Polynomial Models in Finance

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What this course is about

- Polynomial models provide an analytically tractable and statistically flexible framework for financial modeling
- New factor process dynamics, beyond affine, enter the scene
- Definition of polynomial jump-diffusions and basic properties
- Existence and building blocks
- Polynomial models in finance: option pricing, portfolio choice, risk management, economic scenario generation,..
- Examples: stochastic volatility, interest rates, credit risk

Course Outline

Some Literature

- Polynomial processes: [Wong, 1964], [Mazet, 1997], [Forman and Sørensen, 2008],[Cuchiero, 2011], [Cuchiero et al., 2012], [Filipović and Larsson, 2016], and others
- Polynomial models in finance: [Zhou, 2003], [Delbaen and Shirakawa, 2002], [Larsen and Sørensen, 2007], [Gouriéroux and Jasiak, 2006], [Eriksson and Pistorius, 2011], [Filipović et al., 2016], [Filipović et al., 2014], [Ackerer and Filipović, 2015], [Ackerer et al., 2015], [Filipović and Larsson, 2017], and others

This course is based on the highlighted papers. Most results in Parts I–III are from [Filipović and Larsson, 2017].

Part I

Definition and Basic Properties

Outline

Polynomial Jump-Diffusions

Affine Jump-Diffusions

Outline

Polynomial Jump-Diffusions

Affine Jump-Diffusions

Setup

- Filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$
- State space $E \subseteq \mathbb{R}^d$
- E-valued jump-diffusion X_t with extended generator given by

$$\mathcal{G}f(x) = \frac{1}{2}\mathrm{tr}(a(x)\nabla^2 f(x)) + b(x)^\top \nabla f(x) + \int_{\mathbb{R}^d} \Big(f(x+\xi) - f(x) - \xi^\top \nabla f(x) \Big)\, \nu(x, d\xi)$$

for measurable $a : \mathbb{R}^d \to \mathbb{S}^d$, $b : \mathbb{R}^d \to \mathbb{R}^d$, and Lévy transition kernel $\nu(x, d\xi)$ from $\mathbb{R}^d$ into $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} \|\xi\| \wedge \|\xi\|^2 \nu(x, d\xi) < \infty$

Definition of Jump-Diffusion

- That is, X_t is E-valued special semimartingale, such that

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{G}f(X_s)\,ds \quad \text{is a local martingale}$$

for any bounded C^2 function $f(x)$, [Jacod and Shiryaev, 2003, Thm II.2.42]

- Note: this holds for any C^2 function $f(x)$ such that, for any finite t,

$$\int_0^t \int_{\mathbb{R}^d} \left| f(X_s + \xi) - f(X_s) - \xi^\top \nabla f(X_s) \right| \nu(X_s, d\xi)\, ds < \infty.$$

Indeed, then the term is in $\mathcal{A}_{loc}^+$, see [Jacod and Shiryaev, 2003, Thm II.1.8 and proof of Thm II.2.42]

Polynomials on E

- Polynomial on E: restriction $p = q|_E$ of a polynomial q on $\mathbb{R}^d$
- Degree $\deg p = \min\{\deg q : p = q|_E,\ q \text{ polynomial on } \mathbb{R}^d\}$
- Space of polynomials of degree n or less

$$\mathrm{Pol}_n(E) = \{p \text{ polynomial on } E \text{ with } \deg p \le n\}$$

  has $\dim \mathrm{Pol}_n(E) \le \binom{n+d}{n}$ with equality if $\mathrm{int}(E) \neq \emptyset$
- Ring of polynomials

$$\mathrm{Pol}(E) = \cup_{n\ge 1} \mathrm{Pol}_n(E)$$

- Multi-index notation

$$\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d, \quad x^{\mathbf{k}} = x_1^{k_1} \cdots x_d^{k_d}, \quad |\mathbf{k}| = \sum_{i=1}^d k_i$$

Definition of Polynomial Jump-Diffusion (PJD)

Definition 1.1.
$\mathcal{G}$ is well-defined on $\mathrm{Pol}(E)$ if

1. Jump measure of X_t admits moments of all orders,

$$\int_{\mathbb{R}^d} \|\xi\|^n \nu(x, d\xi) < \infty \text{ for all } x \in E \text{ and } n \geq 2;$$

2. $\mathcal{G}f(x) = 0$ on E for any $f \in \mathrm{Pol}(\mathbb{R}^d)$ with $f(x) = 0$ on E.

Definition 1.2.
$\mathcal{G}$ is polynomial on E if it is well-defined on $\mathrm{Pol}(E)$ and

$$\mathcal{G}\mathrm{Pol}_n(E) \subseteq \mathrm{Pol}_n(E) \text{ for all } n \in \mathbb{N}.$$

In this case, we call X_t a polynomial jump-diffusion (PJD) on E.

Example

- State space $E = \mathbb{R} \times \{0\}$, $d = 2$
- The partial differential operator

$$\mathcal{G}f(x,y) = \frac{1}{2}\partial_{xx}f(x,y) + \partial_y f(x,y)$$

is not well-defined on $\mathrm{Pol}(E)$: y vanishes on E but $\mathcal{G}y = 1$

- Note $\mathcal{G}$ is generator of $dX_t = (dB_t, dt)$, which leaves E
- Positive maximum principle implies: $\mathcal{G}$ is well-defined on E if for any $X_0 = x$ in E there exists E-valued jump-diffusion X_t with generator $\mathcal{G}$, see [Filipović and Larsson, 2016, Lemma 2.3].

Characterization of Polynomial Jump-Diffusions

Lemma 1.3.
Assume $\mathcal{G}$ is well-defined on $\mathrm{Pol}(E)$. The following are equivalent:

1. *$\mathcal{G}$ is polynomial on E.*
2. *$a(x)$, $b(x)$, and $\nu(x, d\xi)$ satisfy*

$$
\begin{aligned}
b_i(x) &\in \mathrm{Pol}_1(E), && \textit{drift} \\
a_{ij}(x) + \int_{\mathbb{R}^m} \xi_i \xi_j \nu(x, d\xi) &\in \mathrm{Pol}_2(E), && \textit{modified 2nd characteristic} \\
\int_{\mathbb{R}^m} \xi^{\boldsymbol{\alpha}} \nu(x, d\xi) &\in \mathrm{Pol}_{|\boldsymbol{\alpha}|}(E), && \textit{jumps}
\end{aligned}
$$

   for all $i, j = 1, \ldots, d$ and all $|\boldsymbol{\alpha}| \geq 3$.

In this case, the polynomials on E listed in property 2 are uniquely determined by the action of $\mathcal{G}$ on $\mathrm{Pol}(E)$.

Characterization of Polynomial Jump-Diffusions

Proof.

Plug in polynomials p in $\mathcal{G}p$ and collect and match terms ... □

Properties of Polynomial Jump-Diffusions

Let X_t be a PJD with generator $\mathcal{G}$ on E.

Lemma 1.4.

For any $f \in \mathrm{Pol}(E)$ the process

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{G} f(X_s)\, ds$$

is a local martingale.

Sketch of proof.

Lemma 1.3 implies that

$$\int_{\mathbb{R}^d} \underbrace{\Big(f(x+\xi) - f(x) - \xi^\top \nabla f(x)\Big)^2}_{\text{minimal degree} \geq 4} \nu(x, d\xi) \in \mathrm{Pol}(E).$$

The lemma follows from [Jacod and Shiryaev, 2003, Thm II.1.33 and proof of Thm II.2.42]. □

Properties of Polynomial Jump-Diffusions cont'd

Lemma 1.5.
For any $k \in \mathbb{N}$ there exists a finite C_k such that

$$\mathbb{E}[1 + \|X_t\|^{2k} \mid \mathcal{F}_0] \leq \left(1 + \|X_0\|^{2k}\right) \mathrm{e}^{C_k t}, \quad t \geq 0.$$

Sketch of proof.

Using arguments from [Cuchiero et al., 2012, Thm 2.10] or [Filipović and Larsson, 2016, Lemma B.1]. □

Properties of Polynomial Jump-Diffusions cont'd

Lemma 1.6.
For any $f \in \mathrm{Pol}(E)$ and finite c the process $M_t^f 1_{\{\|X_0\| \leq c\}}$ is a martingale.

Sketch of proof.

The compensator of quadratic variation of M_t^f is given by

$$\langle M^f, M^f \rangle_t = \langle f(X), f(X) \rangle_t = \int_0^t \Gamma(f,f)(X_s)\, ds$$

and $\Gamma(f,f) \in \mathrm{Pol}(E)$, for the carré-du-champ operator Γ. The lemma follows from Lemmas 1.4 and 1.5. □

Carré-du-Champ Operator

The carré-du-champ operator $\Gamma(f,g)$ is defined by

$$\begin{aligned}\Gamma(f,g)(x) &= \mathcal{G}(fg)(x) - f(x)\mathcal{G}g(x) - g(x)\mathcal{G}f(x) \\ &= \nabla f(x)^\top a(x)\nabla g(x) \\ &\quad + \int_{\mathbb{R}^d}(f(x+\xi)-f(x))(g(x+\xi)-g(x))\nu(x,d\xi).\end{aligned}$$

It is related to the co-variation of $f(X)$ and $g(X)$,

$$\begin{aligned}[f(X),g(X)]_t &= \int_0^t \nabla f(X_s)^\top a(X_s)\nabla g(X_s)\,ds \\ &\quad + \sum_{s\le t}(f(X_s)-f(X_{s-}))(g(X_s)-g(X_{s-})),\end{aligned}$$

and its compensator by

$$\langle f(X),g(X)\rangle_t = \int_0^t \Gamma(f,g)(X_s)\,ds.$$

Towards the Moment Transform Formula

- Let $\mathcal{G}$ be polynomial on E
- Fix $n \in \mathbb{N}$, denote $1 + N = \dim \mathrm{Pol}_n(E) \leq \binom{n+d}{n} < \infty$
- $\mathcal{G}$ restricts to linear operator on $\mathrm{Pol}_n(E)$
- Fix a basis $1, h_1(x), \dots, h_N(x)$ of $\mathrm{Pol}_n(E)$, denote

$$H(x) = (h_1(x), \dots, h_N(x))$$

- Coordinate representation $\vec{p}$ of $p \in \mathrm{Pol}_n(E)$:

$$p(x) = (1, H(x))\vec{p}$$

- Matrix representation G of $\mathcal{G}$: $\mathcal{G}(1, H(x)) = (1, H(x))G$,

$$\mathcal{G}p(x) = (1, H(x))G\vec{p}$$

Moment Transform Formula

Theorem 1.7.
For any $p \in \mathrm{Pol}_n(E)$ *we have that*

$$\mathbb{E}[p(X_T) \mid \mathcal{F}_t] = (1, H(X_t))\mathrm{e}^{(T-t)G}\vec{p}$$

is a polynomial in X_t *of degree* $\leq n$, *for all* $T \geq t$.

Moment Transform Formula: Proof

Sketch of proof.

Fix finite c and write $A = \{\|X_0\| \leq c\}$. By Lemma 1.6, the function $F(s) = \mathbb{E}[(1, H(X_s))1_A \mid \mathcal{F}_t]$ satisfies

$$F(s) = (1, H(X_t))1_A + \int_t^s \mathbb{E}[\mathcal{G}(1, H(X_u))1_A \mid \mathcal{F}_t]\, du$$
$$= F(t) + \int_t^s F(u) G\, du,$$

thus $\mathbb{E}[(1, H(X_T))1_A \mid \mathcal{F}_t] = (1, H(X_t))\mathrm{e}^{(T-t)G} 1_A$.
Now let $c \uparrow \infty$. □

Example: Scalar Polynomial Diffusions

- Generic scalar polynomial diffusion on interval $E \subseteq \mathbb{R}$

$$dX_t = (b + \beta X_t)\, dt + \sqrt{a + \alpha X_t + A X_t^2}\, dW_t$$

- Basis $\{1, x, x^2, \cdots, x^n\}$ of $\mathrm{Pol}_n(E)$

- Coordinate representation of $p(x) = \sum_{k=0}^n p_k x^k$:

$$\vec{p} = (p_0, \ldots, p_n)^\top$$

- Matrix representation of $\mathcal{G}$: $(n+1) \times (n+1)$-matrix

$$G = \begin{pmatrix} 0 & b & 2\frac{a}{2} & 0 & \cdots & 0 \\ 0 & \beta & 2\left(b + \frac{\alpha}{2}\right) & 3 \cdot 2\frac{a}{2} & 0 & \vdots \\ 0 & 0 & 2\left(\beta + \frac{A}{2}\right) & 3\left(b + 2\frac{\alpha}{2}\right) & \ddots & 0 \\ 0 & 0 & 0 & 3\left(\beta + 2\frac{A}{2}\right) & \ddots & n(n-1)\frac{a}{2} \\ \vdots & & & 0 & \ddots & n\left(b + (n-1)\frac{\alpha}{2}\right) \\ 0 & & \cdots & & 0 & n\left(\beta + (n-1)\frac{A}{2}\right) \end{pmatrix}$$

More Examples of Polynomial Jump-Diffusions

- Any affine process is a PJD
- Lévy driven GARCH diffusion:

$$dX_t = (b + \beta X_t)\, dt + X_{t-}\, dL_t$$

  for a Lévy martingale L_t

- Jacobi type processes on $E =$ unit ball

$$dX_t = (b + \beta X_t)\, dt + \sqrt{(1 - \|X_t\|^2)}\, \Sigma dW_t$$

  and more general diffusions on quadric (compact) sets in $\mathbb{R}^d$

Outline

Polynomial Jump-Diffusions

Affine Jump-Diffusions

Definition of Affine Jump-Diffusion (AJD)

Let X_t be jump-diffusion on $E \subseteq \mathbb{R}^d$ with generator $\mathcal{G}$

Definition 2.1.
$\mathcal{G}$ is affine on E if, for all $x \in E$, $u \in \mathrm{i}\mathbb{R}^d$

$$\mathcal{G} \exp(u^\top x) = (F(u) + R(u)^\top x) \exp(u^\top x),$$

for functions $F : \mathrm{i}\mathbb{R}^d \to \mathbb{C}$ and $R = (R_1, \dots, R_d)^\top : \mathrm{i}\mathbb{R}^d \to \mathbb{C}^d$. In this case, we call X_t an affine jump-diffusion (AJD) on E.

Note: this is a relaxed definition compared to [Duffie et al., 2003]

Characterization of Affine Jump-Diffusions

Lemma 2.2.

The following are equivalent:

1. *$\mathcal{G}$ is affine on E.*
2. *$a(x)$, $b(x)$, and $\nu(x, d\xi)$ are affine on E,*

$$a(x) = a_0 + \sum_{i=1}^{d} x_i a_i,$$
$$b(x) = b_0 + \sum_{i=1}^{d} x_i b_i,$$
$$\nu(x, d\xi) = \nu_0(d\xi) + \sum_{i=1}^{d} x_i \nu_i(d\xi),$$

for some $a_i \in \mathbb{S}^d$, $b_i \in \mathbb{R}^d$, and signed measures $\nu_i(d\xi)$ on $\mathbb{R}^d$.

In this case, $F(u)$ and $R(u)$ are given by (write $F(u) \equiv R_0(u)$)

$$R_i(u) = \frac{1}{2} u^\top a_i u + b_i^\top u + \int_{\mathbb{R}^d} \left(e^{u^\top \xi} - 1 - u^\top \xi \right) \nu_i(d\xi).$$

Characterization of Affine Jump-Diffusions: Proof

Sketch of Proof.

Observe that

$$\frac{\mathcal{G}\mathrm{e}^{u^\top x}}{\mathrm{e}^{u^\top x}} = \frac{1}{2}u^\top a(x)u + b(x)^\top u + \int_{\mathbb{R}^d}\left(\mathrm{e}^{u^\top \xi} - 1 - u^\top \xi\right)\nu(x,d\xi)$$

and match terms.. □

Affine are Polynomial Jump-Diffusions

Corollary 2.3.

If X_t is an AJD and $\mathcal{G}$ is well-defined on $\mathrm{Pol}(E)$ then X_t is a PJD.

Affine Transform Formula

Theorem 2.4.

Let X_t be an AJD on E, $u \in \mathrm{i}\mathbb{R}^d$, $T > 0$, and let $\phi(\tau)$ and $\psi(\tau) = (\psi_1(\tau), \ldots, \psi_d(\tau))^\top$ solve the generalized Riccati equations

$$\begin{aligned} \phi'(\tau) &= F(\psi(\tau)), & \phi(0) &= 0 \\ \psi'(\tau) &= R(\psi(\tau)), & \psi(0) &= u \end{aligned}$$

for $0 \le \tau \le T$. If

$$\mathrm{Re}\,\phi(\tau) + \mathrm{Re}\,\psi(\tau)^\top x \le 0, \qquad 0 \le \tau \le T, \quad x \in E,$$

then the affine transform formula holds,

$$\mathbb{E}[\exp(u^\top X_T) \mid \mathcal{F}_t] = \exp(\phi(T-t, u) + \psi(T-t, u)^\top X_t).$$

Affine Transform Formula: Proof

Sketch of proof.

