Primal-dual methods for nonlinear pricing problems

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Outline

Backward dynamic program – setting

On a filtered probability space in discrete time $(\Omega, \mathcal{F}, \mathcal{F}_i, P)_{i=0,\dots,n}$ we consider a dynamic program of the form

$$Y_i = F(i, E_i[\beta_{i+1} Y_{i+1}]), \quad 0 \le i \le n-1; \quad Y_n = \xi$$

where

- ξ is an $\mathcal{F}_n$-measurable integrable random variable;
- $(\beta_i)_{i=1,\dots,n}$ is an adapted bounded, $\mathbb{R}^{1+D}$-valued process;
- $F(i, \omega, z)$ is Lipschitz in z (with constant independent of ω) and $\mathcal{F}_i$-measurable for fixed $z \in \mathbb{R}^{1+D}$.
- $E_i[\cdot]$ denotes conditional expectation given $\mathcal{F}_i$.

Example 1: Bermudan options

- Denote by S_i the discounted payoff of a Bermudan option at the ith exercise date, $i = 1, \dots, n$.
- The holder of the option can choose the time, at which she exercises the option, which leads to an optimal stopping problem for the Bermudan option price

$$Y_0 = \sup_{\tau} E[S_i],$$

  where τ runs over the set of $\{0, \dots, n\}$ valued stopping times and expectation is taken with respect to a fixed risk-neutral pricing measure.
- Y_0 can be represented via the dynamic program

$$Y_i = \max\{E_i[Y_{i+1}], S_i\}.$$

- Here: $D = 0, \quad \beta_{0,i} = 1, \; F(i, z) = \max\{z, S_i\}.$

Example 2: Parabolic PDEs

- Suppose W is a $\mathcal{D}$-dimensional Brownian motion on $[0, T]$. Define $h = T/n$ and $t_i = ih, i = 0, 1, \ldots, n$, and $\Delta W_{i+1} = W_{t_{i+1}} - W_{t_i}$. Consider the dynamic program

$$Y_j = E_j[Y_{j+1}]$$
$$+f\left(t_j, W_{t_j}, E_j[Y_{j+1}], E_j\left[\frac{\Delta W_{j+1}}{h} Y_{j+1}\right], E_j\left[\frac{\Delta W_{j+1}\Delta W_{j+1}^{\top} - hI_{\mathcal{D}}}{h^2} Y_{j+1}\right]\right) h$$

  with terminal condition $Y_n = g(W_T)$ for appropriate deterministic functions f, g.

- Then there are determinstic functions $u_j(x)$ such that $Y_j = u_j(W_{t_j})$.

Example 2: Parabolic PDEs

- Given u_{j+1} we can compute u_j as follows. Denote by $\bar{u}^j$ the solution of the heat equation

$$\begin{aligned} \frac{\partial}{\partial t}\bar{u}^j(t,x) + \frac{1}{2}\Delta\bar{u}^j(t,x) &= 0, \ (t,x) \in [t_j, t_{j+1}) \times \mathbb{R}^{\mathcal{D}} \\ \bar{u}^j(t_{j+1}, x) &= u_{j+1}(x) \end{aligned}$$

  Let

$$u_j(x) = \bar{u}^j(t_j, x) + f(t_j, x, \bar{u}^j(t_j, x), \bar{u}^j_x(t_j, x), \bar{u}^j_{xx}(t_j, x))h$$

- Under appropriate conditions (ellipticity of the nonlinearity, etc.), $Y_0 = Y_0(h)$ converges to $v(0,0)$, where v is the (viscosity) solution of the fully nonlinear parabolic Cauchy problem

$$\frac{\partial}{\partial t}v + \frac{1}{2}\Delta v + f(t,x,v,v_x,v_{xx}) = 0, \quad v(T,\cdot) = g,$$

  see Fahim, Touzi, Warin (2011) and Tan (2014).

Example 2: Parabolic PDEs

- In this example the weight vector β consists of 1, the $\mathcal{D}$ Delta weights
$$\frac{\Delta W_{j+1}}{h},$$
for the first spatial derivative, and the $\mathcal{D}^2$ Gamma weights
$$\frac{\Delta W_{j+1}\Delta W_{j+1}^{\top} - hI_{\mathcal{D}}}{h^2}$$
for the second spatial derivative.
- Many pricing problems under credit risk and funding costs can be formulated in terms of such type of second order parabolic PDEs, see the minicourse by Damiano Brigo.

