UiO : Department of Mathematics
University of Oslo

Lecture V

Modelling energy forward prices and ambit fields

Fred Espen Benth

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Overview

Presentation of work in collaboration with Ole Barndorff-Nielsen (Aarhus), Heidar Eyjolfsson (Reykjavik) and Almut Veraart (Imperial)

1. **Ambit fields: direct modelling of forward prices**
2. **Hilbert-valued ambit fields**
3. **Hambit fields as Lévy semistationary (LSS) processes**
4. **Hambit fields and SPDEs**

Overview

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Forward modelling by ambit fields

- Extension of the HJMM approach
- Random field model for the smooth forward curve
  - by direct modelling rather than as the solution of some dynamic equation
- Simple arithmetic model could be (in the risk-neutral setting)

$$F(t,x)=\int_{-\infty}^{t}\int_{0}^{\infty} g(t-s,x,y)\sigma(s,y)L(dy,ds)$$

- x is "time-to-maturity"

Definition of "classical" ambit fields

$$X(t,x) = \int_{-\infty}^{t} \int_{S} g(t-s,x,y)\sigma(s,y)\,L(ds,dy)$$

- L is a *Lévy basis*
- g non-negative deterministic function, $g(u,x,y) = 0$ for $u < 0$.
- Stochastic volatility process σ independent of L, stationary
- S a Borel subset of $\mathbb{R}^d$: "ambit" set

- L is a *Lévy basis* on $\mathbb{R}^d$ if
  1. the law of $L(A)$ is infinitely divisible for all bounded sets A
  2. if $A \cap B = \emptyset$, then $L(A)$ and $L(B)$ are independent
  3. if $A_1, A_2, \ldots$ are disjoint bounded sets, then

$$L(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} L(A_i)\,, a.s$$

- We restrict to zero-mean, and square integrable Lévy bases L
- Use Walsh' definition of the stochastic integral (Walsh (1986))

- Walsh' definition:
  - Extension of Itô integration theory to temporal-spatial setting
  - In time: integration as usual
  - In space: martingale properties from independence and additivity
  - Isometry by square-integrability hypothesis
- Suppose g and σ integrable
  - Essentially square-integrability in time and space
- Application of ambit fields: turbulence, tumor growth, energy finance, fixed-income markets, weather

Martingale condition

- Our model: classical ambit field with $d = 1$ and $S = [0, \infty)$
- Forwards are tradeable
- $t \mapsto F(t, T - t), t \leq T,$ must be a martingale

Proposition

$F(t, T)$ is a martingale if and only if

$$g(t - s, T - t, y) = \widetilde{g}(s, T, y)$$

- Note: g controls the Samuelson effect and the spatial correlation structure

- Example I: exponential damping function

$$g(u, x, y) = \exp\left(-\alpha(u + x + y)\right)$$

- Example II: the Musiela SPDE specification
  - $L = W$, $\mathbb{R}$-valued Brownian motion

$$dF(t, x) = \partial_x F(t, x)\, dt + g(x)\sigma(t)\, dW(t)$$

- Solution of the SPDE

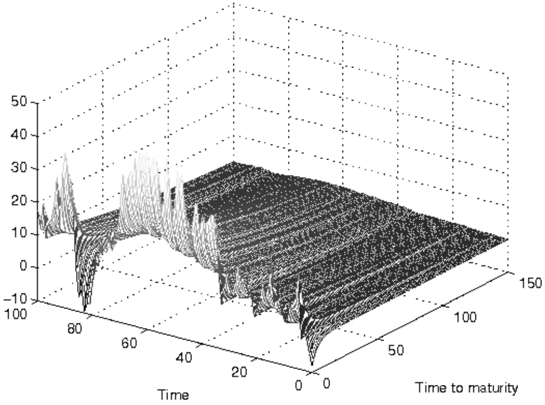
$$F(t, x) = F_0(x + t) + \int_0^t g(x + (t - s))\sigma(s)\, dW(s)$$

Simulation example

- Suppose g is a weighted sum of two exponentials
  - Motivated by a study of spot prices on the German EEX
  - ARMA(2,1) in continuous time

$$
\begin{aligned}
g(t-s,x,y) = w\exp(-\alpha_1(t-s+x+y)) \\
+(1-w)\exp(-\alpha_2(t-s+x-y))
\end{aligned}
$$