Drift of $M_t = \exp(\phi(T-t) + \psi(T-t)^\top X_t)$ is

$$\mathcal{G}\mathrm{e}^{\phi+\psi^\top X_t} = (-\phi' + F(\psi) - \psi' + R(\psi)^\top X_t)M_t = 0$$

and $|M_t| \leq 1$, hence M_t is a martingale. □

Affine Transform Formula: Extension beyond $\mathrm{i}\mathbb{R}^d$

Fact: If $\phi(T-t,u)$ and $\psi(T-t,u)$ have an analytic extension in u on $U \subset \mathbb{C}^d$, the affine transform formula

$$\mathbb{E}[\exp(u^\top X_T) \mid \mathcal{F}_t] = \exp(\phi(T-t,u) + \psi(T-t,u)^\top X_t).$$

holds for all $u \in U$, see [Duffie et al., 2003, Thm 2.16].

Part II

Existence and Building Blocks

Outline

Polynomial Diffusions [Filipović and Larsson, 2016]

Invariance Properties: Exponentiation

Invariance Properties: Subordination

Outline

Polynomial Diffusions [Filipović and Larsson, 2016]

Invariance Properties: Exponentiation

Invariance Properties: Subordination

Overview

- PJDs have great potential for financial applications
- What do we know about their existence? Uniqueness?
- This section shows results for polynomial diffusions
- Based on [Filipović and Larsson, 2016]

Polynomial Diffusions: Ingredients

Ingredients:

- Drift function $b : \mathbb{R}^d \to \mathbb{R}^d$ with $b_i \in \mathrm{Pol}_1(\mathbb{R}^d)$
- Diffusion function $a : \mathbb{R}^d \to \mathbb{S}^d$ with $a_{ij} \in \mathrm{Pol}_2(\mathbb{R}^d)$
- "Polynomial" operator on $\mathbb{R}^d$

$$\mathcal{G}f(x) = \frac{1}{2}\mathrm{tr}(a(x)\nabla^2 f(x)) + b(x)^\top \nabla f(x)$$

- State space $E \subseteq \mathbb{R}^d$

Polynomial Diffusions: Issues

Find conditions on a, b, E for

- Existence of E-valued solutions to corresponding SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \tag{3.1}$$

  for continuous $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ with $\sigma\sigma^\top = a$ on E

- Uniqueness in law for E-valued solutions to (3.1)
- Boundary (non-)attainment of E

For applications: find large parametric classes of such a, b, E

Example: Scalar Polynomial Diffusions

- Generic scalar polynomial diffusion on interval $E \subseteq \mathbb{R}$

$$dX_t = (b + \beta X_t)\, dt + \sqrt{a + \alpha X_t + A X_t^2}\, dW_t$$

- Basis $\{1, x, x^2, \cdots, x^n\}$ of $\mathrm{Pol}_n(E)$

- Coordinate representation of $p(x) = \sum_{k=0}^{n} p_k x^k$:

$$\vec{p} = (p_0, \ldots, p_n)^\top$$

- Matrix representation of $\mathcal{G}$: $(n+1) \times (n+1)$-matrix

$$G = \begin{pmatrix} 0 & b & 2\frac{a}{2} & 0 & \cdots & 0 \\ 0 & \beta & 2\left(b + \frac{\alpha}{2}\right) & 3 \cdot 2\frac{a}{2} & 0 & \vdots \\ 0 & 0 & 2\left(\beta + \frac{A}{2}\right) & 3\left(b + 2\frac{\alpha}{2}\right) & \ddots & 0 \\ 0 & 0 & 0 & 3\left(\beta + 2\frac{A}{2}\right) & \ddots & n(n-1)\frac{a}{2} \\ \vdots & & & 0 & \ddots & n\left(b + (n-1)\frac{\alpha}{2}\right) \\ 0 & & \cdots & & 0 & n\left(\beta + (n-1)\frac{A}{2}\right) \end{pmatrix}$$

Towards Uniqueness: determinacy of moment problem

- Determinacy of moment problem: moments determine distribution
- Sufficient condition: finite exponential moments (analyticity of characteristic function at zero)

Exponential moments

Theorem 3.1.
If

$$\mathbb{E}\left[\mathrm{e}^{\delta\|X_0\|}\right] < \infty \quad \textit{for some} \quad \delta > 0 \tag{3.2}$$

and the diffusion coefficient satisfies the linear growth condition

$$\|a(x)\| \leq C(1 + \|x\|) \quad \textit{for all} \quad x \in E \tag{3.3}$$

for some constant C*, then for each* $t \geq 0$ *there exists* $\varepsilon > 0$ *with*

$$\mathbb{E}\left[\mathrm{e}^{\varepsilon\|X_t\|}\right] < \infty.$$

Uniquess in law from moment problem

Theorem 3.2.

Let X be an E-valued solution to (3.1)*. If for each $t \geq 0$ there exists $\varepsilon > 0$ with $\mathbb{E}[\exp(\varepsilon \|X_t\|)] < \infty$, then any E-valued solution to* (3.1) *with the same initial law as X has the same law as X. In particular, this holds if* (3.2) *and* (3.3) *are satisfied:*

$$\mathbb{E}\left[e^{\delta \|X_0\|}\right] < \infty \quad \textit{for some} \quad \delta > 0$$

$$\|a(x)\| \leq C(1 + \|x\|) \quad \textit{for all} \quad x \in E.$$

Limits and an open problem

- Theorem 3.2 does not apply for geometric Brownian motion

$$dX_t = X_t\, dB_t$$

- Open problem: Can one find a process Y, essentially different from geometric Brownian motion, such that all joint moments of all finite-dimensional marginal distributions,

$$\mathbb{E}[Y_{t_1}^{\alpha_1} \cdots Y_{t_m}^{\alpha_m}]$$

coincide with those of geometric Brownian motion?

Pathwise uniqueness for $d = 1$

Theorem 3.3.
If the dimension is $d = 1$, then uniqueness in law for E-valued solutions to (3.1) holds.

Combined result

Assume SDE (3.1) decomposes for $X = (Y, Z)$ as

$$
\begin{aligned}
dY_t &= b_Y(Y_t)\,dt \;+\; \sigma_Y(Y_t)\,dW_t \\
dZ_t &= b_Z(Y_t, Z_t)\,dt \;+\; \sigma_Z(Y_t, Z_t)\,dW_t
\end{aligned} \tag{3.4}
$$

Theorem 3.4.
Assume that uniqueness in law for E_Y-valued solutions to (3.4) holds, and that σ_Z is locally Lipschitz in z locally in y on E: for each compact subset $K \subseteq E$, there exists a constant κ such that for all $(y, z, y', z') \in K \times K$,

$$\|\sigma_Z(y, z) - \sigma_Z(y', z')\| \leq \kappa \|z - z'\|.$$

Then uniqueness in law for E-valued solutions to (3.1) holds.

Stochastic invariance problem

- Existence of $\mathbb{R}^d$-valued solution to (3.1) holds due to continuity and linear growth of b and σ
- Existence of E-valued solution to (3.1) thus boils down to stochastic invariance of E
- Assume E is basic closed semialgebraic set

$$E = \{p \geq 0 \mid p \in \mathcal{P}\} \cap M$$

where

$$M = \{q = 0 \mid q \in \mathcal{Q}\}$$

for finite collections of polynomials $\mathcal{P}$ and $\mathcal{Q}$

Examples

- $E = \mathbb{R}^d_+$:

$$\mathcal{P} = \{p_i(x) = x_i \mid i = 1..d\}, \quad \mathcal{Q} = \emptyset$$

- $E = [0,1]^d$:

$$\mathcal{P} = \{p_i(x) = x_i,\ p_{d+i}(x) = 1 - x_i \mid i = 1..d\}, \quad \mathcal{Q} = \emptyset$$

- $E =$ unit ball:

$$\mathcal{P} = \left\{p(x) = 1 - \|x\|^2\right\}, \quad \mathcal{Q} = \emptyset$$

- $E = \mathbb{S}^m_+$:

$$\mathcal{P} = \{p_I(x) = \det x_{II} \mid I \subset \{1, \ldots, m\}\}, \quad \mathcal{Q} = \emptyset$$

- $E = \{x \in \mathbb{R}^d_+ \mid x_1 + \cdots + x_d = 1\}$ unit simplex:

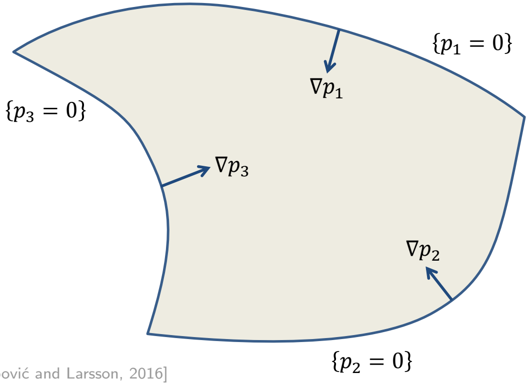
$$\mathcal{P} = \{p_i(x) = x_i \mid i = 1..d\}, \quad \mathcal{Q} = \{q(x) = 1 - x_1 - \cdots - x_d\}$$

Necessary conditions

Theorem 3.5.

Suppose there exists an E-valued solution to (3.1) with $X_0 = x$, for any $x \in E$. Then

1. $a\nabla p = 0$ *and* $\mathcal{G}p \geq 0$ *on* $E \cap \{p = 0\}$ *for each* $p \in \mathcal{P}$;
2. $a\nabla q = 0$ *and* $\mathcal{G}q = 0$ *on* E *for each* $q \in \mathcal{Q}$.



Sufficient conditions

Geometry of E:

(G1) $\nabla r(x)$, $r \in \mathcal{Q}$, are linearly independent for all $x \in M$

(G2) the ideal generated by $\mathcal{Q} \cup \{p\}$ is real for each $p \in \mathcal{P}$

Conditions on a, b:

(A0) $a \in \mathbb{S}^d_+$ on E

(A1) $a \nabla p = 0$ and $\mathcal{G}p > 0$ on $M \cap \{p = 0\}$ for each $p \in \mathcal{P}$

(A2) $a \nabla q = 0$ and $\mathcal{G}q = 0$ on M for each $q \in \mathcal{Q}$

Some interpretations

(G1) $\nabla r(x)$, $r \in \mathcal{Q}$, are linearly independent for all $x \in M$

implies that M is submanifold of dimension $d - |\mathcal{Q}|$.

(G2) the ideal generated by $\mathcal{Q} \cup \{p\}$ is real for each $p \in \mathcal{P}$
(A1) $a \nabla p = 0$ and $\mathcal{G} p > 0$ on $M \cap \{p = 0\}$ for each $p \in \mathcal{P}$

together imply that $a \nabla p = h\, p$ on M for some vector of polynomials h (real Nullstellensatz).

Lemma 3.6.
Let $p \in \mathrm{Pol}(\mathbb{R}^d)$ be irreducible. The ideal generated by p is real if and only if p changes sign on $\mathbb{R}^d$: $p(x)p(y) < 0$ for some x, y.

Existence theorem

Theorem 3.7.
Suppose (G1)–(G2) and (A0)–(A2) hold. Then $\mathcal{G}$ is polynomial on E, and there exists a continuous $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ such that $a = \sigma\sigma^\top$ on E and SDE (3.1) admits an E-valued solution X for any initial law of X_0, which spends zero time at the boundary of E:

$$\int_0^t \mathbf{1}_{\{p(X_s)=0\}} ds = 0 \text{ for all } t \geq 0 \text{ and } p \in \mathcal{P}. \tag{3.5}$$

Boundary attainment

Theorem 3.8.
Let X be an E-valued solution to (3.1) satisfying (3.5). Let $p \in \mathcal{P}$ and h be a vector of polynomials such that $a \nabla p = h\, p$ on M.

1. *If there exists a neighborhood U of $E \cap \{p = 0\}$ such that*

$$2\,\mathcal{G}p - h^\top \nabla p \geq 0 \quad \text{on} \quad E \cap U \tag{3.6}$$

   then $p(X_t) > 0$ for all $t > 0$.

2. *Let $\overline{x} \in E \cap \{p = 0\}$ and assume*

$$\mathcal{G}p(\overline{x}) \geq 0 \qquad \text{and} \qquad 2\,\mathcal{G}p(\overline{x}) - h(\overline{x})^\top \nabla p(\overline{x}) < 0.$$

   Then there exists $\varepsilon > 0$ such that if $\|X_0 - \overline{x}\| < \varepsilon$ almost surely, then X hits $\{p = 0\}$ with positive probability.

Example

- Square-root diffusion on $E = \mathbb{R}_+$

$$dX_t = b\,dt + \sigma\sqrt{X_t}\,dB_t$$

- $a(x) = \sigma^2 x$, $b(x) = b$
- $\mathcal{P} = \{p\}$ with $p(x) = x$, $\mathcal{Q} = \emptyset$
- $a(x)p'(x) = \sigma^2 p(x)$, hence $h(x) = \sigma^2$ and

$$2\mathcal{G}p(x) - \sigma^2 p'(x) = 2b - \sigma^2$$

$\rightarrow$ Feller condition $2b \geq \sigma^2$ for boundary non-attainment

Outline

Motivation

- Build PJDs from basic PJDs
- Introduce nonlinearities into financial models
- Idea: start from simple building blocks (Gaussian process, Lévy process, ..), exponentiate or subordinate
- This works thanks to invariance of polynomial property!

Exponentiation of Polynomial Jump-Diffusion

- Let X_t be a PJD with generator $\mathcal{G}$ on $E \subseteq \mathbb{R}^d$
- Fix $n \in \mathbb{N}$, let $1 + N = \dim \mathrm{Pol}_n(E)$, and $(1, H(x))$ be a basis of $\mathrm{Pol}_n(E)$ where we write

$$H(x) = (h_1(x), \ldots, h_N(x))$$

- Let G be matrix representing $\mathcal{G}$ on $\mathrm{Pol}_n(E)$

Theorem 4.1.
The process $\bar{X}_t = H(X_t)$ is a PJD on $H(E) \subseteq \mathbb{R}^N$.

- Fact: the drift of $(1, \bar{X}_t)$ is $(1, \bar{X}_t)G\,dt$ (why?)
- We next characterize the generator $\bar{\mathcal{G}}$ of $\bar{X}_t$

Some Facts about $\mathrm{Pol}_m(H(E))$

- Fact: $H : E \to H(E)$ is injective: there exists $L : \mathbb{R}^N \to \mathbb{R}^d$ with $L_i \in \mathrm{Pol}_1(\mathbb{R}^N)$ such that

$$L_i(H(x)) = x_i, \quad x \in E$$

- Pullback ϕ^* defined by $\phi^* f = f \circ \phi$ for any function f

Lemma 4.2.

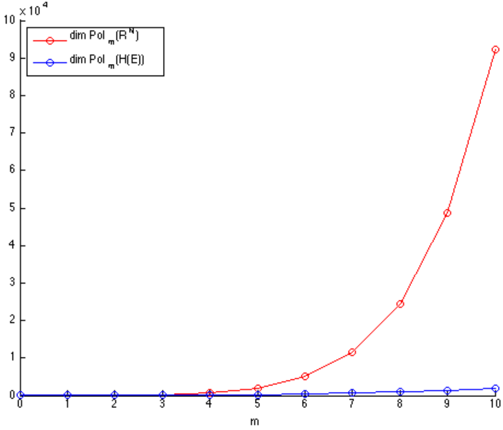
For every $m \in \mathbb{N}$ the pullback $H^* : \mathrm{Pol}_m(H(E)) \to \mathrm{Pol}_{mn}(E)$ is a linear isomorphism with inverse L^*.

Numerically very useful consequence:

$$\underbrace{\dim \mathrm{Pol}_m(H(E))}_{=\dim \mathrm{Pol}_{mn}(E)} \leq \binom{mn+d}{mn} < \binom{m+N}{m} = \dim \mathrm{Pol}_m(\mathbb{R}^N)$$

Dimension Reduction

Illustration for $d = 3$, $E = \mathbb{R}^3$, $n = 2$, such that $N = 9$,



$\dim \mathrm{Pol}_{10}(H(E)) = 1771$, $\dim \mathrm{Pol}_{10}(\mathbb{R}^N) \approx 10^5$, $\dim \mathrm{Pol}_{20}(\mathbb{R}^N) \approx 10^7$.

Action of $\bar{\mathcal{G}}$ on $\mathrm{Pol}_m(H(E))$

- Fact: the generator of $\bar{X}_t = H(X_t)$ is $\bar{\mathcal{G}} = L^* \mathcal{G} H^*$
- Fix $m \in \mathbb{N}$, let $1 + \bar{N} = \dim \mathrm{Pol}_{mn}(E)$ and

$$h_0(x) = 1, h_1(x), \ldots, h_N(x), h_{N+1}(x), \ldots, h_{\bar{N}}(x)$$

  be a basis of $\mathrm{Pol}_{mn}(E)$
- Gives basis $\bar{h}_i = L^* h_i$ on $\mathrm{Pol}_m(H(E))$
- Let $\bar{G}$ be matrix representing $\mathcal{G}$ on $\mathrm{Pol}_{mn}(E)$

Lemma 4.3.
The matrix representing $\bar{\mathcal{G}}$ of $\mathrm{Pol}_m(H(E))$ is $\bar{G}$.