Example 3: Uncertain volatility

- Suppose g is the discounted payoff function of a European option. Under uncertain volatility, the European option pricing problem becomes

$$\mathcal{Y}_0 = \sup_{\sigma} E\left[g\left(S_0 \exp\left\{ \int_0^T \sigma_t dW_t - \frac{1}{2}\int_0^T \sigma_t^2 dt \right\} \right) \right],$$

where σ runs over the set of progressively measurable processes such that $\underline{\sigma} \leq \sigma_t \leq \overline{\sigma}$. The constants $\underline{\sigma} \leq \overline{\sigma}$ determine the possible range of the uncertain volatility.

Example 3: Uncertain volatility

- The Hamilton-Jacobi-Bellmann equation for this control problem can be transformed into the form

$$v_t + \frac{1}{2} v_{xx} + \frac{1}{2} \max_{\rho \in \{\underline{\sigma}, \overline{\sigma}\}} \left(\left(\frac{\rho^2}{\hat{\sigma}^2} - 1 \right) (v_{xx} - \hat{\sigma} v_x) \right) = 0,$$
$$v(T, x) = g(S_0 e^{\hat{\sigma} x - \hat{\sigma}^2 T/2})$$

  for any fixed reference volatility $\hat{\sigma} > 0$.

- Then $\mathcal{Y}_0 = v(0, 0)$.
- Hence, the previous example leads to the discretization

$$Y_i = E_i[Y_{i+1}] + \frac{1}{2} \max_{\rho \in \{\underline{\sigma}, \overline{\sigma}\}} ((\frac{\rho^2}{\hat{\sigma}^2} - 1) E_i[Y_{i+1}(\frac{\Delta W_{i+1}^2}{h} - \hat{\sigma} \Delta W_{i+1} - 1)])$$
$$Y_n = g(S_0 \exp\{\hat{\sigma} W_T - \hat{\sigma}^2 T/2\}),$$

cp. Guyon, Henry-Labordere (2011).

Backward dynamic program

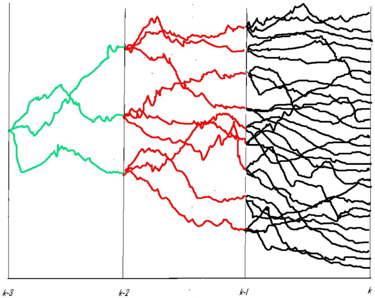
- Recall the dynamic programming equation:

$$Y_i = F(i, E_i[\beta_{i+1} Y_{i+1}]), \quad 0 \le i \le n-1; \quad Y_n = \xi$$

- Aim: Approximate Y_0 numerically.
- Approximate dynamic programming: Replace the conditional expectation by some approximation, which can be computed.
- Primal-dual methodology: Take the solution of the approximate dynamic program as an input in order to construct a confidence interval for the price Y_0.

Approximate dynamic programming

- Difficulties: High order nesting of conditional expectations within the dynamic program (when n is large).
- Plain Monte Carlo is not applicable ...



Approximate dynamic programming

- Typical situation: There is a deterministic function $z(i,\cdot):\mathbb{R}^{\mathcal{D}}\to\mathbb{R}^{1+D}$ and a $\mathbb{R}^{\mathcal{D}}$-valued stochastic process such that
$$E_i[\beta_{i+1}Y_{i+1}]=z(i,X_i).$$
- Example: X_i Markovian, $F(i,\cdot)$ and ξ depend on ω through X_i (resp. X_n) only, β_{i+1} independent of $\mathcal{F}_i$.
- Numerical algorithms typically try to approximate $z(i,x)$.
- Conditional expectation must be replaced by an approximate operator which can be nested without exploding cost.
- Throughout the talk we apply least-squares Monte Carlo (Longstaff-Schwartz, 2001; Lemor, Gobet, Warin, 2006), but other choices are possible: quantization (Bally, Pages, 2003), Malliavin Monte Carlo (Bouchard, Touzi, 2004), cubature on Wiener space (Crisan, Manolarakis, 2012), ...