- $L = W$ a Gaussian basis
- $\sigma(s,y)$ again an ambit field
  - Exponential kernel function
  - Driven by inverse Gaussian Lévy basis



- Spot is very volatile
- Rapid convergence to zero when time to maturity increases

Overview

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4. Hambit fields and SPDEs

Recall "classical" ambit fields

$$X(t,x) = \int_{-\infty}^{t} \int_{S} g(t-s,x,y)\sigma(s,y)\, L(ds,dy)$$

- L is a *Lévy basis*, g non-negative deterministic function, $g(u,x,y) = 0$ for $u < 0$, stochastic volatility process σ independent of L being stationary, S a Borel subset of $\mathbb{R}^d$: "ambit" set
- Our goals:
  - Lift the ambit fields to processes in Hilbert space
  - ..and to analyse representations of such!

- Define $\mathcal{H}$-valued process $t \mapsto X(t)$

$$X(t) = \int_0^t \Gamma(t,s)(\sigma(s))\, dL(s)$$

- $\mathcal{U}, \mathcal{V}, \mathcal{H}$ three separable Hilbert spaces
- $s \mapsto L(s)$ $\mathcal{V}$-valued Lévy process
  - Square integrable with mean zero (L is $\mathcal{V}$-martingale)
  - Covariance operator $\mathcal{Q}$ (symmetric, positive definite, trace class)
- $s \mapsto \sigma(s)$ predictable process with values in $\mathcal{U}$
  - Stochastic volatility or intermittency
- $(t,s) \mapsto \Gamma(t,s)$, $s \leq t$, $\mathcal{L}(\mathcal{U}, \mathcal{L}(\mathcal{V}, \mathcal{H}))$-valued measurable mapping
  - Non-random *kernel* function

- Integrability condition for Γ and σ:

$$\mathbb{E}\left[\int_0^t \|\Gamma(t,s)(\sigma(s))\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2\,ds\right] < \infty$$

- We call X a Hambit field
- A sufficient integrability condition:

$$\int_0^t \|\Gamma(t,s)\|_{\mathrm{op}}^2 \mathbb{E}\left[|\sigma(s)|_{\mathcal{U}}^2\right]\,ds < \infty$$

Characteristic functional

Proposition

Suppose that σ is independent of L. For $h \in \mathcal{H}$ it holds

$$\mathbb{E}\left[\exp(\mathrm{i}(h, X(t))_{\mathcal{H}})\right] = \mathbb{E}\left[\exp\left(\int_0^t \Psi_L((\Gamma(t,s)(\sigma(s)))^* h)\right)\, ds\right]$$

where Ψ_L is the characteristic exponent of $L(1)$.

Proof.

Condition on σ, and use the independent increment property of L along with the fact

$$(h, \Gamma(t,s)(\sigma(s))\Delta L(s))_{\mathcal{H}} = ((\Gamma(t,s)(\sigma(s)))^* h, \Delta L(s))_{\mathcal{V}}$$

- Example: $L = W$, $\mathcal{V}$-valued Wiener process
- For $v \in \mathcal{V}$,

$$\Psi_W(v) = -\frac{1}{2}(\mathcal{Q}v, v)_{\mathcal{V}}$$

- Characteristic function of X (Bochner ds-integral)

$$\begin{aligned}&\mathbb{E}\left[\exp(\mathrm{i}(h, X(t))_{\mathcal{H}})\right] \\ &= \mathbb{E}\left[\exp\left(-\frac{1}{2}(h, \int_0^t \Gamma(t,s)(\sigma(s))\mathcal{Q}(\Gamma(t,s)(\sigma(s)))^* \, ds \, h)_{\mathcal{H}}\right)\right]\end{aligned}$$

