Optimal investment under stochastic mortality and stochastic interest rates

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Introduction

Merton's portfolio problem on a finite horizon $[0, T]$:

- Single agent;
- Invest in a bond and a stock starting from initial wealth x following the (admissible) strategy $\pi(t)$ while consuming $c(t)$;
- Agent derives utility from consumption $c(t)$ and from terminal wealth $X^{x,c,\pi}(T)$;
- Objective is to maximize expected utility

$$\sup_{c,\pi} \ \mathbb{E}\left[\int_0^T U_1(t, c(t))dt + U_2(X^{x,c,\pi}(T))\right].$$

Economic motivation for an extension

- Agent derives utility from consumption and from savings at the retirement date;
- Lifetime is uncertain, i.e. the agent may not survive up until retirement;
- Expected lifetime continues to improve (longevity risk) in an unpredictable way, mortality rates are stochastic;
- Exclude cases where the optimal control problem is not well-posed, e.g. if investing all wealth into the money-market account leads to infinite utility.

Relation to existing literature

	Primal method	Dual method	Consumption	Bequest	Terminal wealth	Stoch. short rate	Uncertain lifetime	Stochastic mort. rate	Non-neg. mort. rate	Closed-form solution	Verification
Yaari [Yaa65]	✓		✓	✓			✓			✓	
Ye and Pliska [PY07]	✓		✓	✓	✓		✓		✓	✓	✓
Menoncin [Men08]	✓	✓	✓			✓	✓	✓			
Deelstra et al. [DGK00]		✓			✓	✓				✓	✓
Jeanblanc and Yu [JY10]	✓			✓	✓		✓	✓	✓		✓
Kraft [Kra03]	✓		(✓)		✓	✓				✓	✓
This talk		✓	✓		✓	✓	✓	✓	✓	✓	✓

Outline

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Dynamics of the short rate and mortality rate

Let $W(t) = (W_1(t), W_2(t))$ be standard Brownian motion on $(\Omega, \mathcal{G}, P)$ and let $\mathcal{F}(t)$ be the filtration generated by $W(t)$.

Assume that the short rate $r(\cdot)$ and the mortality rate $\lambda(\cdot)$ follow Cox-Ingersoll-Ross processes under P

$$dr(t) = \kappa_1(\mu_1 - r(t))dt + \nu_1\sqrt{r(t)}dW_1(t) \qquad (2.1)$$

with $r(0) = r_0 > 0$, and

$$d\lambda(t) = \kappa_2(\mu_2 - \lambda(t))dt + \nu_2\sqrt{\lambda(t)}dW_2(t) \qquad (2.2)$$

with $\lambda(0) = \lambda_0 > 0$ given. The coefficients κ_i, ν_i and μ_i are deterministic and positive. Furthermore we require the Feller condition $2\,\kappa_i\mu_i > \nu_i^2$ to hold for $i = 1, 2$.

Tradeable instruments

Assume absence of arbitrage and construct a complete market by introducing the following tradeables:

- A money market account $S_0(\cdot)$ based on the stochastic short rate $r(\cdot)$;
- A zero-coupon bond $P(\cdot, T_1)$ paying 1 unit of currency at maturity T_1;
- A survival bond $F(\cdot, T_1)$ paying at time T_1 the mortality-dependent quantity

$$R = e^{-\int_0^{T_1} \lambda(u)du} .$$

Tradeable instruments

The price process of the **money market account** $S_0(\cdot)$ satisfies

$$S_0(t) = e^{\int_0^t r(u)du} . \tag{2.3}$$

The price of the **zero-coupon bond** $P(t, T_1)$ is given by

$$P(t, T_1) = \mathbb{E}_{\widetilde{P}} \left[e^{-\int_t^{T_1} r(u)du} \middle| \mathcal{F}(t) \right] \tag{2.4}$$

in which $\widetilde{P} \sim P$ is the (unique) risk neutral measure.