Conditional expectations via least squares Monte Carlo

Pseudo-Algorithm

1. Choose $(1+D)$ row vectors of basis functions

$$\psi_d(i,x) = (\psi_{d,1}(i,x),\dots,\psi_{d,K}(i,x));\ x\in\mathbb{R}^{\mathcal{D}},\ d=0,\dots,D;$$

2. Simulate L independent copies of (X_i,β_i):
$(X_i(\lambda),\beta_i(\lambda);\ i=1,\dots,n),\ \lambda=1,\dots,L.$
3. Solve the $(D+1)$ least squares problems

$$\begin{aligned} a_d(i;z) &= \arg\min_{a\in\mathbb{R}^K} \frac{1}{L}\sum_{\lambda=1}^{L}\left(\beta_{i+1}^{(d)}(\lambda)G(X_{i+1}(\lambda)) - \psi_d(i,X_i(\lambda))a\right)^2 \\ &\approx \arg\min_{a\in\mathbb{R}^K} E\left[\left(\beta_{i+1}^{(d)}G(X_{i+1}) - \psi_d(i,X_i)a\right)^2\right]; \end{aligned}$$

4. Define, as approximation for $E[\beta_{i+1}G(X_{i+1})|X_i=x]$,

$$\hat{E}[\beta_{i+1}G(X_{i+1})|X_i=x] = (\psi_0(i,x)a_0(i;z),\dots,\psi_D(i,x)a_D(i;z))^\top.$$

The curse of dimensionality

- In the PDE example suppose that the nonlinearity does not depend on v_{xx} and add some Lipschitz conditions.
- Lemor, Gobet, Warin (2006): Apply least-squares MC with indicator functions of hypercubes as basis functions.
- Problem: The number of regression paths Λ, which is required to make the whole scheme converge at the order of $n^{-1/2}$ increases like $n^{2\mathcal{D}+3}$.
- Infeasible for e.g. $\mathcal{D} = 5$.
- Gobet and Turkedjiev (2014) show that the complexity and the number of paths which need be stored at the same time can be reduced by applying a variant of the forward scheme of Bender and Denk (2007). E.g. for $\mathcal{D} = 5$, $n^{4.5}$ samples need be stored at the same time. ($40^{4.5} \approx 16$ Mio.)
- There is a curse of dimensionality, when a fine time grid is necessary or the dimension of X is large.

The primal-dual approach for Bermudan options

- Recall: The dynamic program for Bermudan option pricing is

$$Y_i = \max\{S_i, E_i[Y_{i+1}]\}, \quad Y_n = S_n,$$

for some adapted process S (discounted payoff).

- Primal problem: Y_i is the value process of the following maximization problem (optimal stopping):

$$Y_i = \operatorname*{esssup}_{\tau} E_i[S_\tau],$$

where τ runs over the set of $\{i, \ldots, n\}$-valued stopping times. An optimal stopping time is given by

$$\tau_i^* = \inf\{j \geq i;\ S_j \geq E_j[Y_{j+1}]\} \wedge n$$

The primal-dual approach for Bermudan option case

- Dual problem: Y_i is the value process of the following minimization problem (information relaxation dual):

$$Y_i = \operatorname*{essinf}_{M} E_i[\max_{j=i,\dots,n}(S_j - M_j)] + M_i$$

where M runs over the set of martingales (Rogers, 2002; Haugh, Kogan, 2004). The martingale part M^* of the Doob decomposition of Y is optimal, even in a pathwise sense:

$$Y_i = \max_{j=i,\dots,n}(S_j - M_j^*) + M_i^*$$

The primal-dual approach for Bermudan options

- Andersen/Broadie-algorithm (2004); very roughly speaking: Apply the approximation $\hat{z}(i,x)$ to $z(i,x) = E[Y_{i+1}|X_i = x]$ in order to construct a stopping time $\hat{\tau}_0$ and a martingale $\hat{M}$, which are 'close' to the optimal ones τ_0^* and M^*.
- Estimate
$$E[S_{\hat{\tau}_0}]$$
by plain Monte Carlo to get a lower confidence bound for Y_0 and
$$E[\max_{j=0,\dots,n}(S_j - \hat{M}_j)] + E[\hat{M}_0]$$
to get an upper confidence bound.
- In practice: Use only a moderate effort to pre-calculate $\hat{z}$, e.g. least-squares MC with just a couple of well-chosen basis functions and a few thousand regression paths. Check whether the confidence interval by the primal-dual approach is sufficiently tight for the application under consideration.