- X is conditional Gaussian

Example: from Hambit to ambit

- Let $\mathcal{A} \subset \mathbb{R}^n$ Borel set, $\mathcal{U}$ a Hilbert space of real-valued functions on $\mathcal{A}$
- Let $(t, s, x, y) \mapsto g(t, s, x, y)$ be a measurable real-valued function for $0 \leq s \leq t \leq T$, $y \in \mathcal{A}$, $x \in \mathcal{B}$, $\mathcal{B} \subset \mathbb{R}^d$
- Suppose $\mathcal{V}$ is a Hilbert space of absolutely continuous functions on $\mathcal{A}$.
- Define for $\sigma \in \mathcal{U}$ the linear operator on $\mathcal{V}$

$$\Gamma(t, s)(\sigma) := \int_{\mathcal{A}} g(t, s, \cdot, y)\sigma(y)$$

acting on $f \in \mathcal{V}$ as

$$\Gamma(t, s)(\sigma)f = \int_{\mathcal{A}} g(t, s, \cdot, y)\sigma(s, y)f(dy) .$$

- Let $\mathcal{H}$ be a Hilbert space of real-valued functions on $\mathcal{B}$
- Let L be a $\mathcal{V}$-valued Lévy process, σ $\mathcal{U}$-valued predictable process
  - Suppose integrability conditions on $s \mapsto \Gamma(t,s)(\sigma(s))$
- $X(t,x)$ is an ambit field

$$X(t,x) = \int_0^t \int_{\mathcal{A}} g(t,s,x,y)\sigma(y)\,L(ds,dy)$$

- Example of Hilbert space?

Realization in Filipovic space

- Let $\mathcal{U} = \mathcal{V} = \mathcal{H}$, $n = d = 1$, $\mathcal{A} = \mathcal{B} = \mathbb{R}_+$
- Let $w \in C^1(\mathbb{R}_+)$ be non-decreasing, $w(0) = 1$ and $w^{-1} \in L^1(\mathbb{R}_+)$
- Let $\mathcal{U} := H_w$ be the space of absolutely continuous functions on $\mathbb{R}_+$ where

$$|f|_w^2 = f^2(0) + \int_{\mathbb{R}_+} w(y)|f'(y)|^2\,dy < \infty$$

- H_w separable Hilbert space.

Hilbert-valued OU with SV

- Fix $\mathcal{V} = \mathcal{H}$, and let $\mathcal{A}$ unbounded operator on $\mathcal{H}$ with C_0-semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$.
- W $\mathcal{H}$-valued Wiener process with covariance operator $\mathcal{Q}$.
- From Lecture IV: Let $\sigma(t)$ be a $\mathcal{U} := L_{HS}(\mathcal{H})$-valued predictable process,

$$dX(t) = \mathcal{A}X(t)\,dt + \sigma(t)\,dW(t)$$

- Mild solution

$$X(t) = \mathcal{S}(t)X(0) + \int_0^t \mathcal{S}(t-s)\sigma(s)\,dW(s)$$

- X as Hambit field: define $\Gamma(t,s) \in \mathcal{L}(L_{HS}(\mathcal{H}), \mathcal{L}(\mathcal{H}))$

$$\Gamma(t,s) : \sigma \mapsto \mathcal{S}(t-s)\sigma$$

- A BNS SV model: $\sigma(t) = \mathcal{Y}^{1/2}(t)$

$$d\mathcal{Y}(t) = \mathbb{C}\mathcal{Y}(t)\,dt + d\Xi(t)$$

- $\mathbb{C} \in \mathcal{L}(L_{HS}(\mathcal{H}))$, with C_0-semigroup $\{\mathbb{S}(t)\}_{t\geq 0}$
- Ξ is a $L_{HS}(\mathcal{H})$-valued "subordinator"

- $\mathcal{Y}(t)$ symmetric, positive definite, $L_{HS}(\mathcal{H})$-valued process,

$$\mathbb{E}[|\sigma(t)|_{\mathcal{U}}^2] = \sum_{n=1}^{\infty}(\sigma(t)h_k, \sigma(t)h_k)_{\mathcal{H}} = \mathrm{Tr}(\mathcal{Y}(t))$$

- The trace is continuous, and hence the integrability condition for X holds

$$\mathrm{Tr}(\mathcal{Y}(t)) = \mathrm{Tr}(\mathbb{S}(t)\mathcal{Y}_0) + \mathrm{Tr}(\int_0^t \mathbb{S}(s)\, ds \mathbb{E}[\Xi(1)])$$

- Recall Lecture IV!

