ &= \Psi_t dR_t + \Phi_t dS_t \end{aligned}$$

where the last equality follows from the self financing property. Equating coefficients of dS_t we thus have that

$$\Phi_t = \frac{\partial C}{\partial s}(T-t, S_t, K) = \mathcal{N}\left(\frac{\log(S_t/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

where the second equality follows by straightforward differentiation of (18).

Exercise 22 *Evaluate an expression for* Ψ_t *in terms of* $\mathcal{N}(\cdot)$.

5 Other options and extensions of the Black-Scholes model

What we have presented here is the absolute basic theory of option pricing. A lot of the theory given above with some slight adjustment can be made to work for claims other than $(S_T - K)^+$. There are a whole class of options which probabilists calculated the price and hedging strategy for since the original work of Black and Scholes. Examples include the following claim functions:

- The European Put option. This is just a simple variant of the Call option where instead the holder of the option has the right to sell the asset for a fixed price K at time T. The claim is thus $(K - S_T)^+$.
- The American Call and Put Options. This option is a variant of the European Call/Put option. Instead of having the right to buy at a fixed time T, the holder has the right to buy/sell for a fixed price K at *any* time $0 \le t \le T$ during the life of the contract.
- The Asian option. The claim function is written $\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+$. Note that the claim now depends on the entire path of the asset value from time 0 to time T.
- The Lookback options. The claim on this option is $(\sup_{0\le t\le T} S_t - K)^+$. Again this is a path dependent option

There are many other examples; barrier options, double barrier options, Parisian options, Integral options, digital options, Russian options, Israeli options. The markets are now eagerly waiting for the introduction of the Dutch option!

The Black Scholes model is by now well accepted to be a piece of innovative work but none the less not good enough to match reality. Pricing with the Black Scholes formula often over or under prices compared to what market forces dictate the true value of an option really is. There is a big effort amongst financial mathematicians to improve on the Black Scholes model by changing the assumptions. For example, it is possible to make the constants r, μ, and σ vary with time too. Of course they may also be written as stochastic processes.

Generally these extra complications make the mathematics much more difficult. Another currently popular adjustment to the Black Scholes model is to assume that the asset is driven by a process which can have discontinuous jumps, namely Lévy processes. There is a great deal more that can be said as indeed the vast volume of literature that has appeared in the last ten years testifies to.

6 Solutions to Exercises

Solution to 5 (i) Note that

$$\begin{aligned}\mathbb{E}[B_t|B_r : r \leq s] &= \mathbb{E}[B_t - B_s + B_s|B_r : r \leq s] \\ &= \mathbb{E}[B_t - B_s|B_r : r \leq s] + B_s\end{aligned}$$

where the second equality follows from the fact that given $\{B_r : r \leq s\}$, the value of B_s is known. Now use fact (ii) and (iii) in the definition of Brownian motion to deduce that the conditional expectation is zero.

(ii) Begin as above,

$$\begin{aligned}&\mathbb{E}\left[B_t^2 - t|B_r : r \leq s\right] \\ =\; &\mathbb{E}\left[(B_t - B_s + B_s)^2 - t|B_r : r \leq s\right] \\ =\; &\mathbb{E}\left[(B_t - B_s)^2 |B_r : r \leq s\right] \\ &+2B_s\mathbb{E}[(B_t - B_s)|B_r : r \leq s] + B_s^2 - t.\end{aligned}$$

where in the third equality we have written out the square and taken out the terms which are known for the given information. Again using (ii) and (iii) the first term in the last equality has conditional expectation equal to $(t-s)$ and the second zero. The last equality thus simplifies to $B_s^2 - s$.

In both cases I will leave you to check the condition of finite absolute expectation.

Solution to 7 Begin with the definition and break the expectation using the law of total probability in the following way; $t \in [0, T]$

$$\begin{aligned}\mathbb{P}^\lambda(A) &= \mathbb{E}\left(\mathbf{1}_A e^{-\lambda B_T - T\lambda^2/2}\right) \\ &= \mathbb{E}\left[\mathbb{E}\left(\mathbf{1}_A e^{-\lambda B_T - T\lambda^2/2}\Big| B_r : r \leq t\right)\right] \\ &= \mathbb{E}\left[\mathbf{1}_A\mathbb{E}\left(e^{-\lambda B_T - T\lambda^2/2}\Big| B_r : r \leq t\right)\right] \\ &= \mathbb{E}\left[\mathbf{1}_A e^{-\lambda B_t - t\lambda^2/2}\right]\end{aligned}$$

where we have used the martingale property to reach the last equality from the previous one. Let $I = (a, b)$. Now put the event $A = \{B_t + \lambda t \in I\}$ in this new definition and use property (ii) in the definition of Brownian motion together with standard properties of the Normal distribution to get

$$\mathbb{P}^{\lambda}\left(B_t+\lambda t\in I\right)=\frac{1}{\sqrt{2\pi}}\int\limits_{(b-\lambda t)/\sqrt{t}}^{(a-\lambda t)/\sqrt{t}} e^{-\lambda\sqrt{t}z-\lambda^2 t/2}e^{-z^2/2}dz$$
$$=\frac{1}{\sqrt{2\pi}}\int\limits_{(b-\lambda t)/\sqrt{t}}^{(a-\lambda t)/\sqrt{t}} e^{-\frac{1}{2t}\left(z\sqrt{t}+\lambda t\right)^2}dz$$
$$=\frac{1}{\sqrt{2\pi t}}\int_b^a e^{-\frac{1}{2t}u^2}du$$
$$=\mathbb{P}\left(B_t\in I\right).$$