The Bermudan option case – proof of the dual representation

- For any martingale M with $M_0 = 0$ it holds, by optional sampling, that

$$Y_0 = \sup_{\tau} E[S_\tau - M_\tau] \leq E[\max_{i=0,\dots,n} (S_i - M_i)]$$

- Suppose

$$Y_i = Y_0 + M_i^* - A_i^*$$

is the Doob decomposition of Y_i. Then, by the supermartingale property of Y,

$$\begin{aligned} Y_0 &= \max_{i=0,\dots,n} (Y_0 - A_i^*) \\ &= \max_{i=0,\dots,n} (Y_i - M_i^*) \geq \max_{i=0,\dots,n} (S_i - M_i^*) \end{aligned}$$

The Bermudan option case – pathwise dynamic programming

- Alternative approach to the dual representation.
- Define for a martingale M:

$$\theta_i^{up} := \max_{j=i,\dots,n}(S_j - M_j) + M_i := \max\{S_i, \theta_{i+1}^{up} - (M_{i+1} - M_i)\}.$$

- Then for $Y_i^{up} := E_i[\theta_i^{up}]$ by convexity of $z \mapsto \max\{S_i, z\}$

$$Y_i^{up} \geq \max\{S_i, E_i[\theta_{i+1}^{up} - (M_{i+1} - M_i)]\} = \max\{S_i, E_i[Y_{i+1}^{up}]\}.$$

- Hence, Y_i^{up} is a 'supersolution' for the dynamic program, and by backward induction and monotonicity of $z \mapsto \max\{S_i, z\}$,

$$Y_i^{up} \geq \max\{S_i, E_i[Y_{i+1}^{up}]\} \geq \max\{S_i, E_i[Y_{i+1}]\} = Y_i$$

Back to the general setting

- Dynamic program:

$$Y_i = F(i, E_i[\beta_{i+1} Y_{i+1}]), \quad 0 \le i \le n-1; \quad Y_n = \xi$$

- Adapted processes Y_i^{up} (resp. Y_i^{low}) are called supersolution (resp. subsolution) to the above dynamic program, if

$$Y_i^{up} \ge F(i, E_i[\beta_{i+1} Y_{i+1}^{up}]), \quad 0 \le i \le n-1; \quad Y_n^{up} \ge \xi;$$

and with '$\ge$' replaced by '$\le$' for the subsolution.

- (Conv) The map $z \mapsto F(i, z)$ is convex.

Pathwise dynamic programming: The convex case

- Suppose (Conv). Given a $\mathbb{R}^{1+D}$-valued martingale M define $\theta_i^{up} := \theta_i^{up}(M)$ via the pathwise dynamic program

$$\theta_i^{up} = F(i, \beta_{i+1}\theta_{i+1}^{up} - (M_{i+1} - M_i)), \quad 0 \le i \le n-1; \quad \theta_n^{up} = \xi;$$

- Then, by (Conv),

$$E_i[\theta_i^{up}] \geq F(i, E_i[\beta_{i+1} E_{i+1}[\theta_{i+1}^{up}]])$$

  and hence $Y_i^{up} = E_i[\theta_i^{up}]$ is a supersolution.
- Important: Calculating Y_0^{up} only requires the evaluation of one expectation (and not of nested ones), but of course also the choice of a martingale M.

Pathwise dynamic programming: The convex case

- Now take M^* as the Doob martingale of βY, i.e.

$$M^*_{i+1} - M^*_i = \beta_{i+1} Y_{i+1} - E_i[\beta_{i+1} Y_{i+1}].$$

- Then, P-almost surely,

$$\theta_i^{*,up} := \theta_i^{up}(M^*) = Y_i,$$

because by backward induction on i,

$$\begin{aligned} \theta_i^{*,up} &= F(i, \beta_{i+1}\theta_{i+1}^{up,*} - (\beta_{i+1} Y_{i+1} - E_i[\beta_{i+1} Y_{i+1}])) \\ &= F(i, E_i[\beta_{i+1} Y_{i+1}])) = Y_i \end{aligned}$$

- Question: Under which assumptions does a comparison principle hold, i.e. are supersolution above the solution and subsolutions below the solution?

Pathwise dynamic programming: The convex case

Theorem

Suppose (Conv). Then, the following assertions are equivalent:

- *(Comp) For any subsolution Y_i^{low} and any supersolution Y_i^{up}: $Y_i^{up} \geq Y_i^{low}$ P-a.s.*
- *(Mono) If $y, \tilde{y}$ are two integrable random variables such that $y \geq \tilde{y}$ P-a.s., then*

$$F(i, E_i[\beta_{i+1} y]) \geq F(i, E_i[\beta_{i+1} \tilde{y}]) \ P\text{-a.s.}$$

In this case

$$Y_i = \operatorname*{essinf}_M E_i[\theta_i^{up}(M)],$$

where M runs over the set of $\mathbb{R}^{1+D}$-valued martingales. Moreover, the Doob martingale of $\beta_i Y_i$ is a minimizer (and is 'surely optimal').