Affine Property is not invariant under Exponentiation

- Consider the affine (square-root) diffusion

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t$$

- Augmented process $(X_t, Y_t) = (X_t, X_t^2)$ is not affine (why?):

$$\begin{aligned} dX_t &= \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t \\ dY_t &= ((2\kappa\theta + \sigma^2)X_t - 2\kappa Y_t)dt + 2\sigma\sqrt{X_t Y_t}dW_t \end{aligned}$$

- However (X_t, Y_t) is polynomial, consistent with Theorem 4.1

An Extension

As above:

- Let X_t be a PJD with generator $\mathcal{G}^X$ on $E \subseteq \mathbb{R}^d$
- Fix $n \in \mathbb{N}$, let $1 + N = \dim \mathrm{Pol}_n(E)$, and $(1, H(x))$ be a basis of $\mathrm{Pol}_n(E)$ where we write

$$H(x) = (h_1(x), \ldots, h_N(x))$$

New:

- Let Y_t be a semimartingale on $\mathbb{R}^e$ such that $Z_t = (X_t, Y_t)$ is a jump-diffusion with generator

$$\mathcal{G}^Z f(z) = \frac{1}{2}\mathrm{tr}(a^Z(x)\nabla^2 f(z)) + b^Z(x)^\top \nabla f(z) + \int_{\mathbb{R}^{d+e}} \Big(f(z+\zeta) - f(z) - \zeta^\top \nabla f(z) \Big) \, \nu^Z(x, d\zeta)$$

(Y_t has conditionally independent increments given X_t)

Decomposition of Characteristics

- According to decomposition $Z_t = (X_t, Y_t)$ we write

$$a^Z(x) = \begin{pmatrix} a^X(x) & a^{XY}(x) \\ a^{YX}(x) & a^Y(x) \end{pmatrix}, \quad b^Z(x) = \begin{pmatrix} b^X(x) \\ b^Y(x) \end{pmatrix},$$
$$\nu^Z(x, d\zeta) = \nu^Z(x, d\xi \times d\eta), \quad \zeta = (\xi, \eta)$$

- Constituents of polynomial operator $\mathcal{G}^X$ are

$$a^X(x), \quad b^X(x), \quad \nu^X(x, d\xi)$$

for marginal measure of $\nu^Z(x, d\xi \times d\eta)$ given by

$$\nu^X(x, A) = \int_{\mathbb{R}^{d+e}} \mathbf{1}_A(\xi)\nu^Z(x, d\xi \times d\eta)$$

Extension of Polynomial Jump-Diffusion

Theorem 4.4.
The following are equivalent:

1. *The process* $\overline{Z}_t = (H(X_t), Y_t)$ *is a PJD on* $H(E) \times \mathbb{R}^e$*;*
2. $a^Z(x)$, $b^Z(x)$, *and* $\nu^Z(x, d\xi)$ *satisfy*

$$
\begin{aligned}
b_j^Y(x) &\in \mathrm{Pol}_n(E), \\
a_{ij}^Y(x) + \int_{\mathbb{R}^{d+e}} \eta_i \eta_j \nu^Z(x, d\xi \times d\eta) &\in \mathrm{Pol}_{2n}(E), \\
a_{ij}^{XY}(x) + \int_{\mathbb{R}^{d+e}} \xi_i \eta_j \nu^Z(x, d\xi \times d\eta) &\in \mathrm{Pol}_{1+n}(E), \\
\int_{\mathbb{R}^{d+e}} \xi^{\boldsymbol{\alpha}} \eta^{\boldsymbol{\beta}} \nu^Z(x, d\xi \times d\eta) &\in \mathrm{Pol}_{|\boldsymbol{\alpha}|+n|\boldsymbol{\beta}|}(E),
\end{aligned}
$$

for all i, j *and all* $|\boldsymbol{\alpha}| + |\boldsymbol{\beta}| \geq 3$.

Sanity Check

- Theorem 4.4 is trivial for $n = 1$ (why?)

Some Facts about $\mathrm{Pol}_m(H(E) \times \mathbb{R}^e)$

- Fact: $\phi(x,y) = (H(x), y) : E \times \mathbb{R}^e \to H(E) \times \mathbb{R}^e$ is injective:

$$\psi(\phi(x,y)) = (x,y), \quad (x,y) \in E \times \mathbb{R}^e$$

for $\psi(\bar{x}, y) = (L(\bar{x}), y) : \mathbb{R}^N \times \mathbb{R}^e \to \mathbb{R}^d \times \mathbb{R}^e$

Lemma 4.5.
For every $m \in \mathbb{N}$ the pullback $\phi^* : \mathrm{Pol}_m(H(E) \times \mathbb{R}^e) \to V_m$ is a linear isomorphism with inverse ψ^* where

$$\begin{aligned} V_m &= \mathrm{span}\left\{ p(x) y^{\boldsymbol{\beta}} \colon p \in \mathrm{Pol}(E),\ \deg p + n|\boldsymbol{\beta}| \leq nm \right\} \\ &\subseteq \mathrm{Pol}_{mn}(E \times \mathbb{R}^e) \end{aligned}$$

- Fact: the generator of $\bar{Z}_t = (H(X_t), Y_t)$ is $\mathcal{G}^{\bar{Z}} = \psi^* \mathcal{G}^Z \phi^*$

Extension Theorem 4.4 cont'd

Theorem 4.4 (cont'd).

Property 1 or 2 is equivalent to

3. $\mathcal{G}^Z V_m \subseteq V_m$ *for all* $m \in \mathbb{N}$.

- This equivalence is illustrated by

$$\begin{array}{ccc} \mathrm{Pol}_m(H(E) \times \mathbb{R}^d) & \xrightarrow{\mathcal{G}^{\bar{Z}}} & \mathrm{Pol}_m(H(E) \times \mathbb{R}^d) \\ \varphi^* \downarrow \uparrow \psi^* & & \varphi^* \downarrow \uparrow \psi^* \\ V_m & \xrightarrow{\mathcal{G}^Z} & V_m \end{array}$$

- Numerically very useful consequence:

$$\underbrace{\dim \mathrm{Pol}_m(H(E) \times \mathbb{R}^e)}_{= \dim V_m \leq \dim \mathrm{Pol}_{mn}(E \times \mathbb{R}^e)} \leq \binom{mn + d + e}{mn} < \underbrace{\binom{m + N + e}{m}}_{= \dim \mathrm{Pol}_m(\mathbb{R}^N \times \mathbb{R}^e)}$$

Action of $\mathcal{G}^{\bar{Z}}$ on $\mathrm{Pol}_m(H(E) \times \mathbb{R}^e)$

- Assume $\bar{Z}_t$ is a PJD on $H(E) \times \mathbb{R}^e$
- Fix $m \in \mathbb{N}$, let $1 + \bar{N} = \dim \mathrm{Pol}_{mn}(E)$ and

$$h_0(x) = 1, h_1(x), \ldots, h_N(x), h_{N+1}(x), \ldots, h_{\bar{N}}(x)$$

  be a basis of $\mathrm{Pol}_{mn}(E)$
- Gives basis of V_m of the form

$$h_i^Z(x, y) = h_j(x) y^{\boldsymbol{\beta}}, \quad \deg h_j + n|\boldsymbol{\beta}| \leq mn$$

- Gives basis $h_i^{\bar{Z}} = \psi^* h_i^Z$ of $\mathrm{Pol}_m(H(E) \times \mathbb{R}^e)$

Lemma 4.6.
The matrix representing $\mathcal{G}^{\bar{Z}}$ on $\mathrm{Pol}_m(H(E) \times \mathbb{R}^e)$ equals G^Z, the matrix representing $\mathcal{G}^Z$ on V_m.

A Choice of Basis

- Assume $h_i^Z(x, y) = h_i(x)$ for $i = 0 \ldots \bar{N}$ ($\boldsymbol{\beta} = \mathbf{0}$)
- Then G^Z has the form

$$G^Z = \begin{pmatrix} G^{\bar{X}} & * \\ 0 & * \end{pmatrix}$$

- However, we need symbolic calculus to determine G^Z, i.e. $\mathcal{G}^Z h_i^Z(x, y)$ for $h_i^Z(x, y) = h_j(x) y^{\boldsymbol{\beta}}$ with $\boldsymbol{\beta} \neq \mathbf{0}$

Application of the Extension Theorem 4.4

Corollary 4.7.
Let $e = e' + e''$, $P(x) = (p_1(x), \ldots, p_{e'}(x))^\top$ *and* $Q(x) = (q_{ij}(x))$, $1 \leq i \leq e''$, $1 \leq j \leq d$, *with*

$$p_i(x) \in \mathrm{Pol}_n(E), \quad q_{ij}(x) \in \mathrm{Pol}_{n-1}(E).$$

Then

$$dY_t = \begin{pmatrix} P(X_t)\,dt \\ Q(X_{t-})\,dX_t \end{pmatrix}$$

satisfies conditions of Theorem 4.4, such that $Z_t = (H(X_t), Y_t)$ *is a PJD on* $H(E) \times \mathbb{R}^e$.

Co-Variation and Compensator

- Corollary 4.7 covers co-variation

$$d[X_i, X_j]_t = d(X_{i,t}X_{j,t}) - X_{i,t-}dX_{j,t} - X_{j,t-}dX_{i,t}$$

  and its compensator

$$d\langle X_i, X_j\rangle_t = \Gamma^X(x_i, x_j)(X_t)\,dt$$

  for the carré-du-champ operator $\Gamma^X(x_i, x_j) \in \mathrm{Pol}_2(E)$
- Application: variance swaps!

Outline

Polynomial Diffusions [Filipović and Larsson, 2016]

Invariance Properties: Exponentiation

Invariance Properties: Subordination

Markov Setup

- Let X_t be a PJD with generator $\mathcal{G}$ on $E \subseteq \mathbb{R}^d$
- **Assumption:** X_t is Markov with transition kernel $p_t(x, dy)$ on E, such that

$$\mathbb{E}[f(X_{s+t}) \mid \mathcal{F}_s] = \int_E f(y) p_t(X_s, dy)$$

- Let Z_t be an nondecreasing Lévy process (subordinator) with Lévy measure $\nu^Z(d\zeta)$ and drift $b^Z \geq 0$,

$$\mathcal{G}^Z f(z) = b^Z f'(z) + \int_E (f(z+\zeta) - f(z))\, \nu^Z(d\zeta)$$

  see [Sato, 1999, Thm 21.5].
- Fact: distribution $\mu^t(dz)$ of Z_t satisfies $\mu^{t+s} = \mu^t * \mu^s$:

$$\int f(z)\mu^{t+s}(dz) = \int f(z)(\mu^t {*} \mu^s)(dz) := \int \int f(x{+}y)\mu^t(dx)\mu^s(dy)$$

Bochner's Theorem

Theorem 5.1.
The time-changed $\widetilde{X}_t = X_{Z_t}$ is a PJD on E with transition kernel

$$\widetilde{p}_t(x, dy) = \mathbb{E}[p_{Z_t}(x, dy)] = \int_0^\infty p_z(x, dy)\mu^t(dz)$$

and generator on E given by

$$\widetilde{\mathcal{G}} f(x) = b^Z \mathcal{G} f(x) + \int_0^\infty \int_E (f(y) - f(x))\, p_\zeta(x, dy)\nu^Z(d\zeta)$$

Proof.
See [Sato, 1999, Thm 32.1], and also [Linetsky, 2007, Thm 6.2] for more details on characteristics. □

Action of $\widetilde{\mathcal{G}}$ on $\mathrm{Pol}_n(E)$

- Fix $n \in \mathbb{N}$, let $1 + N = \dim \mathrm{Pol}_n(E)$, and $(1, H(x))$ a basis of $\mathrm{Pol}_n(E)$ where

$$H(x) = (h_1(x), \ldots, h_N(x))$$

- Matrix representing $\mathcal{G}$ on $\mathrm{Pol}_n(E)$: $\mathcal{G}(1, H(x)) = (1, H(x))G$
- Matrix $\widetilde{G}$ representing $\widetilde{\mathcal{G}}$ on $\mathrm{Pol}_n(E)$ is then

$$\widetilde{G} = b^Z G + \int_0^\infty \left(\mathrm{e}^{G\zeta} - \mathrm{Id}_N\right) \nu^Z(d\zeta)$$

Affine Property is not invariant under Subordination

- OU process $dX_t = -\kappa X_t\, dt + \sigma\, dW_t$ is affine with normal t.k.

$$p_t(x, dy) \sim \mathcal{N}\left(\mathrm{e}^{-\kappa t}x, \frac{\sigma^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa t}\right)\right)$$

- Poisson subordinator Z_t with $\beta^Z = 0$ and $\nu^Z(d\zeta) = \delta_{\{1\}}(d\zeta)$
- Theorem 5.1: time-changed $\widetilde{X}_t = X_{Z_t}$ is polynomial
- But $\widetilde{X}_t$ is not affine if $\kappa \neq 0$:

$$\widetilde{\mathcal{G}}\mathrm{e}^{ux} = \int_E \left(\mathrm{e}^{uy} - \mathrm{e}^{ux}\right) p_1(x, dy) = \left(\mathrm{e}^{\left(\mathrm{e}^{-\kappa t}-1\right)ux + C(t)} - 1\right)\mathrm{e}^{ux}$$

for $C(t) = \frac{\sigma^2 u^2}{4\kappa}\left(1 - \mathrm{e}^{-2\kappa t}\right)$

Part III

Financial Modeling

Outline

Outline

Polynomial Asset Return Models

Polynomial Expansion Methods

Linear Diffusion Models

Goal

- Construct asset return models based on PJDs for ...
- option pricing ($\mathbb{P} = \mathbb{Q}$)
- portfolio choice
- portfolio risk management
- economic scenario generation
- ...

Polynomial Asset Return Framework

- Let X_t be a PJD with generator $\mathcal{G}$ on $E \subseteq \mathbb{R}^d$
- Let $d = d' + e$ and write $X_t = (X'_t, R_t)$
- e asset price processes $S_{1,t} \dots S_{e,t}$ with returns

$$\frac{dS_{i,t}}{S_{i,t-}} = r_t\, dt + dR_{i,t}$$

- Risk-free rate r_t
- Excess returns $dR_{i,t}$
- Assumption: $\Delta R_{i,t} > -1$ and in fact, write $\xi = (\xi', \xi^R)$,

$$\int_{\mathbb{R}^d} \log(1 + \xi_i^R)^{2k} \nu(x, d\xi) < \infty, \quad i = 1 \dots e$$

Risk-Neutral Dynamics

- Specifying the simple returns allows a simple characterization of risk-neutral dynamics ($\mathbb{P} = \mathbb{Q}$)

Lemma 6.1.
$\mathbb{P} = \mathbb{Q}$ is a risk-neutral measure if and only if R_t has zero drift, $b^R(x) = 0$, such that R_t is a local martingale.

Log Returns

- The logarithmic excess returns Y_t are defined by

$$S_{i,t} = S_{i,0}\,\mathrm{e}^{\int_0^t r_s\,ds + Y_{i,t}}$$

Lemma 6.2.
Stochastic exponential calculus implies

$$dY_{i,t} = \left(b_i^R(X_t) - \tfrac{1}{2}a_{ii}^R(X_t) - \int_{\mathbb{R}^d}\left(\xi_i^R - \log(1+\xi_i^R)\right)\nu(X_t, d\xi)\right)dt \\ + dM_{i,t}$$

where $M_{i,t}$ are local martingales with $d\langle M_i^c, M_j^c\rangle_t = a_{ij}^R(X_t)dt$ and $\Delta M_{i,t} = \log(1+\Delta R_{i,t})$. The jump measure of $Z_t = (X_t, Y_t)$ admits moments of all orders.

Polynomial Log Returns

- Does $Z_t = (X_t, Y_t)$ satisfy Extension Theorem 4.4 ?

Lemma 6.3.
Assume jump measure of X_t is of the mixed type

$$\nu(x, d\xi) = \nu_0(d\xi) + \sum_{i=1}^{d} x_i \nu_i(d\xi) + \sum_{i,j=1}^{d} x_i x_j \nu_{ij}(d\xi) + n(x, d\xi)$$

for signed measures $\nu_0(d\xi), \ldots, \nu_d(d\xi)$ and $\nu_{ij}(d\xi)$, $i, j = 1 \ldots d$, on $\mathbb{R}^d$ and transition kernel $n(x, d\xi)$ from $\mathbb{R}^d$ into $\mathbb{R}^{d'} \times \{0\}^e$. Then Z_t satisfies Extension Theorem 4.4 for $n = 2$, such that $\bar{Z}_t = (H(X_t), Y_t)$ is a PJD on $H(E) \times \mathbb{R}^e$.

Conditional Independent Returns

- If characteristics of $X_t = (X'_t, R_t)$ only depend on X'_t,

$$a(x) = a(x'), \quad b(x) = b(x'), \quad \nu(x, d\xi) = \nu(x', d\xi)$$

- Then $Z_t = (X'_t, Y_t)$ satisfies Extension Theorem 4.4 for $n = 2$, such that $\bar{Z}_t = (H(X'_t), Y_t)$ is a PJD on $H(E') \times \mathbb{R}^e$
- This reduces dimension!