Solution to 8 When $t_k<t\leq t_{k+1}$

$$\int_0^t\Phi_s dB_s=\sum_{1\leq i\leq k}\phi_{i-1}\left(B_{t_i}-B_{t_{i-1}}\right)+\phi_k\left(B_t-B_{t_k}\right)$$

Suppose now that $u\leq t$ such that $t_l<u\leq t_{l+1}\leq t_k$ then

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\Phi_s dB_s\middle| B_r:r\leq u\right]\\
=&\sum_{1\leq i\leq l}\phi_{i-1}\left(B_{t_i}-B_{t_{i-1}}\right)\\
&+\sum_{l+1\leq i\leq k}\left[\phi_{i-1}\left(B_{t_i}-B_{t_{i-1}}\right)|\right]\\
&+\mathbb{E}[\phi_k\left(B_t-B_{t_k}\right)|B_r:r\leq u].
\end{aligned}$$

Now apply the tower property of conditional expectation and the first exercise; for example

$$\begin{aligned}
&\mathbb{E}\left[\,\phi_{i-1}\left(B_{t_i}-B_{t_{i-1}}\right)\middle| B_r:r\leq u\right]\\
=&\ \mathbb{E}\left[\mathbb{E}\left[\,\phi_{i-1}\left(B_{t_i}-B_{t_{i-1}}\right)\middle| B_r:r\leq t_i\right]\middle| B_r:r\leq u\right]\\
=&\ \mathbb{E}\left[\,\phi_{i-1}\mathbb{E}\left[\,\left(B_{t_i}-B_{t_{i-1}}\right)\middle| B_r:r\leq t_{i-1}\right]\middle| B_r:r\leq u\right]\\
=&\ \mathbb{E}\left[\,\phi_{i-1}\left(\mathbb{E}\left[B_{t_i}\middle| B_r:r\leq t_{i-1}\right]-\right)\middle| B_r:r\leq u\right]\\
=&\ \mathbb{E}\left[\,\phi_{i-1}\left(B_{t_{i-1}}-B_{t_{i-1}}\right)\middle| B_r:r\leq u\right]\\
=&\ 0.
\end{aligned}$$

The case that $t_k<u\leq t\leq t_{k+1}$ can be handled more easily. Note that the absolute expectation is bounded by the second moment.

Solution to 15 Note that

$$d\left(X_t^{(1)}+X_t^{(2)}\right)=\left(\Psi_t^{(1)}+\Psi_t^{(2)}\right)dt+\left(\Phi_t^{(1)}+\Phi_t^{(2)}\right)dB_t$$

hence

$$d\left[\left(X_t^{(1)}+X_t^{(2)}\right)^2\right] = 2\left(X_t^{(1)}+X_t^{(2)}\right) d\left(X_t^{(1)}+X_t^{(2)}\right) + \left(\Phi_t^{(1)}+\Phi_t^{(2)}\right)^2 dt$$

and for $i = 1, 2$

$$d\left(\left(X_t^{(i)}\right)^2\right) = 2X_t^{(i)} dX_t^{(i)} + \left(\Phi_t^{(i)}\right)^2 dt.$$

It follows that

$$\begin{aligned} d\left(X_t^{(1)}X_t^{(2)}\right) &= \frac{1}{2} d\left[\left(X_t^{(1)}+X_t^{(2)}\right)^2 - \left(X_t^{(1)}\right)^2 - \left(X_t^{(2)}\right)^2\right] \\ &= \frac{1}{2}\left[2\left(X_t^{(1)}+X_t^{(2)}\right) d\left(X_t^{(1)}+X_t^{(2)}\right)\right. \\ &\quad + \left(\Phi_t^{(1)}+\Phi_t^{(2)}\right)^2 dt - 2X_t^{(1)} dX_t^{(1)} - \left(\Phi_t^{(1)}\right)^2 dt \\ &\quad \left. -2X_t^{(2)} dX_t^{(2)} - \left(\Phi_t^{(2)}\right)^2 dt\right] \\ &= X_t^{(1)} dX_t^{(2)} + X_t^{(2)} dX_t^{(1)} + \Phi_t^{(1)}\Phi_t^{(2)} dt \end{aligned}$$