Pathwise dynamic programming: The convex case

- Question: How to construct subsolutions?
- Define the convex conjugate of F by

$$F^{\#}(i,\rho) = \sup_{z\in\mathbb{R}^{1+D}} \left(\rho^\top z - F(i,z)\right).$$

- An adapted process ρ is said to be an admissible control, if

$$\sum_{j=0}^{n-1} E[|F^{\#}(j,\rho_j)|] < \infty.$$

- Given an admissible control consider the pathwise linearization $\theta^{low} = \theta^{low}(\rho)$

$$\theta_i^{low} = \rho_i^\top \beta_{i+1}\theta_{i+1}^{low} - F^{\#}(i,\rho_i), \quad 0 \le i \le n-1; \quad \theta_n^{low} = \xi;$$

and define $Y_i^{low} = E_i[\theta_i^{low}]$.

Pathwise dynamic programming: The convex case

- Then, by adaptedness of ρ and noting that $\rho \in [-L, L]^{1+D}$.

$$\begin{aligned} Y_i^{low} &= \rho_i^\top E_i[\beta_{i+1}\theta_{i+1}^{low}] - F^{\#}(i, \rho_i) \\ &= \rho_i^\top E_i[\beta_{i+1} Y_{i+1}^{low}] - F^{\#}(i, \rho_i) \leq F(i, E_i[\beta_{i+1} Y_{i+1}^{low}]), \end{aligned}$$

  because $F^{\#\#} = F$ by convexity and since F is defined on the whole $\mathbb{R}^{1+D}$.

- Applying measurable selection of a subgradient (Cheridito, Kupper, Vogelpoth, 2014), there is an admissible control ρ^*, which satisfies

$$(\rho_i^*)^\top E_i[\beta_{i+1} Y_{i+1}] - F^{\#}(i, \rho_i^*) = F(i, E_i[\beta_{i+1} Y_{i+1}]),$$

  and hence by induction turns the above inequality into equality.

Pathwise dynamic programming: The convex case

Theorem

Suppose (Conv) and (Mono). Then

$$Y_i = \operatorname*{esssup}_{\rho} E_i[\theta_i^{low}(\rho)],$$

where ρ runs over the set of admissible controls. Any admissible ρ^* satisfying

$$(\rho_i^*)^\top E_i[\beta_{i+1} Y_{i+1}] - F^{\#}(i, \rho_i^*) = F(i, E_i[\beta_{i+1} Y_{i+1}])$$

is optimal and such maximizer exists.

Information relaxation

- Solving the linear program for $\theta_0^{low}(\rho)$ explicitly yields the primal problem

$$Y_0 = \sup_{(\rho_j)_{j=0,\ldots,n-1}} E[\xi \prod_{k=0}^{n-1} \rho_k^\top \beta_{k+1} - \sum_{i=0}^{n-1} F^\#(i,\rho_i) \prod_{k=0}^{i-1} \rho_k^\top \beta_{k+1}],$$

  where the sup runs over admissible controls ρ.
- The information relaxation dual due to Brown, Smith, Sun (2010), cp. also Rogers (2007), states that

$$Y_0 = \inf_{\mathfrak{p}} E\left[\sup_{(r_i)_{i=0,\ldots,n-1}} \left(\xi \prod_{k=0}^{n-1} r_k^\top \beta_{k+1} - \sum_{i=0}^{n-1} F^\#(i,r_i) \prod_{k=0}^{i-1} r_k^\top \beta_{k+1} - \mathfrak{p}(r)\right)\right]$$

  where $\mathfrak{p}$ in general runs over a huge class of 'penalties'.
- Our results show: one can restrict to 'martingale penalties':

$$\mathfrak{p}(r) = \sum_{i=0}^{n-1} r_i^\top (M_{i+1} - M_i) \prod_{k=0}^{i-1} r_k^\top \beta_{k+1}.$$

Monte Carlo implementation

- Step 1: Approximate dynamic programming: Pre-compute an approximation $\hat{z}(i,x)$ of

  $$z(i,x) = E[\beta_{i+1} Y_{i+1} | X_i = x] = E[\beta_{i+1} F(i{+}1, z_{i+1}(X_{i+1})) | X_i = x],$$

  initiated at $z(n,x) = 0$ with the convention $F(n,0) := \xi$. E.g.:

  $$\hat{z}(i,x) = \hat{E}[\beta_{i+1} F(i+1, \hat{z}_{i+1}(X_{i+1})) | X_i = x], \quad \hat{z}(n,x) = 0,$$

  where $\hat{E}[\cdot | X_i = x]$ approximates the conditional expectation operator. Let $\hat{Y}_0 := F(0, z(0, X_0))$.
- We apply least-squares Monte Carlo, i.e. we have to choose basis functions and the number of regression paths.