Example: Factor Models

- Factor models assume excess return is

$$dR_{i,t} = \beta_i^\top dX_t^F + dX_{i,t}^{idio}, \quad i = 1 \dots e$$

where

  - X_t^F is d^F-dimensional factor process
  - β_i loading vector of ith excess return
  - $dX_{i,t}^{idio}$ idiosyncratic component of ith excess return

- Put in polynomial asset return framework as

$$X_t = (X_t^F, X_t^{idio}, X_t')$$

with $d = d^F + e + d'$, such that (X_t, R_t) is a PJD with conditionally independent returns dR_t given X_t

Towards Real-World Dynamics

- Assume we have specified PJD X_t under $\mathbb{Q}$ (a, b, ν)
- **Goal:** equivalent change of measure $\mathbb{P} \sim \mathbb{Q}$ such that $\mathbb{P}$-characteristics of X_t are

$$
\begin{aligned}
a^{\mathbb{P}}(x) &= a(x), \\
b^{\mathbb{P}}(x) &= b(x) + a(x)\phi(x) + \int_{\mathbb{R}^d} (\psi(\xi) - 1)\xi \, \nu(x, d\xi), \\
\nu^{\mathbb{P}}(x, d\xi) &= \psi(\xi)\nu(x, d\xi)
\end{aligned}
\tag{6.1}
$$

where

- $\phi(x) \in \mathbb{R}^d$ is market price of diffusion risk
- $\psi(\xi) > 0$ is market price of risk of the jump event of size ξ

Equivalent Change of Measure

Assumption: $\mathcal{E}(L)$ is a true martingale for

$$dL_t = \phi(X_t)^\top dX_t^c + \int_{\mathbb{R}^d} (\psi(\xi) - 1) \left(\mu^X(d\xi, dt) - \nu(X_t, d\xi)dt\right),$$

where X_t^c is the continuous local martingale part of X_t and $\mu^X(d\xi, dt)$ the integer-valued random measure associated to the jumps of X_t.

Lemma 6.4.
$\mathbb{P} \sim \mathbb{Q}$ *with Radon-Nikodym density process* $\mathcal{E}(L)$ *and* X_t *has* $\mathbb{P}$*-characteristics given by* (6.1)*.*

Polynomial Property under Real-World Measure

Corollary 6.5.

Assume jump measure of X_t is of the mixed type as in Lemma 6.3. Then X_t is a PJD under $\mathbb{P}$ if and only if

$$(a(x)\phi(x))_i + \int_{\mathbb{R}^d} (\psi(\xi) - 1)\xi_i \left(\sum_{k,l=1}^{d} x_k x_l \nu_{kl}(d\xi) + n(x, d\xi) \right) \in \mathrm{Pol}_1(E), \quad i = 1 \ldots d.$$

In this case, Z_t satisfies Extension Theorem 4.4 for $n = 2$, such that $\bar{Z}_t = (H(X_t), Y_t)$ is a PJD on $H(E) \times \mathbb{R}^e$ also under $\mathbb{P}$.

Pricing European Call Options

- Call option on S_i with strike K and maturity T has price

$$\mathbb{E}\left[e^{-\int_0^T r_s ds}(S_{i,T}-K)^+ \mid \mathcal{F}_0\right] = \mathbb{E}\left[\left(S_{i,0}\mathrm{e}^{Y_{i,T}} - Ke^{-\int_0^T r_s ds}\right)^+ \mid \mathcal{F}_0\right]$$

- **Assumption:** deterministic interest rates r_t
- Pricing boils down to computing expectation of the form

$$\mathbb{E}\left[F(Y_{i_T}) \mid \mathcal{F}_0\right]$$

for discounted payoff function $F(y_i) = (e^{y_i} - c)^+$

Pricing Path-Dependent Options

Barrier and fader options on S_i have payoff of the form $P_T f(S_{i,T})$ at maturity T where

- $f(S_{i,T})$ is some European style nominal payoff function
- P_T is path-dependent variable of the form

$$P_T = \begin{cases} 1_{\{\inf_{t\leq T} S_{i,t}\geq b\}}, & \text{barrier type} \\ \frac{1}{T}\int_0^T 1_{\{S_{i,t}\geq b\}}dt, & \text{fader type.} \end{cases}$$

  for some barrier b

Such options do not admit closed form prices and need to be numerically approximated.

Pricing Path-Dependent Options: Approximation

- Discretising the time interval $0 = t_0 < t_1 < \cdots < t_m = T$ leads to

$$P_T \approx \begin{cases} \prod_{j=1}^m 1_{\{S_{i,t_{j-1}} \geq b\}}, & \text{barrier type} \\ \sum_{j=1}^m 1_{\{S_{i,t_{j-1}} \geq b\}} \frac{t_j - t_{j-1}}{T}, & \text{fader type.} \end{cases}$$

- Pricing boils down to computing expectations of the form

$$\mathbb{E}\left[F(Y_{i,t_1}, \ldots, Y_{i,t_m}) \mid \mathcal{F}_{t_0}\right]$$

for discounted payoff function F

Outline

Generic Pricing Problem in Finance

Let X_t be a PJD with generator $\mathcal{G}$ on $E \subseteq \mathbb{R}^d$.

Pricing an (path-dependent) option boils down to compute conditional expectation

$$I_{t_0} = \mathbb{E}[F(\mathbf{X}) \mid \mathcal{F}_{t_0}]$$

for some

- time partition $0 \leq t_0 < t_1 < \cdots < t_m$
- (polynomial) projection $\mathbf{X} = P(X_{t_1}, \ldots, X_{t_m})$ on $\mathbf{E} = P(E^m)$
- discounted payoff function $F(\mathbf{x})$ with $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbf{E}$

The following method extends [Filipović et al., 2013]

Weighted L^2 Space

- Denote $g(d\mathbf{x})$ regular conditional distribution of $\mathbf{X}$ given $\mathcal{F}_{t_0}$
- Let $w(d\mathbf{x})$ be auxiliary probability kernel from $(\Omega, \mathcal{F}_{t_0})$ to $\mathbf{E}$ such that

$$g(d\mathbf{x}) \ll w(d\mathbf{x}) \quad \mathbb{P}\text{-a.s.} \tag{7.1}$$

  with likelihood ratio function denoted by $\ell(\mathbf{x})$ such that

$$g(d\mathbf{x}) = \ell(\mathbf{x}) w(d\mathbf{x}).$$

- Define $L^2_w = L^2_w(\mathbf{E})$ with norm given by

$$\|f\|^2_w = \int_{\mathbf{E}} f(\mathbf{x})^2 w(d\mathbf{x})$$

  and corresponding scalar product

$$\langle f, h \rangle_w = \int_{\mathbf{E}} f(\mathbf{x}) h(\mathbf{x}) w(d\mathbf{x}).$$

Orthogonal Polynomials

- Assumption: L^2_w contains all polynomials on $\mathbf{E}$,

$$\mathrm{Pol}(\mathbf{E}) \subset L^2_w \tag{7.2}$$

- Let $\{h_0(\mathbf{x}) = 1, h_1(\mathbf{x}), \dots\}$ be an orthonormal set of polynomials spanning the closure $\overline{\mathrm{Pol}(\mathbf{E})}$ in L^2_w.
- Assumption: the likelihood ratio function lies in L^2_w,

$$\ell(\mathbf{x}) \in L^2_w. \tag{7.3}$$

- As a consequence, its Fourier coefficients

$$\ell_k = \langle h_k, \ell \rangle_w = \int_{\mathbf{E}} h_k(\mathbf{x})\ell(\mathbf{x})w(d\mathbf{x}) = \mathbb{E}\left[h_k(\mathbf{X}) \mid \mathcal{F}_{t_0}\right]$$

  are in closed form by moment transform formula Theorem 1.7.

Projected Price

- Assumption: the discounted payoff function lies in L^2_w,

$$F(\mathbf{x}) \in L^2_w.$$

- Denote $\bar{F}$ the orthogonal projection of F onto $\overline{\mathrm{Pol}(\mathbf{E})}$ in L^2_w.
- Elementary functional analysis implies that the projected price

$$\bar{I}_{t_0} = \mathbb{E}[\bar{F}(\mathbf{X}) \mid \mathcal{F}_{t_0}]$$

equals

$$\bar{I}_{t_0} = \int_{\mathbf{E}} \bar{F}(\mathbf{x}) g(d\mathbf{x}) = \left\langle \bar{F}, \ell \right\rangle_w = \sum_{k \geq 0} F_k \ell_k \qquad (7.4)$$

with Fourier coefficients given by

$$F_k = \langle h_k, \bar{F} \rangle_w = \langle h_k, F \rangle_w = \int_{\mathbf{E}} h_k(\mathbf{x}) F(\mathbf{x}) w(d\mathbf{x}). \qquad (7.5)$$

Proxy Price

- Fact: $\bar{I}_{t_0} = I_{t_0}$ if the projection $\bar{F} = F$ in L^2_w.
- Note: $\bar{F} = F$ if $\overline{\mathrm{Pol}(\mathbf{E})} = L^2_w$, which depends on $w(d\mathbf{x})$.
- Proxy price: approximate the price by truncating series (7.4),

$$I_{t_0}^{(K)} = \sum_{k=0}^{K} F_k \ell_k,$$

for finite K, such that the pricing error is

$$\epsilon^{(K)} = I_{t_0} - I_{t_0}^{(K)} = \underbrace{I_{t_0} - \bar{I}_{t_0}}_{\text{projection bias}} + \underbrace{\bar{I}_{t_0} - I_{t_0}^{(K)}}_{\text{truncation error}}$$

with truncation error $\bar{I}_{t_0} - I_{t_0}^{(K)} \to 0$ for $K \to \infty$.

Proxy Measures

- Computation of $I_{t_0}^{(K)}$ as numerical integration over $\mathbf{E}$,

$$I_{t_0}^{(K)} = \sum_{k=0}^{K} \langle F, \ell_k h_k \rangle_w = \int_{\mathbf{E}} F(\mathbf{x}) g^{(K)}(d\mathbf{x}), \tag{7.6}$$

  for the proxy measure

$$g^{(K)}(d\mathbf{x}) = \left(\sum_{k=0}^{K} \ell_k h_k(\mathbf{x}) \right) w(d\mathbf{x}).$$

- Fact: $g^{(K)}(\mathbf{E}) = 1$ because $\langle h_k, h_0 = 1 \rangle_w = 0$ for $k \geq 1$
- But $g^{(K)}(d\mathbf{x})$ is only a signed measure in general.
- Fact: $g^{(K)}(d\mathbf{x}) \to g(d\mathbf{x})$ in a L^2_w-weak sense: for all $f \in L^2_w$

$$\lim_{K \to \infty} \int_{\mathbf{E}} f(\mathbf{x}) g^{(K)}(d\mathbf{x}) = \int_{\mathbf{E}} f(\mathbf{x}) g(d\mathbf{x}).$$

Choice of Auxiliary Kernel

- In specific cases: closed-form Fourier coefficients F_k, e.g. [Ackerer et al., 2015] for call options
- In general: numerical integration of (7.5), or equivalently (7.6)
- Depends on the choice of auxiliary kernel $w(d\mathbf{x})$
- How to choose $w(d\mathbf{x})$?
- Either good guessing, e.g. mixture of normals

  $$w(d\mathbf{x}) = (1-\lambda) n_{\mu_1,\sigma_1}(\mathbf{x}) d\mathbf{x} + \lambda n_{\mu_2,\sigma_2}(\mathbf{x}) d\mathbf{x}$$

  matching first two moments of $g(d\mathbf{x})$
- Or via simulation, see next..

Simulation Approach: Markov Setup

- Assume Markov setup: parametric family of probability measure $\{\mathbb{P}^\theta\}_{\theta\in\Theta}$ on $(\Omega, \mathcal{F})$ such that X_t is a PJD with generator $\mathcal{G}^\theta$ under any $\mathbb{P}^\theta$
- Denote $g^\theta(d\mathbf{x})$ the $\mathbb{P}^\theta$-regular conditional distribution of $\mathbf{X}$ given $\mathcal{F}_{t_0}$
- Fix baseline parameter $\theta_0 \in \Theta$, fix initial $x_0 \in E$, and set

$$w(d\mathbf{x}) = \mathbb{E}^{\theta_0}\left[\mathbf{X} \in d\mathbf{x} \mid X_{t_0} = x_0\right]$$

- Assume

$$g^\theta(d\mathbf{x}) \ll w(d\mathbf{x}) \quad \mathbb{P}^\theta\text{-a.s.}$$

  with likelihood ratio function $\ell^\theta(\mathbf{x}) \in L^2_w$ $\mathbb{P}^\theta$-a.s. for all $\theta \in \Theta$

Simulation Approach: Orthonormal Polynomials

Obtain ONB $\{h_0(\mathbf{x}) = 1, h_1(\mathbf{x}), \dots\}$ of $\overline{\mathrm{Pol}(\mathbf{E})}$ in L^2_w without numerical integration:

- Let $\tilde{h}_0(\mathbf{x}) = 1,\ \tilde{h}_1(\mathbf{x}), \dots$ be any basis of $\mathrm{Pol}(\mathbf{E})$.
- Moment transform formula Theorem 1.7: scalar products

$$\langle \tilde{h}_k, \tilde{h}_l \rangle_w = \mathbb{E}^{\theta_0}\left[\tilde{h}_k(\mathbf{X})\tilde{h}_l(\mathbf{X}) \mid X_{t_0} = x_0\right]$$

  in closed form
- Perform exact Gram–Schmidt orthonormalization gives orthonormal basis $\{h_0 = 1, h_1, \dots\}$ of $\overline{\mathrm{Pol}(\mathbf{E})}$ in L^2_w
- Yields closed-form Fourier coefficients

$$\ell^\theta_k = \langle h_k, \ell^\theta \rangle_w = \int_{\mathbf{E}} h_k(\mathbf{x})\ell^\theta(\mathbf{x}) w(d\mathbf{x}) = \mathbb{E}^\theta\left[h_k(\mathbf{X}) \mid \mathcal{F}_{t_0}\right]$$

Simulation Approach: Fourier Coefficients of $F(\mathbf{x})$

- Approximate $w(d\mathbf{x})$ by simulating $\mathbf{X}$ under $\mathbb{P}^{\theta_0}$ given $X_{t_0} = x_0$
- Estimate the Fourier coefficients

$$F_k = \mathbb{E}^{\theta_0}\left[h_k(\mathbf{X})F(\mathbf{X}) \mid X_{t_0} = x_0\right]$$

  by Monte-Carlo method
- Numerical efficiency: pre-compute and store simulation; using polynomial expansion above allows to compute proxies $I_{t_0}^{(K)}$ efficiently for various $\theta \in \Theta$ and thus calibrate θ to data

Alternative Approach: Edgeworth Expansion

- Use an Edgeworth expansion of the characteristic function

$$\mathbb{E}\left[\mathrm{e}^{zF(\mathbf{X})} \mid \mathcal{F}_{t_0}\right] = \mathrm{e}^{\sum_{n=1}^{\infty} C_n \frac{z^n}{n!}}$$
$$= \mathrm{e}^{C_1 z + C_2 \frac{z^2}{2}} \left(1 + C_3 \frac{z^3}{3!} + O(z^4)\right)$$

  where C_n refers to the nth cumulant of $g(d\mathbf{x})$
- Moment transform formula Theorem 1.7 gives closed-form expressions for C_n
- Apply standard Fourier inversion to infer I_{t_0}, e.g. [Carr and Madan, 1998] for at-the-money call options and [Fang and Oosterlee, 2008] for out-of-the-money call options

Outline

Specification Problem

- We have seen how to change measure and how to price options in a general polynomial asset return framework
- How shall we specify the polynomial factor process X_t?
- Example: every affine model falls into the polynomial framework
- Example: factor models with conditionally independent returns
- Here we focus on (novel) non-affine polynomial models

Linear Diffusion Models: Framework

- A novel flexible class of diffusion based models
- Assume $X_t = (X_t', R_t)$ is a linear diffusion (hence polynomial)

$$dX_t = (b + \beta X_t)dt + (C + X_{1,t}\Gamma_1 + \cdots + X_{d,t}\Gamma_d)dW_t$$

  for some m-dimensional standard Brownian motion W_t
- Nice (in contrast to affine models):
  - a priori no constraints on parameters
  - unique strong solution always exists in $\mathbb{R}^d$
- Allows for stochastic volatility and correlations $\langle X_i, X_j \rangle$

Alternative Volatility Representation

- Linear volatility

$$(C + X_{1,t}\Gamma_1 + \cdots + X_{d,t}\Gamma_d)dW_t$$

can alternatively be represented as

$$\sum_{k=1}^{m} (c_k + \gamma_k X_t)\, dW_{k,t}$$

where c_k are column vectors of C and ith column of γ_k is kth column of Γ_i: $\gamma_{k,i} = \Gamma_{i,k}$

Linear Diffusion Models: Cond. Independent Returns

Start with an observation:

Lemma 8.1.
Let X_t be a linear diffusion on E and $(1, H(x))$ a basis of $\mathrm{Pol}_n(E)$ for some $n \in \mathbb{N}$. Then $H(X_t)$ is a linear diffusion on $H(E)$.