Tradeable instruments

The price of the **survival bond** is given by

$$F(t, T_1) = \mathbb{E}_{\widetilde{P}} \left[R\, e^{-\int_t^{T_1} r(u) du} \,\middle|\, \mathcal{F}_t \right] . \tag{2.5}$$

Outline

Main problem

Let τ (time of death) be a stopping time satisfying

$$\overline{F}(t) \ := \ P(\tau > t \mid \mathcal{F}_t) \ = \ e^{-\int_0^t \lambda(s)ds} \ . \tag{3.1}$$

Find the consumption and terminal wealth maximising power utility on the horizon $[0, T \wedge \tau]$ where $0 < T < T_1$.

This problem can be formulated as:

$$\sup_{c,\pi \ admissible} \mathbb{E}\left[\int_0^T U_1(t, c(t))dt + U_2(X^{x,c,\pi}(T))\right] \tag{3.2}$$

where $U_1(t,x) = \overline{F}(t)\widehat{U}_1(t,x)$ and $U_2(x) = \overline{F}(T)\widehat{U}_2(x)$ and $\widehat{U}_1(t,x) = \widehat{U}_2(x) = \frac{1}{p}x^p$, $p \in (-\infty, 1) \setminus \{0\}$.

Outline

Solution method:

- **Step 1.** Use martingale method and call on results by Karatzas and Shreve [KS98] to establish existence of solution provided that an 'implicit' integrability condition is satisfied;
- **Step 2.** Replace 'implicit' integrability condition by a condition stated explicitly in terms of model parameters using results by Kraft [Kra03] and Hurd et al. [HK08] on the Laplace transform of a CIR process;
- **Step 3.** Exploit affine structure of CIR process to derive closed-form solution for optimal consumption and wealth;
- **Step 4.** Derive dynamics of optimal wealth process and determine optimal strategy.

Step 1: Martingale method

The following result, which applies to a continuous-time diffusion setting, is due to Karatzas and Shreve [KS98]. Suppose that

$$\mathbb{E}\left[\int_0^T H_0(u)du \ + H_0(T)\right] \ < \ \infty \quad \text{and} \quad \mathcal{X}(1) \ < \ \infty \tag{4.1}$$

where $H_0(\cdot)$ denotes the state price density and

$$\mathcal{X}(y) \quad = \quad \mathbb{E}\left[\int_0^T H_0(t)I_1(t, yH_0(t))dt + H_0(T)I_2(yH_0(T))\right] \ , \tag{4.2}$$

in which $I_1(t,\cdot)$ and $I_2(\cdot)$ are the inverse functions of $U_1'(t,\cdot)$ and $U_2'(\cdot)$. Then the optimal consumption and terminal wealth are given by

$$c(t) \quad = \quad I_1(t, \mathcal{Y}(x)H_0(t)) \ , \tag{4.3}$$

$$X(T) \quad = \quad I_2(\mathcal{Y}(x)H_0(T)) \ . \tag{4.4}$$

where

$$\mathcal{Y}(x) \ = \ \left(\frac{x}{\mathcal{X}(1)}\right)^{p-1}$$

is the inverse of $\mathcal{X}(\cdot)$.

Step 2: Laplace transform of CIR process

In our economy the state price density is given by

$$H_0(t) = \exp\left\{ -\int_0^t r(u)du - \int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t ||\theta(u)||^2 du \right\}, \tag{4.5}$$

in which, for positive constants $\overline{\theta}_1$, $\overline{\theta}_2$, ν_1, ν_2, the market price of risk is

$$\theta(t) = \left(-\frac{\overline{\theta}_1}{\nu_1}\sqrt{r(t)}, -\frac{\overline{\theta}_2}{\nu_2}\sqrt{\lambda(t)} \right). \tag{4.6}$$

Step 2: Laplace transform of CIR process

It can be shown that for some constants K_1 and K_2

$$\mathbb{E}\left[H_0(t)I_1(t, H_0(t))\right] = \mathbb{E}_{\widetilde{P}_0}\left[\exp\left\{-K_1\int_0^t r(u)du - K_2\int_0^t \lambda(u)du\right\}\right] . \tag{4.7}$$

in which, $\frac{d\widetilde{P}_0}{dP} = Z(T)$ and, for every $p < 1,\ p \neq 0$, the stochastic exponential

$$Z(t) = \mathcal{E}\left(\frac{p}{1-p}\,\theta(t)\cdot W(t)\right) ,$$

is a martingale (see Revuz and Yor [RY99, Ch. VIII, Ex. 1.40]).