Monte Carlo implementation

- Step 2: Lower confidence bound: Given $\hat{z}(i,x)$ there is a $\hat{\rho}_i(x)$ which solves
$$\hat{\rho}_i(x)^\top \hat{z}(i,x) - F^{\#}(i,\hat{\rho}_i(x)) = F(i,\hat{z}(i,x))$$
such that $\hat{\rho}_i(X_i)$ is admissible.
- Simulate Λ^{out} 'outer' sample paths $(\beta_i(\lambda), X_i(\lambda))$ of (β_i, X_i) (independent of whatever paths might have been used to compute $\hat{z}$) and define
$$\hat{\theta}_i^{low}(\lambda) = \hat{\rho}_i(X_i(\lambda))^\top \beta_{i+1}(\lambda)\theta_{i+1}^{low}(\lambda) - F^{\#}(i,\hat{\rho}_i(X_i(\lambda)))$$
initiated at the terminal condition of the dynamic program.
- Then, the plain MC estimator
$$\frac{1}{\Lambda^{out}}\sum_{\lambda=1}^{\Lambda^{out}} \hat{\theta}_0^{low}(\lambda)$$
is biased downwards for Y_0 and an asymptotic confidence bound can be constructed in the usual way.

Monte Carlo implementation

- Step 3: Upper confidence bound: Given $\hat{z}(i,x)$ define along each outer paths

$$\begin{aligned} \Delta \hat{M}_i(\lambda) &= \beta_{i+1}(\lambda) F(i+1, \hat{z}(i+1, X_{i+1}(\lambda))) \\ &\quad - E[\beta_{i+1} F(i+1, \hat{z}(i+1, X_{i+1}))|X_i = X_i(\lambda)]. \end{aligned}$$

  If the conditional expectation is not available in closed form, replace it by an unbiased estimator, e.g. one layer of nested simulation with Λ^{in} 'inner' sample paths.

- Define

$$\hat{\theta}_i^{up}(\lambda) = F(i, \beta_{i+1}(\lambda)\hat{\theta}_{i+1}^{up}(\lambda) - \Delta \hat{M}_i(\lambda))$$

  initiated at the the terminal condition of the dynamic program and proceed analogously to the 'lower bound' in order to construct an estimator with bias upwards and an upper confidence bound.

Example: option pricing with different interest rates

D independent, identically distributed Black-Scholes stock:

$$X_t^d = x_0 \exp\{(\mu - \sigma^2/2)t + \sigma W_t^d\}$$

Interest rates: R_b for borrowing money, R_l for lending; $R_b > R_l$.

Pricing problem of a European option with payoff $\varphi(X_T)$:

The price process after time discretization is given by $Y_n = \varphi(X_T)$,

$$\begin{aligned} Y_j &= (1 - rh)E_j[Y_{j+1}] - \frac{\mu - R_l}{\sigma} \sum_d Z_{d,j} h \\ &\quad + (R_b - R_l)h \left(E_j[Y_{j+1}] - \frac{1}{\sigma} \sum_d Z_{d,j} \right)_- \\ Z_{d,j} &= E_j\left[\frac{W_{d,t_{j+1}} - W_{d,t_j}}{h} Y_{j+1} \right], \end{aligned}$$

where the Brownian increments can be suitably truncated to meet condition (Mono).

Numerical example

- Parameters $D = 5$, $X_0 = 100$, $\mu = 0.05$, $\sigma = 0.2$, $R_b = 0.06$, $R_l = 0.01$, $T = 0.25$, $h = T/n$, $t_i = ih$,
- European Max-Call-Spread Option:

$$\varphi(X_T) = \left(\max_d X_{d,T} - 95\right)^+ - 2\left(\max_d X_{d,T} - 115\right)^+$$

- Bermudan version of this option with 4 exercise dates.

Numerical example

- Approximate dynamic programming: LSMC with 2 or 7 basis functions: 1, $E_j[\varphi(X_T)]$ and the five underlyings for Y-part. Derivatives of these for "Z-part", 10^5 regression paths. Similar choice for Bermudan option.
- Lower and upper bounds calculated with 10^4 outer paths. 100 inner paths along each outer paths to approximate the Doob martingales of $\hat{Y}$, $\beta\hat{Y}$.
- Control variates... see paper.