Build up linear diffusion models with cond. independent returns:

1. Let X_t be d-dim. linear diffusion on $E \subseteq \mathbb{R}^d$
2. Specify excess returns

$$dR_t = Q(X_t)\, dW_t$$

   for $Q(x) \in \mathbb{R}^{e \times m}$ with $q_{ij} \in \mathrm{Pol}_n(E)$ for some $n \in \mathbb{N}$
3. Let $(1, H(x))$ be a basis of $\mathrm{Pol}_n(E)$. Then $(H(X_t), R_t)$ is a linear diffusion on $H(E) \times \mathbb{R}^e$

Examples for $d = e = 1$

- Revisit some examples for $d = e = 1$

$$\begin{aligned} dX_t &= (b + \beta X_t)dt + (c + \gamma X_t)\, dW_t^X \\ dR_t &= X_t\, dW_t^R \end{aligned}$$

  with leverage $d\langle W^X, W^R\rangle = \rho\, dt$
- extended Stein and Stein (1991): OU (affine)

$$dX_t = (b + \beta X_t)dt + c\, dW_t^X$$

- extended Hull–White (1987): log-normal (not affine)

$$dX_t = (b + \beta X_t)dt + \gamma X_t\, dW_t^X$$

  see also [Sepp, 2016]

Example for $d = e = 1$: Quadratic Volatility

- Quadratic volatility, [Filipović et al., 2016]:

$$
\begin{aligned}
dX_t &= (b + \beta X_t)dt + (c + \gamma X_t)\, dW_t^X \\
dR_t &= X_t^2\, dW_t^R
\end{aligned}
$$

  with leverage $d\langle W^X, W^R\rangle = \rho\, dt$
- Lemma 8.1: (X_t, X_t^2) is a linear diffusion on $\{(x, x^2)\}$
- Extension Theorem 4.4: (X_t, X_t^2, R_t) is a linear diffusion on $\{(x, x^2)\} \times \mathbb{R}$
- Lemma 6.3: (X_t, X_t^2, Y_t) is a linear diffusion on $\{(x, x^2)\} \times \mathbb{R}$ for log-excess return Y_t
- For OU ($\gamma = 0$): (X_t, X_t^2) is affine but (X_t, X_t^2, Y_t) is not affine if mean-reversion level is non-zero, $b \neq 0$ (why?)

Stochastic Volatility and Correlation Models

- Let $X_t = (X_t^\ell, X_t')$ be linear diffusion, $d = d^\ell + d'$
- Specify excess returns

$$dR_{i,t} = \sigma_{i,t}\, \ell_{i,t}^\top\, dW_t$$

for volatility process $\sigma_{i,t}$ and loadings process $\ell_{i,t}$

- Volatility process linear in X_t,

$$\sigma_{i,t} = k_i + \kappa_i^\top X_t,$$

for parameters $k_i \in \mathbb{R}$ and $\kappa_i \in \mathbb{R}^d$

- Loadings process linear in X_t^ℓ,

$$\ell_{i,t} = \lambda_i + \Lambda_i X_t^\ell,$$

for parameters $\lambda_i \in \mathbb{R}^m$ and $\Lambda_i \in \mathbb{R}^{m \times d^\ell}$, $m = \dim W_t$

Unit Sphere-Valued Diffusion

Denote $\mathcal{S} = \{\|x\| = 1\}$ the unit sphere in $\mathbb{R}^{d^\ell}$

Lemma 8.2.

Assume X_t^ℓ is autonomous with $X_0 \in \mathcal{S}$ and of the form

$$dX_t^\ell = \beta^\ell X_t^\ell \, dt + \sum_{k=1}^{m} \gamma_k^\ell X_t^\ell \, dW_{k,t}$$

for $\gamma_k^\ell \in \mathrm{Skew}_{d^\ell}$ and $\beta^\ell + \frac{1}{2}\sum_{k=1}^m \gamma_k^{\ell\top}\gamma_k^\ell \in \mathrm{Skew}_{d^\ell}$. Then $X_t^\ell \in \mathcal{S}$.

- **Assumption:** Conditions of Lemma 8.2 hold and

$$\|\lambda_i\| \leq 1, \quad \Lambda_i^\top \Lambda_i = (1 - \|\lambda_i\|) Id_{d^\ell}$$

- Then $\|\ell_{i,t}\| \equiv 1$

Obtain Stochastic Volatility and Correlation Model

As above: $(H(X_t), R_t)$ and $(H(X_t), Y_t)$ are linear diffusions, where $(1, H(x))$ is a basis of $\mathrm{Pol}_2(\mathcal{S} \times \mathbb{R}^{d'})$, with

- stochastic volatility of returns

$$\sqrt{\frac{d\langle R_i, R_i \rangle_t}{dt}} = |\sigma_{i,t}|$$

- stochastic instantaneous correlation between returns

$$\frac{d\langle R_i, R_j \rangle_t}{|\sigma_{i,t}||\sigma_{j,t}|\, dt} = \ell_{i,t}^\top \ell_{j,t} = \lambda_i^\top \lambda_j + X_t^{\ell\top} \Lambda_i^\top \Lambda_j X_t^\ell$$

Part IV

Stochastic Volatility Models

Outline

Jacobi Stochastic Volatility Model [Ackerer et al., 2015]

- Motivation and model specification
- Log-price density
- Density approximation and pricing algorithm
- Numerical aspects
- Exotic option pricing
- Conclusion

Quadratic Variance Swap Models [Filipović et al., 2016]

Outline

Outline

Stochastic volatility models

The volatility of stock price log-returns is stochastic

	Black-Scholes	Heston (affine SVJD)
volatility	constant	stochastic $\in \mathbb{R}_+$
calls and puts	closed-form	Fourier transform
exotic options	closed-form	...

Black-Scholes model $\subset$ Jacobi model $\rightarrow$ Heston model

- **stochastic volatility** on a parametrized **compact support**
- vanilla and exotic **option prices** have a **series representation**
- **fast and accurate** price approximations

Jacobi Stochastic Volatility model

Fix $0 \leq v_{min} < v_{max}$. Define the quadratic function

$$Q(v) = \frac{(v - v_{min})(v_{max} - v)}{(\sqrt{v_{max}} - \sqrt{v_{min}})^2} \leq v$$

Jacobi Model

Stock price dynamics $S_t = e^{X_t}$ given by

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)\,dt + \sigma\sqrt{Q(V_t)}\,dW_{1t} \\ dX_t &= (r - V_t/2)\,dt + \rho\sqrt{Q(V_t)}\,dW_{1t} + \sqrt{V_t - \rho^2 Q(V_t)}\,dW_{2t} \end{aligned} \tag{9.1}$$

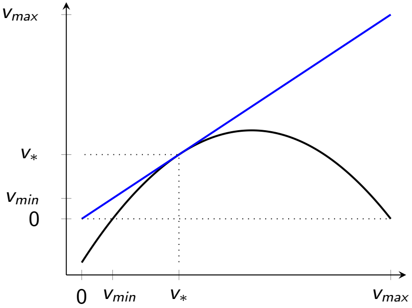
for $\kappa, \sigma > 0$, $\theta \in [v_{min}, v_{max}]$, interest rate r, $\rho \in [-1, 1]$, and 2-dimensional BM $W = (W_1, W_2)$

Remark: $\mathrm{e}^{-rt} S_t = \mathrm{e}^{-rt + X_t}$ is a martingale

Some properties

The function $Q(v)$

$v \geq Q(v)$, $v = Q(v)$ if and only if $v = \sqrt{v_{min} v_{max}}$, and $Q(v) \geq 0$ for all $v \in [v_{min}, v_{max}]$



Instantaneous variance

$d\langle X, X\rangle_t = V_t \in [v_{min}, v_{max}]$ is a **Jacobi process**

Some properties (cont.)

Instantaneous correlation

$$\frac{d\langle V, X\rangle_t}{\sqrt{d\langle V, V\rangle_t}\sqrt{d\langle X, X\rangle_t}} = \rho\sqrt{Q(V_t)/V_t}$$

Polynomial model

(V_t, X_t) is a polynomial diffusion – **efficient calculation of moments**

Black-Scholes model nested

Take $v_{min} = v_{max} = \sigma^2_{\mathrm{BS}}$

Heston model as a limit case

If $v_{min} \to 0$ and $v_{max} \to \infty$ then (V_t, X_t) converges weakly in the path space to the Heston model

Implied volatility

Bounded implied volatility

Option with positive BS gamma ($\Leftrightarrow$ convex payoff for Europ.)

$$\sqrt{v_{min}} \leq \sigma_{\mathrm{IV}} \leq \sqrt{v_{max}}$$

$\Rightarrow$ **Forward start option** σ_{IV} does not explode
(Jacquier and Roome 2015)

Outline

Log-price density

We define

$$C_T = \int_0^T \left(V_t - \rho^2 Q(V_t)\right) dt$$

Theorem 9.1.
Let $\epsilon < 1/(2v_{max}T)$. If $C_T > 0$ then the distribution of X_T admits a density $g_T(x)$ on $\mathbb{R}$ that satisfies

$$\int_{\mathbb{R}} \mathrm{e}^{\epsilon x^2} g_T(x)\, dx < \infty \tag{9.2}$$

If

$$\mathbb{E}\left[{C_T}^{-1/2}\right] < \infty \tag{9.3}$$

then $g_T(x)$ and $\mathrm{e}^{\epsilon x^2} g_T(x)$ are uniformly bounded and continuous on $\mathbb{R}$. A sufficient condition for (9.3) is $v_{min} > 0$ and $\rho^2 < 1$.

Remark: The Heston model does not satisfy (9.2) for any $\epsilon > 0$

A crucial corollary

Corollary 9.2.

Assume (9.3) *holds. Then* $\ell(x) = \frac{g_T(x)}{w(x)} \in L^2_w$*, where*

$$L^2_w := \left\{ h : \int_{\mathbb{R}} |h(x)|^2 w(x)\, dx \right\}$$

and $w(x)$ *is any Gaussian density with variance* σ^2_w *satisfying*

$$\sigma^2_w > \frac{v_{max}\, T}{2} \tag{9.4}$$

- (Filipovic, Mayerhofer, Schneider 2013) For the Heston model we have that $\ell(x) = \frac{g_T(x)}{w(x)} \in L^2_w$, where $w(x)$ **is a (bilateral) Gamma density**

Outline

Weighted L^2–space

The weight function

$$w(x) = \text{Gaussian density with mean } \mu_w \text{ and variance } \sigma_w^2$$

The weighted Hilbert space

$$L^2_w = \left\{ f(x) \mid \|f\|^2_w = \int_{\mathbb{R}} f(x)^2\, w(x)dx < \infty \right\}$$

which is a Hilbert space with scalar product

$$\langle f, g \rangle_w = \int_{\mathbb{R}} f(x)g(x)\, w(x)dx$$

Orthonormal basis – Generalized Hermite polynomials

$$H_n(x) = \frac{1}{\sqrt{n!}} \mathcal{H}_n \left(\frac{x - \mu_w}{\sigma_w} \right)$$

where $\mathcal{H}_n(x)$ are the standard Hermite polynomials

Price approximation

Pricing problem

Assume that X_T has a density $g_T(x)$

$$\pi_f = \mathbb{E}[f(X_T)] = \int_{\mathbb{R}} f(x) g_T(x) dx$$

Price series expansion

Suppose $\ell(x) = g_T(x)/w(x) \in L^2_w$ and $f(x) \in L^2_w$. Then

$$\pi_f = \langle f, \ell \rangle_w = \sum_{n \geq 0} f_n \ell_n \tag{9.5}$$

for the **Fourier coefficients and Hermite moments**

$$f_n = \langle f, H_n \rangle_w, \quad \ell_n = \langle \ell, H_n \rangle_w = \int_{\mathbb{R}} H_n(x) g_T(x)\, dx$$

Price approximation

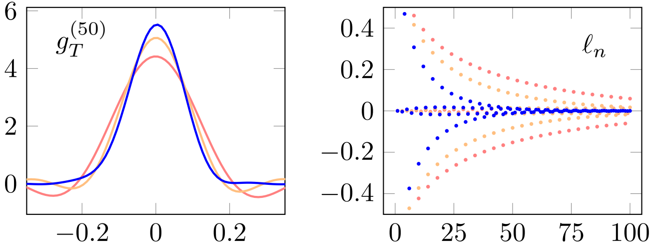
$$\pi_f \approx \pi_f^{(N)} = \sum_{n=0}^{N} f_n \ell_n = \sum_{n=0}^{N} \langle f, \ell_n H_n \rangle_w = \int_{\mathbb{R}} f(x) g_T^{(N)}(x)\, dx \tag{9.6}$$

Density approximation

"Gram-Charlier A expansion"

$$g_T^{(N)}(x) = w(x) \sum_{n=0}^{N} \ell_n H_n(x)$$

Gram-Charlier expansions of prices: Jarrow and Rudd (1982), Corrado and Su (1996) ... Drimus, Necula, and Farkas (2013), Heston and Rossi (2015)...



$\sigma_w \in \{1\nu, 1.5\nu, 2\nu\}$ with $\nu = \sqrt{v_{max} T/2} + \epsilon$, $T = 1/12$, $X_0 = 0$, $\kappa = 0.5$, $\theta = V_0 = (0.25)^2$, $\sigma = 0.25$, $v_{min} = (0.10)^2$, $\rho = -0.5$, and $v_{max} = 1$

European calls and puts - Fourier coefficients

Theorem 9.3.
Consider the discounted payoff function for a call option with log strike k,

$$f(x) = \mathrm{e}^{-rT}\left(\mathrm{e}^{x} - \mathrm{e}^{k}\right)^{+}$$

Its Fourier coefficients f_n for $n \geq 1$ are given by

$$f_n = \mathrm{e}^{-rT+\mu_w}\frac{1}{\sqrt{n!}}\sigma_w I_{n-1}\left(\frac{k-\mu_w}{\sigma_w};\sigma_w\right)$$

The functions $I_n(\mu;\nu)$ are defined recursively by

$$I_0(\mu;\nu) = \mathrm{e}^{\frac{\nu^2}{2}}\Phi(\nu-\mu);$$
$$I_n(\mu;\nu) = \mathcal{H}_{n-1}(\mu)\mathrm{e}^{\nu\mu}\phi(\mu) + \nu I_{n-1}(\mu;\nu), \quad n \geq 1$$

where $\mathcal{H}_n(x)$ are the standard Hermite polynomials, $\Phi(x)$ denotes the standard Gaussian distribution function, and $\phi(x)$ its density

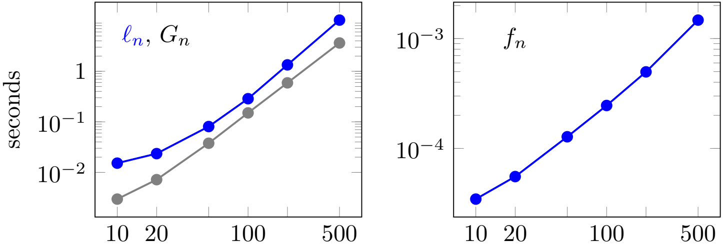
Outline

Computational cost

Theorem 9.4.
The coefficients ℓ_n are given by

$$\ell_n = [h_1(V_0, X_0), \dots, h_M(V_0, X_0)] \, \mathrm{e}^{TG_n} \, \mathbf{e}_{\pi(0,n)}, \quad 0 \leq n \leq N$$

where $\mathbf{e}_i$ is the i–th standard basis vector in $\mathbb{R}^M$ and $h_0, \dots, h_M$ is a basis of polynomials. G_n is the $(M \times M)$–matrix representing the infinitesimal generator of (V_t, X_t) on Pol_N – sparse matrix



Example: Call option pricing

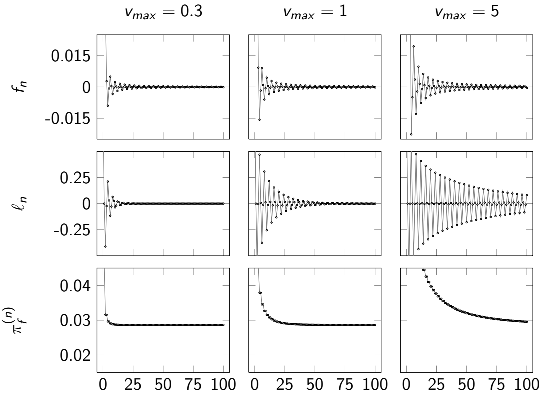


Figure: The Fourier coefficients (first row), the Hermite coefficients (second row), and the price expansion (third row) as a function of the order n. The parameters values are $T = 1/12$, $X_0 = k = 0$, $\kappa = 0.5$, $\theta = V_0 = (0.25)^2$, $\sigma = 0.25$, $v_{min} = (0.10)^2$, $\rho = -0.5$, and $v_{max} \in \{0.3, 1, 5\}$

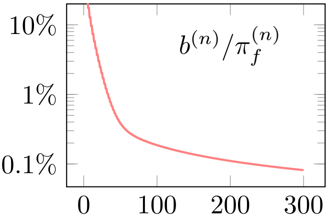
Error bounds

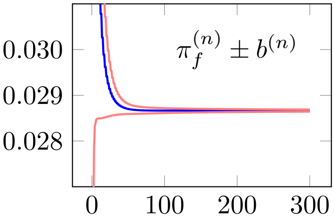
Pricing error $\pi_f - \pi_f^{(N)} = \epsilon^{(N)}$

$$\left|\epsilon^{(N)}\right| = \left|\sum_{n>N} f_n \ell_n\right| \leq \sqrt{\left(\sum_{n>N} f_n^2\right)\left(\sum_{n>N} \ell_n^2\right)}$$