Hence by changing the probability measure, the evaluation of $\mathcal{X}(y)$ can be recast into the evaluation of a Laplace transform of $\int r(u)du$ and $\int \lambda(u)du$.

Step 2: Laplace transform of CIR process

Let $\kappa > 0$, $\theta > 0$ and $\nu > 0$ such that $2\kappa\theta > \nu^2$. Suppose that $\zeta(\cdot)$ follows a Cox-Ingersoll-Ross process

$$d\zeta(t) \;=\; \kappa(\theta - \zeta(t))\, dt + \nu\sqrt{\zeta(t)}\, dW(t)\,. \tag{4.8}$$

The Laplace transform

$$\phi(t, T, \zeta) \;=\; \mathbb{E}\left[\exp\left\{-\alpha\zeta(T) - \beta\int_t^T \zeta(s)ds\right\} \,\middle|\, \zeta(t) = \zeta\right]\,, \tag{4.9}$$

of $\left(\zeta(T), \int_t^T \zeta(s)ds\right)$ is finite-valued if

$$\text{(i)} \quad -\beta < \frac{\kappa^2}{2\nu^2} \qquad \text{and} \qquad \text{(ii)} \quad -\alpha < \frac{\kappa + a}{\nu^2}\,,$$

where $a = \sqrt{\kappa^2 + 2\beta\nu^2}$. Moreover, $\phi(t, T, \zeta)$ has an affine representation $\phi(t, T, \zeta) \;=\; e^{-A(t,T)-B(t,T)r}$.

Proof: Combine results in Kraft [Kra03] and Hurd et al. [HK08].

Step 3: Optimal consumption and terminal wealth

If the following conditions are satisfied

$$p\left[2\nu_1^2 + (\kappa_1 + \overline{\theta}_1)^2\right] < \kappa_1^2 , \quad (4.10)$$

$$-2\nu_2^2 + p\,(\kappa_2 + \overline{\theta}_2)^2 < \kappa_2^2 \quad \text{and} \quad \widetilde{\kappa}_2 + \widetilde{a}_2 > 0 , \quad (4.11)$$

Then the optimal consumption strategy is given by

$$c(t) = \frac{m(t)}{n(t)} X(t) , \quad (4.12)$$

in which

$$X(t) = \frac{x\, n(t)\Lambda(t)}{n(0)H_0(t)} , \quad (4.13)$$

is the optimal wealth process, and value function is given by

$$V(x) = \frac{1}{p} n(0)^{1-p} x^p , \quad (4.14)$$

where

$$\Lambda(t) = \exp\left\{ \frac{p}{1-p} \int_0^t \theta(u) dW(u) - \frac{1}{2} \left(\frac{p}{1-p}\right)^2 \int_0^t ||\theta(u)||^2 du \right\}$$

$$m(t) = \exp\left\{ -\frac{1}{1-p} \int_0^t \lambda(u) du + \frac{p}{1-p} \int_0^t r(u) du + \frac{p}{2(1-p)^2} \int_0^t ||\theta(u)||^2 du \right\}$$

$$n(t) = \int_t^T L(t,u) du + L(t,T)$$

and $L(\cdot, T)$ has an affine representation

$$L(t,T) = e^{-\overline{A}_1(t,T) - \overline{A}_2(t,T) - \overline{B}_1(t,T) r(t) - \overline{B}_2(t,T) \lambda(t)}$$

in which

$$\overline{A}_i(t,T) = \frac{-\kappa_i \mu_i (\tilde{\kappa}_i - \widetilde{a}_i)(T-t)}{\nu_i^2} + \frac{2\kappa_i \mu_i}{\nu_i^2} \log\left(\frac{1 - q_i e^{-\widetilde{a}_i (T-t)}}{1 - q_i} \right)$$

$$\overline{B}_i(t,T) = 2K_i \frac{e^{\widetilde{a}_i (T-t)} - 1}{e^{\widetilde{a}_i (T-t)} (\widetilde{\kappa}_i + \widetilde{a}_i) - \widetilde{\kappa}_i + \widetilde{a}_i}$$

for $i = 1, 2$ and

$$q_i = \frac{\widetilde{\kappa}_i - \widetilde{a}_i}{\widetilde{\kappa}_i + \widetilde{a}_i} \quad , \quad \widetilde{a}_i = \sqrt{\widetilde{\kappa}_i^2 + 2K_i \nu_i^2} \, , \quad \widetilde{\kappa}_i = \kappa_i - \frac{p}{1-p} \overline{\theta}_i \, .$$