	40		80		120	
Eur. 2	13.7783 (0.0022)	13.8172 (0.0024)	13.7817 (0.0022)	13.8443 (0.0027)	13.7848 (0.0024)	13.8682 (0.0029)
Eur. 7	13.7818 (0.0020)	13.8140 (0.0021)	13.7767 (0.0020)	13.8321 (0.0022)	13.7789 (0.0022)	13.8560 (0.0025)
Ber.	15.5362 (0.0028)	15.5664 (0.0028)	15.5441 (0.0037)	15.6160 (0.0035)	15.5246 (0.0041)	15.6396 (0.0042)

Table: Price bounds for n time steps. Standard deviations are in brackets.

Beyond the monotonicity condition

- Suppose (Conv), but that (Mono) is violated and thus the comparison principle is not in force: This can easily happen in the presence of gamma weights.
- Given an $\mathbb{R}^{1+D}$-valued martingale M with increments ΔM and an admissible control ρ define $\theta^{up} = \theta^{up}(M, \rho)$ and $\theta^{low} = \theta^{low}(M, \rho)$ via

$$\begin{aligned} \theta_i^{up} &= \max\{F(i, \beta_{i+1}\theta_{i+1}^{up} - \Delta M_{i+1}),\ F(i, \beta_{i+1}\theta_{i+1}^{low} - \Delta M_{i+1})\} \\ \theta_i^{low} &= (\rho_i^\top \beta_{i+1})_+ \theta_{i+1}^{low} - (\rho_i^\top \beta_{i+1})_- \theta_{i+1}^{up} - \rho_i^\top \Delta M_{i+1} - F^\#(i, \rho_i), \end{aligned}$$

with $\theta_n^{up} = \theta_n^{low} = \xi$.

- Then, $E_i[\theta_i^{up}(M, \rho)]$ is a supersolution, $E_i[\theta_i^{low}(M, \rho)]$ is a subsolution and

$$E_i[\theta_i^{low}(M, \rho)] \le Y_i \le E_i[\theta_i^{up}(M, \rho)].$$

Beyond the monotonicity condition

Theorem

Suppose (Conv). Then

$$Y_i = \operatorname*{esssup}_{M,\rho} E_i[\theta_i^{low}(M,\rho)] = \operatorname*{essinf}_{M,\rho} E_i[\theta_i^{up}(M,\rho)]$$

where ρ runs over the set of admissible controls and M over the set of $\mathbb{R}^{1+D}$-valued martingales M . Optimizers (M^*, ρ^*) for both problems are given by the Doob martingale M^* of βY and any admissible ρ^* satisfying

$$(\rho_i^*)^\top E_i[\beta_{i+1} Y_{i+1}] - F^\#(i, \rho_i^*) = F(i, E_i[\beta_{i+1} Y_{i+1}]).$$

These optimizers are 'surely' optimal, i.e. P-almost surely

$$Y_i = \theta_i^{low}(M^*, \rho^*) = \theta_i^{up}(M^*, \rho^*)$$

A numerical example: uncertain volatility

- Recall the pricing problem under uncrtain volatility

$$\sup_{\sigma} E\left[g\left(S_0 \exp\left\{\int_0^T \sigma_s dW_s - \int_0^T \frac{\sigma_s^2}{2} ds\right\}\right)\right],$$

  where σ runs over the progressively measurable processes with values in $[\underline{\sigma}, \overline{\sigma}]$.

- Time discretization with reference volatility $\hat{\sigma}$

$$Y_i = E_i[Y_{i+1}] + \frac{1}{2} \max_{\rho \in \{\underline{\sigma}, \overline{\sigma}\}} ((\frac{\rho^2}{\hat{\sigma}^2} - 1) E_i[Y_{i+1}(\frac{\Delta W_{i+1}^2}{h} - \hat{\sigma}\Delta W_{i+1} - 1)])$$

$$Y_n = g(S_0 \exp\{\hat{\sigma} W_T - \hat{\sigma}^2 T/2\}),$$

- Least-squares Monte Carlo for the approximate dynamic program suffers from large variances due to the gamma weight, see the discussion in Alanko and Avellaneda (2013).

A variant of least-squares Monte Carlo

- LSMC approximates the true regression functions via

$$\hat{z}(i,x) = \hat{E}[\beta_{i+1} F(i+1, \hat{z}_{i+1}(X_{i+1}))|X_i = x], \quad \hat{z}(n,x) = 0,$$

where $\hat{E}$ denotes a empirical regression on a set of basis functions.