Type of bounds

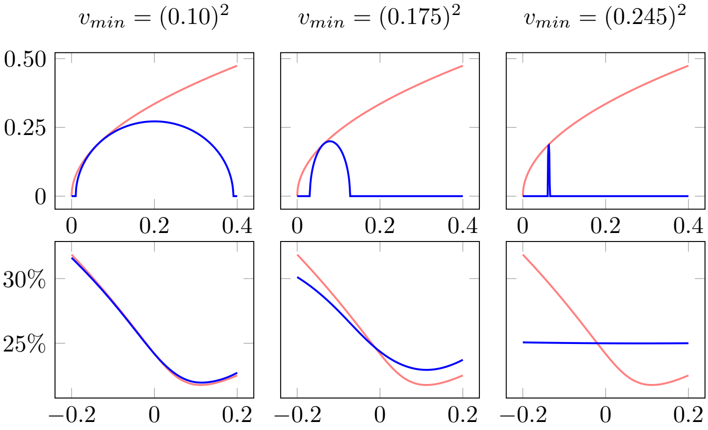
1. Analytic: $\ell_n^2, f_n^2 \leq C \times n^{-k}$ for some $k > 1$ and $C > 0$
2. Numeric: $\sum_{n>N} \ell_n^2 = \|\ell\|_w^2 - \sum_{n=0}^{N} \ell_n^2$





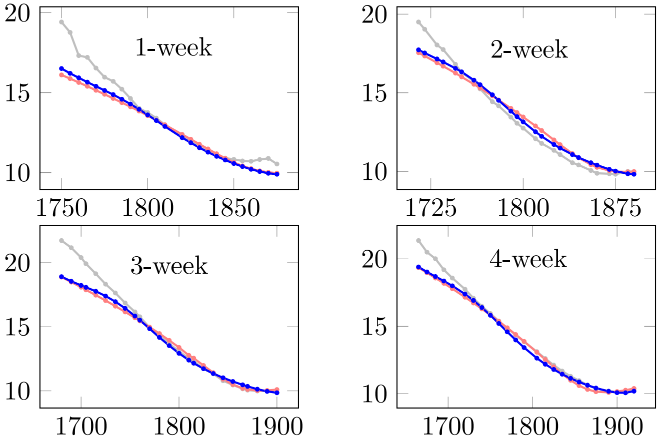
Volatility smiles - Call option

Fix $\theta = \sqrt{v_{min} v_{max}} = v_*$ and scale up v_{min}



Diffusion function $\sigma\sqrt{Q(v)}$ (1st row) and smile (2nd row)

SPX implied volatility calibration



	$\sqrt{\theta}$	κ	σ	ρ	$\sqrt{V_0}$	$\sqrt{V_{min}}$	$\sqrt{V_{max}}$	RMSE
Jacobi	0.3660	0.7507	1.0072	-0.6057	0.1178	0.0499	0.4476	0.8461
Heston	0.3655	0.7498	0.8573	-0.6047	0.1178			0.9447

Outline

Key corollary revisited

Log-returns density

$$Y_{t_i} = X_{t_i} - X_{t_{i-1}}$$

for $0 \leq t_0 < t_1 < t_2 < \cdots < t_n$, $Y = (Y_{t_i})$ has a density $g_{t_0,\ldots,t_n}(y)$

Weighting with Gaussians

Define $w(y) = \prod_{i=1}^{n} w_i(y_i)$ where $w_i(y_i)$ is a Gaussian density with variance $\sigma_{w_i}^2$, then $\frac{g_{t_0,\ldots,t_n}(y)}{w(y)} \in L_w^2$ if

$$\sigma_{w_i}^2 > \frac{v_{max}(t_i - t_{i-1})}{2}$$

Forward start call option

Payoff function $e^{-rt_2}(S_{t_2} - e^k S_{t_1})^+$ with $0 = t_0 < t_1 < t_2$

$$\tilde{f}(y_1, y_2) = e^{-rt_2}(e^{X_0+y_1+y_2} - e^{k+X_0+y_1})^+$$

Fourier coefficients

$$\begin{aligned}\tilde{f}_{m_1,m_2} &= \int_{\mathbb{R}^2} \tilde{f}(y) H_{m_1}(y_1) H_{m_2}(y_2) w(y) dy \\ &= f_{m_2}^{(0,k)} \frac{\sigma_w^{m_1}}{\sqrt{m_1!}} \mathrm{e}^{X_0 - rT + \mu_{w_1} + \sigma_{w_1}^2/2}\end{aligned}$$

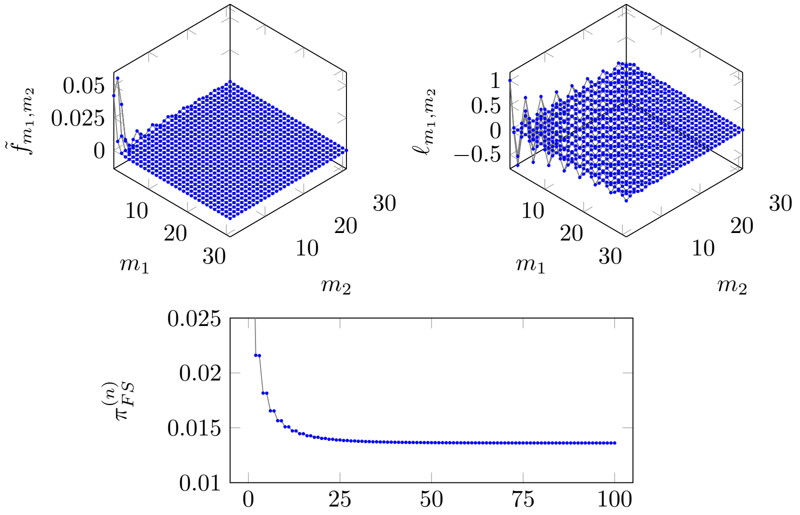
Hermite moments

$$\begin{aligned}\ell_{m_1,m_2} &= \mathbb{E}[H_{m_1}(Y_{t_1}) H_{m_2}(Y_{t_2})] \\ &= \mathbb{E}\left[H_{m_1}(Y_{t_1})\, \mathbb{E}\left[H_{m_2}(Y_{t_2}) \mid \mathcal{F}_{t_1}\right]\right]\end{aligned}$$

Price approximation

$$\pi_{FS} = \sum_{m_1,m_2 \geq 0} \tilde{f}_{m_1,m_2} \ell_{m_1,m_2} \approx \sum_{m_1,m_2=0}^{m_1+m_2 \leq N} \tilde{f}_{m_1,m_2} \ell_{m_1,m_2} =: \pi_{FS}^{(N)}$$

Forward start call option (cont.)



$t = 1/12$, $T - t = 1/52$, and $k = 0$

Forward start options on the return

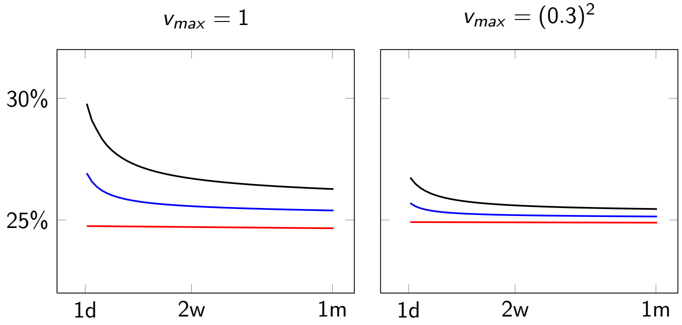


Figure: Implied volatility of a forward start option on the return with maturity $t + T$, and strikes $k = -0.10$ (black line), $k = -0.05$ (blue line), and $k = 0$ (red line) are displayed as a function of maturity T. Here $t = 1/12$, $X_0 = 0$, $\kappa = 0.5$, $V_0 = \theta = (0.25)^2$, $\sigma = 0.25$, $v_{min} = 10^{-4}$, and $\rho = -0.5$

Outline

Conclusion

- new stochastic volatility model, V_t is a **Jacobi process**
- **option price series** representation in weighted L^2_w space
  - Hermite moments (**polynomial model**)
  - Fourier coefficient (**recursive formulas**)
- computationally fast, empirically $\gtrsim$ Heston model, pricing **error bounds**
- methodology applies to **exotic option pricing**

Outline

Variance Swaps

- Underlying price process (e.g. S&P 500 index)

$$\frac{dS_t}{S_{t-}} = r_t\,dt + \sigma_t\,dW_t^* + \int_{\mathbb{R}} (\mathrm{e}^x - 1)\,(\mu(dt, dx) - \nu_t(dx)dt)$$

- The annualized realized variance over $[t, T]$ equals

$$\mathrm{RV}(t, T) = \frac{1}{T - t}\left(\int_t^T \sigma_s^2\,ds + \int_t^T \int_{\mathbb{R}} x^2\,\mu(ds, dx)\right)$$

- A variance swap initiated at t with maturity T pays

$$\mathrm{RV}(t, T) - \mathrm{VS}(t, T)$$

- $\mathrm{VS}(t, T)$: variance swap rate fixed at t

Forward Variance

- Fair valuation:

$$\mathrm{VS}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{RV}(t,T)\right]$$

- Define the spot variance

$$v_t = \sigma_t^2 + \int_{\mathbb{R}} x^2\,\nu_t(dx)$$

- Define the forward variance

$$f(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[v_T\right]$$

- Then the variance swap rate equals

$$\mathrm{VS}(t,T) = \frac{1}{T-t}\int_t^T f(t,s)\,ds$$

Quadratic Variance Swap Model

- Bivariate **PP** diffusion state process

$$dX_{1t} = (b_1 + \beta_{11}\, X_{1t} + \beta_{12}\, X_{2t})\, dt + \sqrt{a_1 + \alpha_1 X_{1t} + A_1 X_{1t}^2}\, dW_{1t}^*$$

$$dX_{2t} = (b_2 + \beta_{22}\, X_{2t})\, dt + \sqrt{a_2 + \alpha_2 X_{2t} + A_2 X_{2t}^2}\, dW_{2t}^*$$

- Spot variance is specified by

$$v_t = \phi_0 + \psi_0 X_{1t} + \pi_0 X_{1t}^2$$

Explicit Forward Variance Curve

- $f(t,T) = \phi(T-t) + \psi(T-t)^\top X_t + X_t^\top \pi(T-t)\, X_t$
- Linear ODEs for ϕ, ψ, and π can be vectorized by setting

$$q(\tau) = \begin{pmatrix} \phi(\tau) & \psi_1(\tau) & \psi_2(\tau) & \pi_{11}(\tau) & \pi_{12}(\tau) & \pi_{22}(\tau) \end{pmatrix}^\top$$

- The linear system then reads

$$\frac{dq(\tau)}{d\tau} = \begin{pmatrix} 0 & b_1 & b_2 & a_1 & 0 & a_2 \\ 0 & \beta_{11} & \beta_{12} & 2b_1 + \alpha_1 & 2b_2 & 0 \\ 0 & 0 & \beta_{22} & 0 & 2b_1 & 2b_2 + \alpha_2 \\ 0 & 0 & 0 & 2\beta_{11} + A_1 & 2\beta_{12} & 0 \\ 0 & 0 & 0 & 0 & \beta_{11} + \beta_{22} & \beta_{12} \\ 0 & 0 & 0 & 0 & 0 & 2\beta_{22} + A_2 \end{pmatrix} q(\tau)$$

$$q(0) = \begin{pmatrix} \phi_0 & \psi_0 & 0 & \pi_0 & 0 & 0 \end{pmatrix}^\top .$$

Data

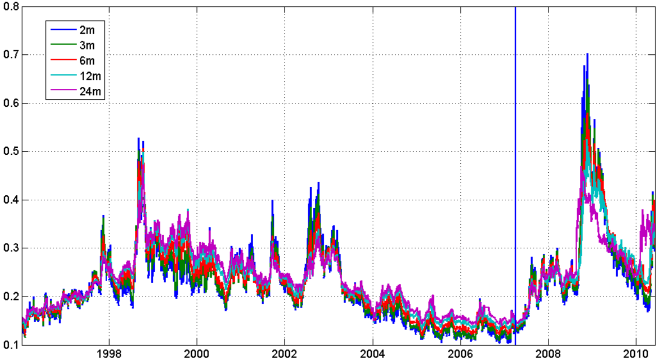


Figure: Variance swap rates $\sqrt{\mathrm{VS}(t, t+\tau)}$ on the S&P 500 index from Jan 4, 1996 to Jun 7, 2010. Source: Bloomberg

- In-sample (pre-crisis): Jan 4, 1996 to Apr 2, 2007
- Out-of-sample: Apr 3, 2007 to Jun 7, 2010

Estimation Results: Bivariate Model

- Best fit for

$$dX_{1t} = (\ell + (\lambda + \beta_{11})\, X_{1t} + \beta_{12}\, X_{2t})\, dt + \sqrt{1 + A_1 X_{1t}^2}\, dW_{1t}$$

$$dX_{2t} = (b_2 + \beta_{22}\, X_{2t})\, dt + \sqrt{X_{2t} + A_2 X_{2t}^2}\, dW_{2t}$$

- Recall spot variance $v_t = \phi_0 + \psi_0 X_{1t} + \pi_0 X_{1t}^2$

β_{11}	β_{12}	b_2	β_{22}	A_1	A_2
-5.1720	4.2324	0.1824	-0.2483	3.3895	0.0985
(0.0903)	(0.2346)	(0.0322)	(0.0021)	(0.1206)	(0.0001)

ϕ_0	ψ_0	π_0	MPR	ℓ	λ
0.0175	0.0130	0.0283		-0.1770	-0.0021
(0.0002)	(0.0008)	(0.0004)		(0.0190)	(0.0868)

Table: Estimated parameters (robust standard errors into parentheses)

In-Sample Analysis: Filtered Factors

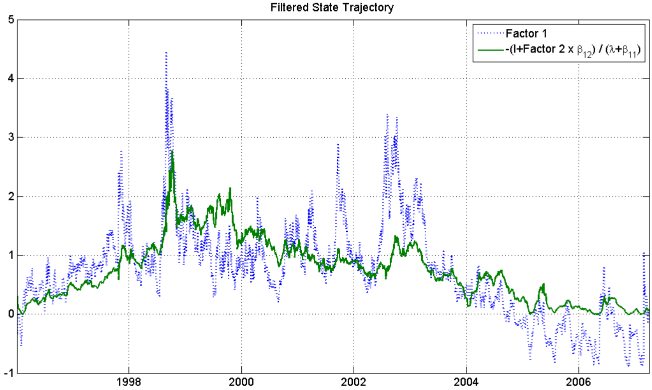


Figure: Filtered factors X_1 vs. stochastic mean reversion level $\frac{\ell+\beta_{12}X_2}{-(\lambda+\beta_{11})}$.

Out-of-Sample Analysis: Predicted VS

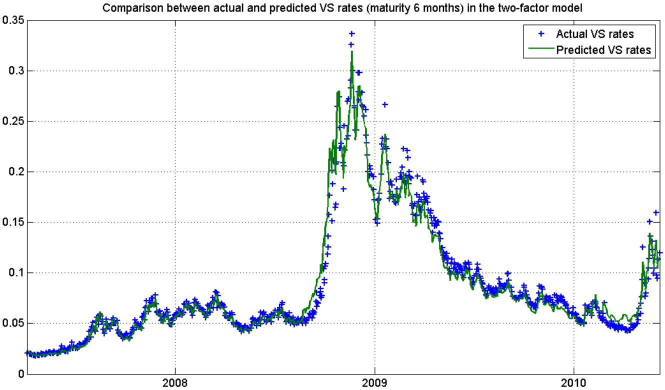


Figure: Out-of-sample predicted variance swap rates vs. data for 6 months maturity. The quadratic diffusion model captures extreme movements and spikes.