Step 4: Optimal investment strategy

For all $0 < T < \infty$ define the process $L(\cdot, T)$ by

$$dL(t, T) = \widetilde{\mu}(t, T)\, dt + \widetilde{\nu}(t, T) dW(t)$$

where $W(t) = (\, W_1(t), W_2(t)\,)$ is a Brownian motion, $\widetilde{\mu}$ is an adapted, non-negative function $[0, T] \times [0, T] \times \Omega \to R$ and $\widetilde{\nu}$ is an adapted, non-negative function $[0, T] \times [0, T] \times \Omega \to R^2$.
If for $i = 1, 2$

$$\int_0^t \{\widetilde{\nu}_i(u, s)\}^2 du \; < \; \infty \;\text{ a.s. for all } t \in [0, T] \text{ and } s \in [0, T] \;, \tag{4.15}$$

$$\int_0^t \left\{ \int_0^T \widetilde{\nu}_i(u, s)\, ds \right\}^2 du \; < \; \infty \;\text{ a.s. for all } t \in [0, T] \;, \tag{4.16}$$

Step 4: Optimal investment strategy

If furthermore

$$t \mapsto \int_0^T \left\{ \int_0^t \widetilde{\nu}_i(u,s)\, dW_i(u) \right\}^2 ds$$

is almost surely continuous then

$$d\left(\int_t^T L(t,s)ds\right) = \left\{-L(t,t) + \int_t^T \widetilde{\mu}(t,s)ds\right\} dt + \sum_{i=1}^{2} \left\{\int_t^T \widetilde{\nu}_i(t,s)ds\right\} dW_i(t)\ . \qquad (4.17)$$

Proof: Combine results in Munk [Mun03, Thm. 3.3] and Heath et al. [HJM92, Appendix A].

Step 4: Optimal investment strategy

Apply Itô's lemma and Leibniz' rule to the dynamics of optimal wealth process to obtain:

$$\frac{d\left(X(t)+\int_0^t c(u)du\right)}{X(t)} = \Big(1-Q_1(t)\Big)\frac{dS_0(t)}{S_0(t)} + \Big(Q_1(t)-Q_2(t)\Big)\frac{dP(t,T_1)}{P(t,T_1)} + Q_2(t)\frac{dF(t,T_1)}{F(t,T_1)} \,,$$

where, for $i=1,2$, the hedging strategies take the following explicit form:

$$Q_i(t) = \frac{1}{B_i(t,T_1)}\left(\frac{1}{1-p}\frac{\overline{\theta}_1}{\nu_i^2} + \frac{\Xi_i(t)}{n(t)}\right)$$

in which

$$\Xi_i(t) = \overline{B}_i(t,T)L(t,T) + \int_t^T \overline{B}_i(t,u)L(t,u)\,du \,.$$

The short rate and mortality rate are observable due to the affine relation between the (observable) bond price and S-forward price.

More results

- Bounds on the hedging demand can be derived in terms of model parameters.
- A rolling bond can be used (instead of a long-term bond) to trade interest rate risk.
- Stocks can be added to the asset mix.
- The proportionality constants in the market price of risk can be assumed to be time-dependent, provided that the resulting Riccati equation has a continuous solution; conditions under which this holds are left for future research.

Outline

Conclusion

- A solution to the optimal consumption and investment problem with (non-negative) Cox-Ingersoll-Ross short rate and mortality rate exists under conditions which can be expressed explicitly in terms of model parameters;
- The optimal consumption and investment strategy has been derived in closed-form.

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