- Suppose that basis functions $\psi_k(i,x)$, $k = 1, \ldots, K$ are chosen such that

$$E[\beta_{i+1}\psi_k(i+1, X_{i+1})|X_i = x]$$

is available in closed form.

- Define an alternative approximation by

$$\tilde{z}(i,x) = E[\beta_{i+1}\hat{E}[F(i+1, \tilde{z}_{i+1}(X_{i+1}))|X_{i+1}]|X_i = x], \ \tilde{z}(n,x) = 0,$$

where $\hat{E}[\cdot|X_{i+1}]$ denotes empirical regression on the basis functions at time $i+1$, cp. Glasserman, Yu (2004), Bender, Steiner (2012).

Back to the numerical example

- Choice of the parameters:

$$\underline{\sigma} = 0.3, \quad \overline{\sigma} = 0.4, \quad T = 1, \quad S_0 = 100$$

- Call spread option:

$$g(x) = (x - 90)_+ - (x - 110)_+$$

- Approximate dynamic programming: above variant of least-squares MC, 100.000 regression paths, basis functions: $1, x$, and the prices of call options with 160 different strikes between 20.5 and 230.5.
- Computation of the confidence bounds: 100.000 outer paths, martingale available in closed form.

A numerical example

- $\hat{\sigma} = 0.35$:

n	$\hat{Y}_0$	$\hat{Y}_0^{low}$	$\hat{Y}_0^{up}$	$\hat{Y}_0^{LGW}$
20	9.6924	9.6922 (<0.0001)	9.6930 (<0.0001)	9.8013
40	9.7409	9.7406 (<0.0001)	9.7435 (0.0001)	10.1383
60	9.7574	9.7571 (0.0001)	9.7707 (0.0006)	10.5239
80	9.7653	9.7644 (0.0002)	9.8651 (0.0088)	10.8792

- $\hat{\sigma} = 0.4/\sqrt{3} \approx 0.23$:

n	$\hat{Y}_0$	$\hat{Y}_0^{low}$	$\hat{Y}_0^{up}$	$\hat{Y}_0^{LGW}$
6	9.7183	9.7181 (0.0003)	9.7202 (0.0003)	9.6165
12	9.7584	9.7589 (0.0003)	9.7663 (0.0022)	9.8452
18	9.7686	9.7685 (0.0004)	9.7799 (0.0014)	10.3520
24	9.7745	9.7749 (0.0004)	9.8073 (0.0065)	11.2847

- True option price (Vanden, 2006): 9.7906

Beyond the convexity condition

- Suppose (Mono), which ensures the comparison pinciple, even if (Conv) is violated. Denote by L a Lipschitz constant of F.
- Fix an integrable process $\tilde{Z}_i$, which we think of an approximation of $Z_i = E_i[\beta_{i+1} Y_{i+1}]$
- Consider the auxiliary convex dynamic program

$$Y_i^{\tilde{Z}} = F(i, \tilde{Z}_i) + L \left| \tilde{Z}_i - E_i[\beta_{i+1} Y_{i+1}^{\tilde{Z}}] \right|, \quad Y_n^{\tilde{Z}} = \xi.$$

  Then $Y^{\tilde{Z}}$ is a supersolution to the original dynamic program for any $\tilde{Z}$ and $Y^Z = Y$.
- Hence

$$Y_i = \operatorname*{essinf}_{\tilde{Z}} \operatorname*{essinf}_{M} E_i[\theta_i^{up,\tilde{Z}}(M)].$$

- For the 'lower bound' one considers the auxiliary concave dynamic program

$$\underline{Y}_i^{\tilde{Z}} = F(i, \tilde{Z}_i) - L \left| \tilde{Z}_i - E_i[\beta_{i+1} \underline{Y}_{i+1}^{\tilde{Z}}] \right|, \quad \underline{Y}_n^{\tilde{Z}} = \xi.$$

Conclusion

- Generalizing from the Bermudan option case, we presented a methodology which complements any numerical method for approximating discrete time dynamic programs with a confidence interval for the quantity of interest Y_0 (e.g. an option price).
- In particular, a 'cheap' approximation method for the conditional expectations in the approximate dynamic program can be justified a-posteriori, if the confidence interval is sufficiently tight for the application under consideration.

Thank you for your attention

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- Bender, Schweizer, Zhuo, A primal-dual algorithm for BSDEs, under revision. Preprint available on arXiv.
- work in progress with C. Gärtner and N. Schweizer.

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