Part V

Interest Rate and Credit Risk Models

Outline

Linear Credit Risk Model [Ackerer and Filipović, 2015]

- The linear framework
- Bonds and credit default swap pricing
- Empirical results
- CDS option price approximation

Linear-Rational Term Structure Models [Filipović et al., 2014]

- The linear-rational framework
- The Linear-Rational Square-Root (LRSQ) model
- Empirical analysis

Outline

Motivation

Dynamic credit risk models

- Security pricing (bonds and CDSs $\sim$\$XX billions daily vol.)
- Risk management (portfolio, XVA, Basel III, IFRS 9)

Reduced form models (v.s. structural models)

- Simplicity: exogenous defaults driven by market factors
  (Jarrow and Turnbull 1995, Lando 1998, Elliott, Jeanblanc, and Yor 2000)
- Affine default intensity models (Duffie and Singleton 1999, ...)
- Limitations: high dimension, non-vanilla pricing problems

This paper

- New flexible class of (linear) credit risk models
  (related to Gabaix 2009, Filipović, Trolle, and Larsson 2016)
- Tractable: explicit bond and CDS pricing formulas
- Versatile: simple price approximation with moments

Outline

Cox construction of default time

- Default intensity process λ_t driven by some factors X_t

$$\lambda_t = f(X_t) \geq 0$$

  $\approx$ probability of default over a small period dt is $\lambda_t dt$

- Default time τ is defined by

$$\tau = \inf\left\{ t \geq 0 : \int_0^t \lambda_s ds \geq E \right\}$$

  where E is an exponential random variable with mean 1

- Conditional survival probability

$$\mathbb{P}\left[\tau > t \mid (X_s)_{0\leq s\leq t}\right] = \exp\left(-\int_0^t f(X_s) ds \right)$$

  positive non-increasing function of t starting at 1

Alternative construction

- Let S_t be a positive non-increasing process starting at 1
- Default time τ is defined by

$$\tau = \inf\{t \geq 0 : S_t \leq U\}$$

  where U is a uniform variable on $(0, 1)$

- When S_t is driven by some factors X_t we obtain

$$\mathbb{P}[\tau > t \mid (X_s)_{0 \leq s \leq t}] = S_t$$

- Two filtrations
  - $\mathcal{F}_t$ = all the information about X_t up to time t
  - $\mathcal{G}_t = \mathcal{F}_t$ and whether default occurred by time t

The linear framework

Specification

Model directly the survival process S_t! Linear drift

$$dS_t = -\gamma^\top X_t dt - dM_t^S$$
$$dX_t = (\beta S_t + BX_t)dt + dM_t^X$$

$\gamma, \beta \in \mathbb{R}^m$, $B \in \mathbb{R}^{m\times m}$, $\mathcal{F}_t$-martingales $M_t^S \in \mathbb{R}$ and $M_t^X \in \mathbb{R}^m$

Conditions to verify

- non-increasing process: $-\gamma^\top X_t dt - dM_t^S \leq 0$
- positive process: $S_t > 0$

When $M_t^S = 0$ the default intensity is given by

$$\lambda_t = \frac{\gamma^\top X_t}{S_t}$$

One-factor model

Set $m = 1$, $M_t^S = 0$, and M_t^X such that $X_t \in [0, S_t]$

$$
\begin{aligned}
dS_t &= -\gamma X_t dt \\
dX_t &= (\beta S_t + BX_t)dt + \sigma\sqrt{X_t(S_t - X_t)}dW_t
\end{aligned}
$$

Conditions are verified by construction for any $\gamma > 0$

- $dS_t \leq 0$ since $X_t \geq 0$
- $S_t \geq e^{-\gamma t} > 0$ since $\lambda_t = \frac{\gamma X_t}{S_t} \in [0, \gamma]$

Lemma

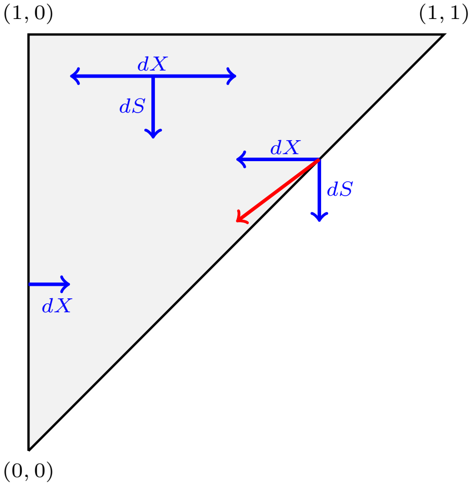
The process (S_t, X_t) is well-defined if and only if

$$\beta \geq 0 \quad \text{and} \quad (\gamma + B + \beta) \leq 0$$

One-factor model II

Inward pointing condition

The state space of the process (S_t, X_t) is of the form



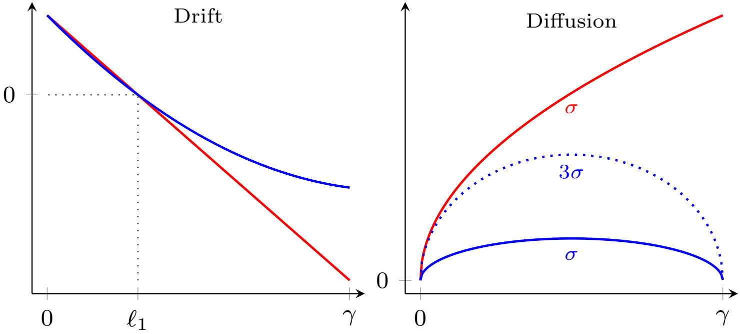
One-factor model III

The default intensity has an autonomous dynamics

$$d\lambda_t = (\ell_1 - \lambda_t)(\lambda_t - \ell_2)\, dt + \sigma\sqrt{\lambda_t(\gamma - \lambda_t)}\, dW_t$$

One-factor affine default intensity model

$$d\lambda_t = \ell_2(\lambda_t - \ell_1)\, dt + \sigma\sqrt{\lambda_t}\, dW_t$$



The linear hypercube model

Polynomial diffusion (Filipović and Larsson 2016) with state space

$$E = \left\{(s, x) \in \mathbb{R}^{1+m} : s \in (0,1] \text{ and } x \in [0,s]^m\right\}$$

The process dynamics rewrites

$$\begin{aligned} dS_t &= -\gamma^\top X_t \, dt \\ dX_t &= (\beta S_t + B X_t)\, dt + \Sigma(S_t, X_t)\, dW_t \end{aligned}$$

with $\Sigma(s,x) = \operatorname{diag}\left(\sigma_1\sqrt{x_1(s-x_1)}, \ldots, \sigma_m\sqrt{x_m(s-x_m)}\right)$

The default intensity satisfies $0 \le \lambda_t \le \gamma^\top \mathbf{1}$

Lemma

The process (X_t, S_t) is well defined if and only if

$$\beta_i - \sum_{j \neq i} B_{ij}^- \ge 0 \quad \text{and} \quad \gamma_i + B_{ii} + \beta_i + \sum_{j\neq i} (\gamma_j + B_{ij})^+ \le 0$$

Outline

Defaultable bond

Assume henceforth constant risk-free interest rate r
Security B pays one if $\tau > T$ and zero otherwise

$$\begin{aligned} B^Z(t,T) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\mathrm{e}^{-r(T-t)}\mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\mathrm{e}^{-r(T-t)}\frac{S_T}{S_t} \,|\, \mathcal{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\frac{\mathrm{e}^{-r(T-t)}}{S_t}\psi_Z(t,T)^\top \begin{pmatrix} S_t \\ X_t \end{pmatrix} \end{aligned}$$

with the vector $\psi_Z(t,T)^\top = (1;0_m)^\top \mathrm{e}^{A(T-t)}$ which follows from

$$\mathbb{E}\left[\begin{pmatrix} S_T \\ X_T \end{pmatrix} \,|\, \mathcal{F}_t\right] = \mathrm{e}^{A(T-t)}\begin{pmatrix} S_t \\ X_t \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} 0 & -\gamma^\top \\ \beta & B \end{pmatrix}$$

Affine models require (numerical) resolution of ODEs

Contingent cash-flow

Security C^D pays one at τ if and only if $t < \tau < T$

$$
\begin{aligned}
C^D(t,T) &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}\left[\mathbb{1}_{\{t<\tau<T\}} \mathrm{e}^{-r(\tau-t)} \,|\, \mathcal{G}_t \right] \\
&= \mathbb{1}_{\{\tau>t\}} \int_t^T \mathrm{e}^{-r(s-t)} d\mathbb{P}\left[\tau < s \,|\, \mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} \int_t^T \mathrm{e}^{-r(s-t)} \mathbb{E}\left[\frac{\gamma^\top X_s}{S_t} \,|\, \mathcal{F}_t \right] ds \\
&= \mathbb{1}_{\{\tau>t\}} \frac{1}{S_t} \psi_D(t,T)^\top \begin{pmatrix} S_t \\ X_t \end{pmatrix}
\end{aligned}
$$

with the vector $\psi_D(t,T)^\top = \begin{pmatrix} 0 & \gamma^\top \end{pmatrix} A_*^{-1} \left(\mathrm{e}^{A_*(T-t)} - \mathrm{Id} \right)$ and the matrix $A_* = A - \mathrm{Id} r$

Affine models require numerical integration

Credit default swap

Protection against firm default over the period (T_0, T) in exchange of premium payments until default or maturity

$$V_{\mathrm{CDS}}(t, T_0, T, k) = V_{\mathrm{prot}}(t, T_0, T) - k\, V_{\mathrm{prem}}(t, T_0, T)$$

With constant recovery rate R, protection leg and premium leg are linear combinations of contingent bonds and cash-flows

$$V_{\mathrm{CDS}}(t, T_0, T, k) = \mathbb{1}_{\{\tau > t\}} \frac{1}{S_t} \psi_{\mathrm{CDS}}(t, T_0, T, k)^\top \begin{pmatrix} S_t \\ X_t \end{pmatrix}$$

where the vector $\psi_{\mathrm{CDS}}(t, T_0, T, k)$ is explicit

Bonds and CDS prices do not depend on M_t^S and M_t^X
$\Rightarrow$ Some flexibility in modelling unspanned factors

Outline

Model specification and data

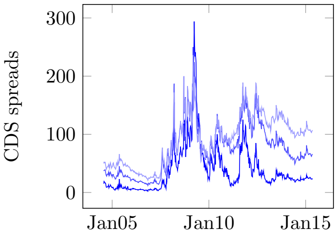
A LHC cascading structure (LHCC)

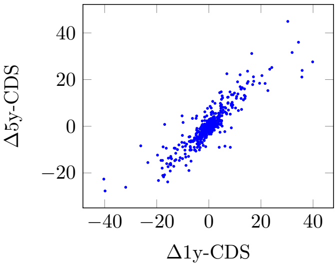
$$dS_t = -\gamma_1 X_{1t} dt$$
$$dX_{it} = \kappa_i(\theta_i X_{(i+1)t} - X_{it})\, dt + \sigma_i \sqrt{X_{it}(S_t - X_{it})}\, dW_{it}$$
$$dX_{mt} = \kappa_m(\theta_m S_t - X_{mt})\, dt + \sigma_m \sqrt{X_{mt}(S_t - X_{mt})}\, dW_{mt}$$

Three fits: $m \in \{2, 3\}$, and $m = 3$ with $\gamma_1 = 25\%$

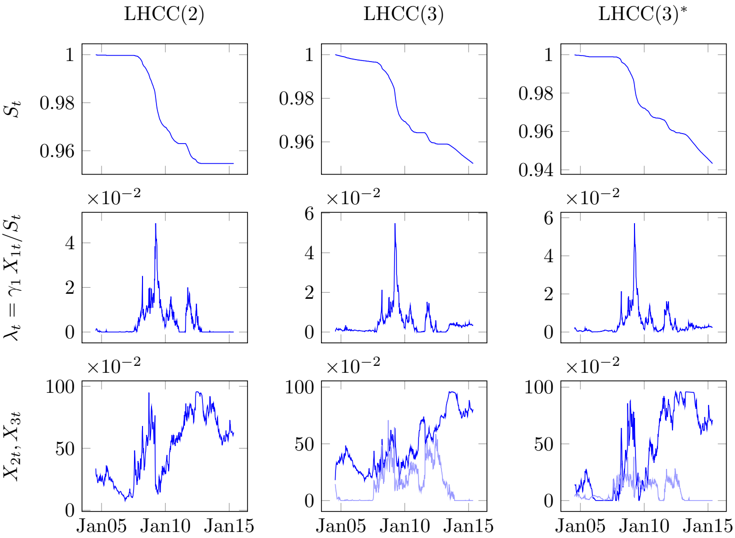
Data

1-year to 10-year CDS spreads on J.P. Morgan, $r = 2.53\%$.

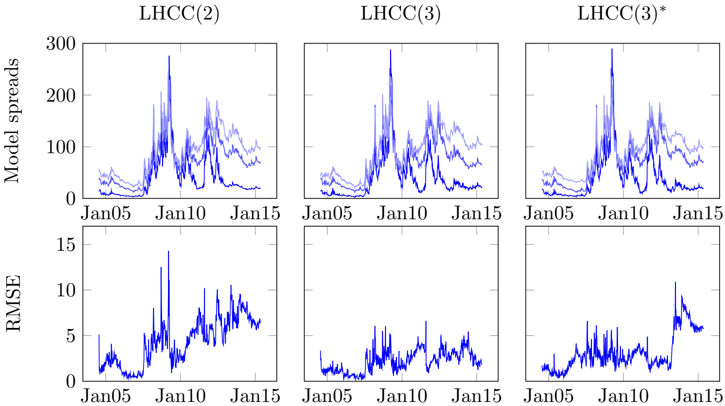




Filtered fitted factors



Fitted spreads and errors



specification / RMSE	**all**	1 yr	2 yrs	3 yrs	4 yrs	5 yrs	7 yrs	10 yrs
two-factor	**5.08**	4.30	4.59	5.36	6.19	5.98	2.67	5.71
three-factor	**2.53**	1.93	2.56	2.36	2.70	3.65	2.21	1.86
three-factor & $\gamma = 25\%$	**3.77**	2.48	2.25	3.59	5.03	4.77	2.43	4.73

Outline

Single-name Europ. CDS Option

$$\mathrm{CDSO}(t, T_0, T, k) = \mathbb{E}\left[\mathrm{e}^{-r(T_0-t)} V_{\mathrm{CDS}}(T_0, T_0, T, k)^+ \,|\, \mathcal{G}_t\right]$$
$$= \mathbb{1}_{\{\tau>t\}} \frac{\mathrm{e}^{-r(T_0-t)}}{S_t} \mathbb{E}\left[Z(T_0, T, k)^+ \,|\, \mathcal{F}_t\right]$$

with $Z(T_0, T, k) = \psi_{\mathrm{CDS}}(T_0, T_0, T, k)^\top \begin{pmatrix} S_{T_0} \\ X_{T_0} \end{pmatrix}$.

LHC model takes values on a compact support

$Z(T_0, T, k) \in [a, b]$ and analytic moments $\mathbb{E}\left[Z(T_0, T, k)^n \,|\, \mathcal{F}_t\right]$

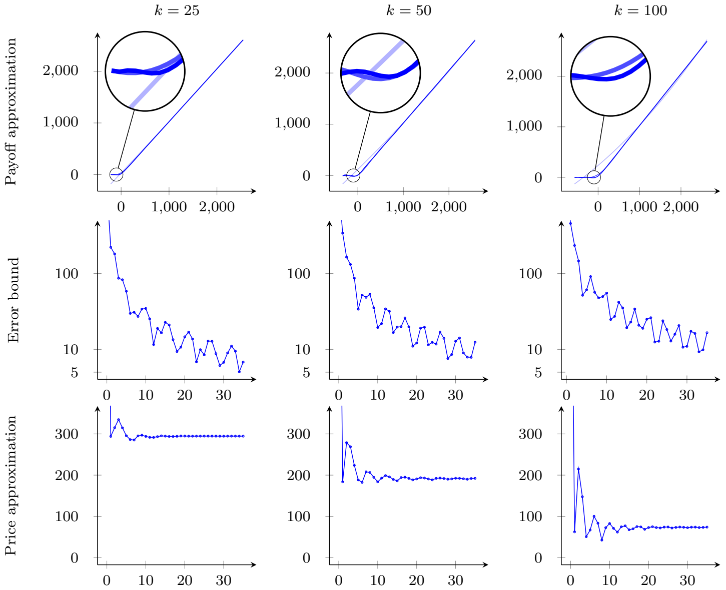
Price approximation

Polynomial series $p_n(z)$ converging to $(z)^+$ on $[a, b]$, then

$$\mathbb{E}\left[p^n(Z(T_0, T, k)) \,|\, \mathcal{F}_t\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[Z(T_0, T, k)^+ \,|\, \mathcal{F}_t\right]$$

with non-tight error upper bound $\|p^n(z) - (z)^+\|_\infty$ on $[a, b]$

CDSO price approximates



Conclusion

- New class of reduced form models for credit-risk
- Model directly the survival process $S_t = \mathbb{P}[\tau > t \mid \mathcal{F}_t]$
- Analytical formulas for defaultable bond and CDS prices
- Accurate CDS option price approximation (LHC model)
- Promising directions: multi-firm models, XVA, ...

Outline

Near-zero short-term interest rates

Contribution

- Existing models that respect zero lower bound (ZLB) on interest rates face limitations:
  - Shadow-rate models do not capture volatility dynamics
  - Multi-factor CIR and quadratic models do not easily accommodate unspanned factors and swaption pricing
- We develop a new class of linear-rational term structure models
  - Respects ZLB on interest rates
  - Easily accommodates unspanned factors affecting volatility and risk premia
  - Admits analytical solutions to swaptions
- Extensive empirical analysis
  - Parsimonious model specification has very good fit to interest rate swaps and swaptions since 1997
  - Captures many features of term structure, volatility, and risk premia dynamics.

Outline

Linear Credit Risk Model [Ackerer and Filipović, 2015]
- The linear framework
- Bonds and credit default swap pricing
- Empirical results
- CDS option price approximation

Linear-Rational Term Structure Models [Filipović et al., 2014]
- The linear-rational framework
- The Linear-Rational Square-Root (LRSQ) model
- Empirical analysis

Outline

State price density

- Filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$
- State price density: positive process ζ_t
- Model price at t of any claim C_T maturing at T:

$$\Pi(t, T) = \frac{1}{\zeta_t}\mathbb{E}\left[\zeta_T C_T \mid \mathcal{F}_t\right]$$

  This gives an arbitrage-free price system.