Overview

- Let $\{u_n\}, \{v_m\}$ and $\{h_k\}$ be ONB in $\mathcal{U}, \mathcal{V}$ and $\mathcal{H}$ resp.
  - Recall separability of the Hilbert spaces
- $L_m := (L, v_m)_{\mathcal{V}}$ are $\mathbb{R}$-valued Lévy processes
  - zero mean, square integrable
  - but, not independent nor zero correlated
- Define LSS processes $Y_{n,m,k}(t)$ by

$$Y_{n,m,k}(t) = \int_0^t g_{m,n,k}(t,s)\sigma_n(s)\,dL_m(s)$$

$$g_{n,m,k}(t,s) := (\Gamma(t,s)(u_n)v_m, h_k)_{\mathcal{H}} \qquad \sigma_n(s) := (\sigma(s), u_n)_{\mathcal{U}}$$

Proposition

Assume

$$\int_0^t \|\Gamma(t,s)\|_{op}^2 \left(\sum_{n=1}^{\infty} \mathbb{E}[\sigma_n^2(s)]^{1/2}\right)^2 ds < \infty$$

then,

$$X(t) = \sum_{n,m,k=1}^{\infty} Y_{n,m,k}(t) h_k$$

Proof.

Expand all elements along the ONB's in their respective spaces. The integrability assumption ensures the commutation of an infinite sum and stochastic integral wrt. L_m (A stochastic Fubini theorem). ∎

- Barndorff-Nielsen et al. (2013): energy spot price modeling using LSS processes
  - Finite factors
  - Implied forward prices become scaled finite sums of LSS processes
- Barndorff-Nielsen et al. (2014): energy forward prices as ambit fields
  - Infinite LSS factor models!
- B. Krühner (2014): HJM forward price dynamics representable as countable scaled sums of OU process
  - Possibly complex valued OU processes

- Integrability condition implies the sufficient condition for existence of Hambit field:
- By Parseval's identity

$$\mathbb{E}[\|\sigma(s)\|_{\mathcal{U}}^2] = \sum_{n=1}^{\infty} \mathbb{E}[(\sigma(s), u_n)_{\mathcal{U}}^2]$$

- Sufficient condition for LSS representation: there exists $a_n > 0$ s.t. $\sum_{n=1}^{\infty} a_n^{-1} < \infty$ and

$$\sum_{n=1}^{\infty} a_n \int_0^t \|\Gamma(t,s)\|_{\text{op}}^2 \mathbb{E}[(\sigma(s), u_n)_{\mathcal{U}}^2]\, ds < \infty$$

Overview

- Known connection between an LSS process and the boundary of a hyperbolic stochastic partial differential equation (SPDE):

$$dZ(t,x) = \partial_x Z(t,x)\,dt + g(t+x,t)\sigma(t)\,dL(t)$$
$$Z_0(t) := Z(t,0) = \int_0^t g(t,s)\sigma(s)\,dL(s)$$

- L $\mathbb{R}$-valued Lévy process, $x \geq 0$
- Goal: show similar result for Hambit fields!
  - Application: B. Eyjolfsson (2015) deviced iterative (finite difference) numerical schemes in the $\mathbb{R}$-valued case using this relationship

- Assume $\widetilde{\mathcal{H}}$ a Hilbert space $\mathcal{H}$-valued functions on $\mathbb{R}_+$
- Suppose $\{\mathcal{S}(\xi)\}_{\xi\geq 0}$ right-shift operator is C_0-semigroup on $\widetilde{\mathcal{H}}$ with generator $\partial_\xi = \partial/\partial\xi$

$$\mathcal{S}(\xi)f := f(\xi + \cdot)\,, \quad f \in \widetilde{\mathcal{H}}$$

- Consider hyperbolic SPDE in $\widetilde{\mathcal{H}}$

$$\mathcal{X}(t) = \partial_\xi \mathcal{X}(t)\,dt + \Gamma(t+\cdot, t)(\sigma(t))\,dL(t)\,, \mathcal{X}(0) \in \widetilde{\mathcal{H}}$$

- Predictable $\widetilde{\mathcal{H}}$-valued unique solution

$$\mathcal{X}(t) = \mathcal{S}(t)\mathcal{X}(0) + \int_0^t \mathcal{S}(t-s)\Gamma(s+\cdot, s)(\sigma(s))\,dL(s)$$

Proposition

Assume that the evaluation map $\delta_x : \widetilde{\mathcal{H}} \to \mathcal{H}$ *defined by* $\delta_x f = f(x) \in \mathcal{H}$ *for every* $x \geq 0$ *and* $f \in \widetilde{\mathcal{H}}$ *is a continuous linear operator. If* $\mathcal{X}(0) = 0$*, Then* $X(t) = \delta_0(\mathcal{X}(t))$*.*

Proof.