Solution to 16 The idea here is that if we compare the previous solution with what we know from normal calculus, we see that the final term would appear to be something extra. Suppose that this extra term is called the 'cross-product' term and denoted '$dX_t^{(1)} \cdot dX_t^{(2)}$' When $\Psi_t^{(i)} = 0$ then the previous exercise tells us

$$d\left(X_t^{(1)}X_t^{(2)}\right) = dX_t^{(1)} \cdot dX_t^{(2)} = \Phi_t^{(1)}\Phi_t^{(2)} dt$$

suggesting that in this special case we could work with the rule of actually multiplying $dX_t^{(1)}$ by $dX_t^{(2)}$ and using the rule that $dB_t \cdot dB_t = dt$. Now suppose that $\Phi_t^{(i)} = 0$ then again the previous exercise tells us that

$$d\left(X_t^{(1)}X_t^{(2)}\right) = X_t^{(1)} dX_t^{(2)} + X_t^{(2)} dX_t^{(1)} + 0$$

and hence $dX_t^{(1)} \cdot dX_t^{(2)} = 0$ suggesting that again in this special case we could work with the rule of actually multiplying $dX_t^{(1)}$ by $dX_t^{(2)}$ and using the rule that $dt \cdot dt = 0$. Finally considering the case that $\Phi_t^{(1)} = 0$ and $\Psi_t^{(2)} = 0$ (and hence by symmetry the same situation with the indices reversed). Again in this case $dX_t^{(1)} \cdot dX_t^{(2)} = 0$ and hence as before we deduce a multiplication rule $dt \cdot dB_t = dB_t \cdot dt = 0$. The given table thus follows.

Warning: be careful not to confuse this multiplication rule with the multiplication of real numbers. You should understand '$\cdot$' to mean 'when writing the process $X_t^{(1)} X_t^{(2)}$ out as a stochastic integral, there is an extra term that is acquired over and above normal calculus. The presence of this term depends on the integrands and is characterized by $\cdot$ as an abstract multiplication rule'.

Solution to 17 Take $X_t = (\mu - \sigma^2/2)t + \sigma B_t$ and $f(x) = c \cdot \exp\{x\}$ and apply Itô remembering that $f' = f'' = f$

$$
\begin{aligned}
df(X_t) &= f(X_t)\, dX_t + \frac{1}{2} f(X_t)\, \sigma^2 dt \\
&= f(X_t) \left[\left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dB_t + \frac{1}{2}\sigma^2 \right] \\
&= f(X_t) \left[\mu dt + \sigma dB_t \right]
\end{aligned}
$$

as required.

Solution to 18 To show that

$$dZ_t = Z_t \left[- \left(\mu - \sigma^2 \right) dt + \sigma dB_t \right]$$

simply note that $Z_t = \exp\{(\mu' - \sigma^2/2)t + \sigma B_t\}$ where $\mu' = -\left(\mu - \sigma^2\right)$ and apply the conclusion of the previous exercise. Suppose that Y is another solution to

$$dY_t = Y_t \left[\mu dt + \sigma dB_t \right]$$

Now apply the conclusions of Exercise 15 and 16 to give

$$
\begin{aligned}
d(Y_t Z_t) &= Y_t dZ_t + Z_t dY_t + dZ_t \cdot dY_t \\
&= Y_t Z_t \left[- \left(\mu - \sigma^2 \right) dt + \sigma dB_t \right] \\
&\quad + Y_t Z_t \left[\mu dt + \sigma dB_t \right] \\
&\quad + Y_t Z_t \sigma^2 dt \\
&= 0.
\end{aligned}
$$

Clearly the only solution to this SDE is a constant and hence uniqueness follows up to a multiplicative constant in the argument.

Solution to 21 Well actually this is a long calculation so I will only bring you into the beginning but the rest is straightforward. Begin by noting that under the measure $\mathbb{P}^{(\mu-r)/\sigma}$ the risky asset evolves as the exponential martingale with multiplying constant s. Thus

$$C(\tau, s, K) = \mathbb{E}\left[e^{-r\tau} \left(s e^{\sigma\sqrt{\tau} Z - \tau\sigma^2/2} - K \right)^+ \right]$$

where Z is a standard Normal variable. It then follows that

$$C(\tau, s, K) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\frac{-\log(s/K)+\tau\sigma^2/2}{\sigma\sqrt{\tau}}}^{\infty} \left(s e^{\sigma\sqrt{\tau}z - \tau\sigma^2/2} - K\right) e^{-\frac{1}{2}z^2} dz.$$

The rest is pure perseverance!

Solution to 22 Simply take the formulae in terms of the Normal distribution function $\mathcal{N}(\cdot)$ and use the fact that

$$\Psi_t = e^{-rt} C(T-t, S_t, K) - e^{-rt} S_t \Phi_t$$

to deduce that

$$\Psi_t = -K e^{-rt} \mathcal{N}\left(\frac{\log(S_t/K) + (T-t)(r - \sigma^2/2)}{\sigma\sqrt{T-t}}\right).$$