Argue that

$$\delta_0 \int_0^t \Gamma(t+\cdot, s)(\sigma(s))\, dL(s) = \int_0^t \Gamma(t,s)(\sigma(s))\, dL(s)$$

■

- Need a space $\widetilde{\mathcal{H}}$ with $\delta_x \in \mathcal{L}(\widetilde{\mathcal{H}}, \mathcal{H})$

Abstract Filipovic space

- $f \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ is *weakly differentiable* if there exists $f' \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ such that

$$\int_{\mathbb{R}_+} f(x)\varphi'(x)\,dx = -\int_{\mathbb{R}_+} f'(x)\varphi(x)\,dx\,, \forall \varphi \in C_c^\infty(\mathbb{R}_+)$$

- Integrals interpreted in Bochner sense
- Let $w \in C^1(\mathbb{R}_+)$ be a non-decreasing function with $w(0) = 1$ and

$$\int_{\mathbb{R}_+} w^{-1}(x)\,dx < \infty$$

- Define $\mathcal{H}_w$ to be the space of $f \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ for which there exists $f' \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ such that

$$\|f\|_w^2 = |f(0)|_{\mathcal{H}}^2 + \int_{\mathbb{R}_+} w(x)|f'(x)|_{\mathcal{H}}^2 \, dx < \infty \, .$$

- $\mathcal{H}_w$ is a separable Hilbert space with inner product

$$\langle f, g \rangle_w = (f(0), g(0))_{\mathcal{H}} + \int_{\mathbb{R}_+} w(x)(f'(x), g'(x))_{\mathcal{H}} \, dx$$

- Fundamental theorem of calculus (FTC): If $f \in \mathcal{H}_w$, then $f' \in L^1(\mathbb{R}_+, \mathcal{H})$, $\|f'\|_1 \leq c\|f\|_w$, and

$$f(x+t) - f(x) = \int_x^{x+t} f'(y)\,dy$$

- Shift-operator $\mathcal{S}(\xi), \xi \geq 0$ is *uniformly bounded*

$$\|\mathcal{S}(\xi)f\|_w^2 \leq 2(1+c^2)\|f\|_w^2$$

- Constant equal to $c^2 = \int_{\mathbb{R}_+} w^{-1}(x)\,dx$

Lemma

Evaluation map $\delta_x : \mathcal{H}_w \to \mathcal{H}$ is a linear bounded operator with

$$|\delta_x f|_{\mathcal{H}} \leq K \|f\|_w$$

Proof.

FTC, Bochner's norm inequality and Cauchy-Schwartz inequality yield

$$|\delta_x f|^2_{\mathcal{H}} = |f(x)|^2_{\mathcal{H}} \leq 2|f(0)|^2_{\mathcal{H}} + 2\int_{\mathbb{R}_+} w^{-1}(y)\,dy \int_{\mathbb{R}_+} w(y)|f'(y)|^2_{\mathcal{H}}\,dy$$

∎

- We have an example $\widetilde{\mathcal{H}} = \mathcal{H}_w$!

Classical and abstract Filipovic space

Proposition

For $\mathcal{L} \in \mathcal{H}^*$, $x \mapsto \mathcal{L} \circ \delta_x(g) = \mathcal{L}(g(x)) \in H_w$ for $g \in \mathcal{H}_w$. Moreover, if $h_x(y) = 1 + \int_0^{x \wedge y} w^{-1}(z)\,dz$ and $\ell_x = \mathcal{L}^*(h_x)$, then

$$\mathcal{L}(g(x)) = \langle g, \ell_x \rangle_w$$

Proof.

Follows from linearity of $\mathcal{L}$, FTC and Bochner's norm inequality. Further, if $\bar{\delta}_x$ is the evaluation map on H_w, then $\bar{\delta}_x(v) = (v, h_x)_w$, $v \in H_w$. ∎

Bedankt voor uw aandacht!

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