- Relation to short rate r_t and pricing measure $\mathbb{Q}$:

$$\frac{\zeta_t}{\zeta_0} = \mathrm{e}^{-\int_0^t r_s ds} \times \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t}$$

Factor model

- Factor process Z with range $E \subset \mathbb{R}^m$ and linear drift:

$$\mathrm{d}Z_t = \kappa\,(\theta - Z_t)\,\mathrm{d}t + \mathrm{d}M_t,$$

  where $\kappa \in \mathbb{R}^{m\times m}$, $\theta \in \mathbb{R}^m$, M_t is a martingale.

- Specify state price density as linear in Z_t

$$\zeta_t = \mathrm{e}^{-\alpha t}\left(\phi + \psi^\top Z_t\right)$$

  where $\alpha \in \mathbb{R}$, $\phi \in \mathbb{R}$, $\psi \in \mathbb{R}^m$, such that

$$\phi + \psi^\top z > 0 \quad \text{on } E$$

Linear-rational term structure

Lemma 12.1.

The $\mathcal{F}_t$-conditional expectation of Z_T is

$$\mathbb{E}[Z_T \mid \mathcal{F}_t] = \theta + \mathrm{e}^{-\kappa(T-t)}(Z_t - \theta)$$

$\Rightarrow$ Linear-rational zero-coupon bond prices

$$P(t,T) = F(T-t, Z_t)$$

where

$$F(\tau, z) = \mathrm{e}^{-\alpha\tau}\,\frac{\phi + \psi^\top\theta + \psi^\top \mathrm{e}^{-\kappa\tau}(z-\theta)}{\phi + \psi^\top z}$$

$\Rightarrow$ Linear-rational short rate

$$r_t = -\partial_T \log P(t,T)|_{T=t} = \alpha - \frac{\psi^\top\kappa\,(\theta - Z_t)}{\phi + \psi^\top Z_t}$$

Choice of α

Define

$$\alpha^* = \sup_{z \in E} \frac{\psi^\top \kappa (\theta - z)}{\phi + \psi^\top z} \quad \text{and} \quad \alpha_* = \inf_{z \in E} \frac{\psi^\top \kappa (\theta - z)}{\phi + \psi^\top z}.$$

- Should arrange so that $\alpha^* < \infty$ to get r_t bounded below
- With $\alpha = \alpha^*$, we get

$$r_t \in [0, \alpha^* - \alpha_*]$$

- For the model to be useful, this range must be wide enough
- If eigenvalues of κ have nonnegative real part then

$$\lim_{T \to \infty} -\frac{1}{T - t} \log P(t, T) = \alpha \qquad \text{infinite maturity ZCB yield}$$

Unspanned stochastic volatility

- Empirical fact: volatility risk cannot be hedged using bonds
  - Collin-Dufresne & Goldstein (02): Interest rate swaps can hedge only 10%–50% of variation in ATM straddles (a volatility-sensitive instrument)
  - Heidari & Wu (03): Level/curve/slope explain 99.5% of yield curve variation, but 59.5% of variation in swaption implied vol
- Phenomenon is called Unspanned Stochastic Volatility (USV)
- Fact: nonnegative exponential-affine term structure models cannot (generically) produce USV

Spanned vs. unspanned factors

- Recall factor dynamics

$$\mathrm{d}Z_t = \kappa\,(\theta - Z_t)\,\mathrm{d}t + \mathrm{d}M_t$$

- Linear-rational ZCB prices $P(t,T) = F(T-t, Z_t)$ where

$$F(\tau, z) = \mathrm{e}^{-\alpha\tau}\,\frac{\phi + \psi^\top\theta + \psi^\top \mathrm{e}^{-\kappa\tau}(z-\theta)}{\phi + \psi^\top z}$$

$\Rightarrow$ $F(\tau, z)$ depends on drift of Z_t only

$\Rightarrow$ Specify exogenous factors U_t feeding in martingale part of Z_t

$\Rightarrow$ U_t unspanned by term structure, give rise to USV

Term structure factors

- The term structure kernel $\mathcal{U}$ is defined as orthogonal complement in $\mathbb{R}^m$ to factor loadings of the term structure

$$\mathcal{U} = \bigcap_{\tau \geq 0,\, z \in E} \ker \nabla_z F(\tau, z)$$

Theorem 12.2.

1. *Identity* $\mathcal{U} = \operatorname{span}\left\{\psi, \kappa^\top \psi, \ldots, \kappa^{(m-1)\top}\psi\right\}^\perp$
2. *After dimension reduction if necessary we can assume* $\mathcal{U} = \{0\}$*, such that* Z_t *become term structure factors*
3. *Term structure* $F(\tau, z)$ *injective if and only if* $\mathcal{U} = \{0\}$*,* κ *is invertible, and* $\phi + \psi^\top \theta \neq 0$

Interest rate swaps

- Exchange a stream of fixed-rate for floating-rate payments
- Consider a tenor structure

$$T_0 < T_1 < \cdots < T_n, \quad T_i - T_{i-1} \equiv \Delta$$

- At T_i, $i = 1 \ldots n$:
  - pay Δk, for fixed rate k
  - receive floating LIBOR $\Delta L(T_{i-1}, T_i) = \frac{1}{P(T_{i-1}, T_i)} - 1$
- Value of payer swap at $t \leq T_0$

$$\Pi_t^{\mathrm{swap}} = \underbrace{P(t, T_0) - P(t, T_n)}_{\text{floating leg}} - \underbrace{\Delta k \sum_{i=1}^{n} P(t, T_i)}_{\text{fixed leg}}$$

- Forward swap rate $S_t = \frac{P(t,T_0) - P(t,T_n)}{\Delta \sum_{i=1}^{n} P(t,T_i)}$

Swaptions

- Payer swaption = option to enter the swap at T_0 paying fixed, receiving floating
- Payoff at expiry T_0 of the form

$$C_{T_0} = \left(\Pi_{T_0}^{\mathrm{swap}}\right)^+ = \left(\sum_{i=0}^{n} c_i P(T_0, T_i)\right)^+ = \frac{1}{\zeta_{T_0}} p_{\mathrm{swap}}(Z_{T_0})^+$$

for the explicit linear function

$$p_{\mathrm{swap}}(z) = \sum_{i=0}^{n} c_i \mathrm{e}^{-\alpha T_i} \left(\phi + \psi^\top \theta + \psi^\top \mathrm{e}^{-\kappa(T_i - T_0)}(z - \theta)\right)$$

- Swaption price at $t \le T_0$ is given by

$$\Pi_t^{\mathrm{swaption}} = \frac{1}{\zeta_t} \mathbb{E}[\zeta_{T_0} C_{T_0} \mid \mathcal{F}_t] = \frac{1}{\zeta_t} \mathbb{E}_t \left[p_{\mathrm{swap}}(Z_{T_0})^+\right]$$

- Efficient swaption pricing via Fourier transform ...!

Fourier transform

- Define

$$\widehat{q}(x) = \mathbb{E}_t \left[\exp\left(x\, p_{\mathrm{swap}}(Z_{T_0})\right)\right]$$

for every $x \in \mathbb{C}$ such that the conditional expectation is well-defined

- Then

$$\Pi_t^{\mathrm{swaption}} = \frac{1}{\zeta_t \pi} \int_0^\infty \mathrm{Re}\left[\frac{\widehat{q}(\mu + \mathrm{i}\lambda)}{(\mu + \mathrm{i}\lambda)^2}\right] d\lambda$$

for any $\mu > 0$ with $\widehat{q}(\mu) < \infty$

- $\widehat{q}(x)$ has semi-analytical solution in LRSQ model

Outline

Linear-Rational Square-Root (LRSQ) model

- Objective: A model with joint factor process (Z_t, U_t), where
  - Z_t: m term structure factors
  - U_t: $n \leq m$ USV factors
- Denoted **LRSQ(m,n)**
- Based on a $(m+n)$-dimensional square-root diffusion process X_t taking values in $\mathbb{R}_+^{m+n}$ of the form

$$\mathrm{d}X_t = (b - \beta X_t)\,\mathrm{d}t + \mathrm{Diag}\left(\sigma_1\sqrt{X_{1t}}, \ldots, \sigma_{m+n}\sqrt{X_{m+n,t}}\right)\mathrm{d}B_t,$$

- Define $(Z_t, U_t) = SX_t$ as linear transform of X_t
- Need to specify a $(m+n) \times (m+n)$-matrix S such that
  - the implied term structure state space is $E = \mathbb{R}_+^m$
  - the drift of Z_t does not depend on U_t, while U_t feeds into the martingale part of Z_t

Linear-Rational Square-Root (LRSQ) model (cont.)

- S given by

$$S = \begin{pmatrix} \mathrm{Id}_m & A \\ 0 & \mathrm{Id}_n \end{pmatrix} \quad \text{with } A = \begin{pmatrix} \mathrm{Id}_n \\ 0 \end{pmatrix}.$$

- β chosen upper block-triangular of the form

$$\beta = S^{-1} \begin{pmatrix} \kappa & 0 \\ 0 & A^\top \kappa A \end{pmatrix} S = \begin{pmatrix} \kappa & \kappa A - AA^\top \kappa A \\ 0 & A^\top \kappa A \end{pmatrix}$$

  for some $\kappa \in \mathbb{R}^{m \times m}$

- b given by

$$b = \beta S^{-1} \begin{pmatrix} \theta \\ \theta_U \end{pmatrix} = \begin{pmatrix} \kappa\theta - AA^\top \kappa A \theta_U \\ A^\top \kappa A \theta_U \end{pmatrix}$$

  for some $\theta \in \mathbb{R}^m$ and $\theta_U \in \mathbb{R}^n$.

Linear-Rational Square-Root (LRSQ) model (cont.)

- Resulting joint factor process (Z_t, U_t):

$$\mathrm{d}Z_t = \kappa\,(\theta - Z_t)\,\mathrm{d}t + \sigma(Z_t, U_t)\mathrm{d}B_t$$
$$\mathrm{d}U_t = A^\top \kappa A\,(\theta_U - U_t)\,\mathrm{d}t + \mathrm{Diag}\left(\sigma_{m+1}\sqrt{U_{1t}}\,\mathrm{d}B_{m+1,t}, \ldots, \sigma_{m+n}\sqrt{U_{nt}}\,\mathrm{d}B_{m+n,t}\right),$$

with dispersion function of Z_t given by

$$\sigma(z, u) = (\mathrm{Id}_m, A)\,\mathrm{Diag}\left(\sigma_1\sqrt{z_1 - u_1}, \ldots, \sigma_{m+n}\sqrt{u_n}\right)$$

- Example: **LRSQ(1,1)**

$$\mathrm{d}Z_{1t} = \kappa\,(\theta - Z_{1t})\,\mathrm{d}t + \sigma_1\sqrt{Z_{1t} - U_{1t}}\mathrm{d}B_{1t} + \sigma_2\sqrt{U_{1t}}\mathrm{d}B_{2t}$$
$$\mathrm{d}U_{1t} = \kappa\,(\theta_U - U_{1t})\,\mathrm{d}t + \sigma_2\sqrt{U_{1t}}\mathrm{d}B_{2t}$$

Example: LRSQ(3, 1)

- $$\beta = \left(\begin{array}{ccc|c} \kappa_{11} & \kappa_{12} & \kappa_{13} & 0 \\ \kappa_{21} & \kappa_{22} & \kappa_{23} & \kappa_{21} \\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{31} \\ \hline 0 & 0 & 0 & \kappa_{11} \end{array}\right)$$
- $$\left(\begin{array}{c} Z_{1t} \\ Z_{2t} \\ Z_{3t} \\ \hline U_{1t} \end{array}\right) = SX_t = \left(\begin{array}{c} X_{1t} + X_{4t} \\ X_{2t} \\ X_{3t} \\ \hline X_{4t} \end{array}\right)$$
- $$\sigma(z, u) = \left(\begin{array}{ccc|c} \sigma_1\sqrt{z_1 - u_1} & 0 & 0 & \sigma_4\sqrt{u_1} \\ 0 & \sigma_2\sqrt{z_2} & 0 & 0 \\ 0 & 0 & \sigma_3\sqrt{z_3} & 0 \\ \hline 0 & 0 & 0 & \sigma_4\sqrt{u_1} \end{array}\right)$$

Example: LRSQ(3, 2)

- $$\beta = \left(\begin{array}{ccc|cc} \kappa_{11} & \kappa_{12} & \kappa_{13} & 0 & 0 \\ \kappa_{21} & \kappa_{22} & \kappa_{23} & 0 & 0 \\ \kappa_{31} & \kappa_{32} & \kappa_{33} & \kappa_{31} & \kappa_{32} \\ \hline 0 & 0 & 0 & \kappa_{11} & \kappa_{12} \\ 0 & 0 & 0 & \kappa_{21} & \kappa_{22} \end{array}\right)$$
- $$\begin{pmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \\ \hline U_{1t} \\ U_{2t} \end{pmatrix} = SX_t = \begin{pmatrix} X_{1t} + X_{4t} \\ X_{2t} + X_{5t} \\ X_{3t} \\ \hline X_{4t} \\ X_{5t} \end{pmatrix}$$
- $$\sigma(z, u) = \left(\begin{array}{ccc|cc} \sigma_1\sqrt{z_1 - u_1} & 0 & 0 & \sigma_4\sqrt{u_1} & 0 \\ 0 & \sigma_2\sqrt{z_2 - u_2} & 0 & 0 & \sigma_5\sqrt{u_2} \\ 0 & 0 & \sigma_3\sqrt{z_3} & 0 & 0 \\ \hline 0 & 0 & 0 & \sigma_4\sqrt{u_1} & 0 \\ 0 & 0 & 0 & 0 & \sigma_5\sqrt{u_2} \end{array}\right)$$

Example: LRSQ(3, 3)

- $\beta = \left(\begin{array}{c|c} \kappa & 0 \\ \hline 0 & \kappa \end{array}\right)$
- $\left(\begin{array}{c} Z_{1t} \\ Z_{2t} \\ Z_{3t} \\ \hline U_{1t} \\ U_{2t} \\ U_{3t} \end{array}\right) = SX_t = \left(\begin{array}{c} X_{1t} + X_{4t} \\ X_{2t} + X_{5t} \\ X_{3t} + X_{6t} \\ \hline X_{4t} \\ X_{5t} \\ X_{6t} \end{array}\right)$
- $\sigma(z, u) = \left(\begin{array}{ccc|ccc} \sigma_1\sqrt{z_1 - u_1} & 0 & 0 & \sigma_4\sqrt{u_1} & 0 & 0 \\ 0 & \sigma_2\sqrt{z_2 - u_2} & 0 & 0 & \sigma_5\sqrt{u_2} & 0 \\ 0 & 0 & \sigma_3\sqrt{z_3 - u_3} & 0 & 0 & \sigma_6\sqrt{u_3} \\ \hline 0 & 0 & 0 & \sigma_4\sqrt{u_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5\sqrt{u_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_6\sqrt{u_3} \end{array}\right)$

Linear-rational vs. exponential-affine framework

	Exponential-affine	Linear-rational
Short rate	affine	LR
ZCB price	exponential-affine	LR
ZCB yield	affine	log of LR
Coupon bond price	sum of exponential-affines	LR
Swap rate	ratio of sums of exponential-affines	LR
ZLB	(✓)	✓
USV	(✓)	✓
Cap/floor valuation	semi-analytical	semi-analytical
Swaption valuation	approximate	semi-analytical
Linear state inversion	ZCB yields	bond prices or swap rates

Linear-rational vs. exponential-affine framework: MPR

Exponential-affine model:

$$P(t,T) = \mathrm{e}^{A(T-t)+B(T-t)^\top Z_t}$$

- Z_t square-root diffusion under risk-neutral measure $\mathbb{Q}$
- Market price of risk λ_t determining $\frac{d\mathbb{Q}}{d\mathbb{P}}$ exogeneous

LRSQ model:

$$P(t,T) = \mathrm{e}^{-\alpha(T-t)} \frac{1 + \mathbf{1}^\top \theta + \mathbf{1}^\top \mathrm{e}^{-\kappa(T-t)}(Z_t - \theta)}{1 + \mathbf{1}^\top Z_t}$$

- Z_t square-root diffusion under historical measure $\mathbb{P}$
- Market price of risk λ_t determining $\frac{d\mathbb{Q}}{d\mathbb{P}}$ endogenous

Linear-rational vs. exponential-affine framework: MPR

Exponential-affine model:

$$P(t,T) = \mathrm{e}^{A(T-t)+B(T-t)^\top Z_t}$$

- Z_t square-root diffusion under risk-neutral measure $\mathbb{Q}$
- Market price of risk λ_t determining $\frac{d\mathbb{Q}}{d\mathbb{P}}$ exogeneous

LRSQ model:

$$P(t,T) = \mathrm{e}^{-\alpha(T-t)} \frac{1 + \mathbf{1}^\top \theta + \mathbf{1}^\top \mathrm{e}^{-\kappa(T-t)}(Z_t - \theta)}{1 + \mathbf{1}^\top Z_t}$$

- Z_t square-root diffusion under **auxiliary measure $\mathbb{A}$**
- Market price of risk λ_t determining $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{A}}\frac{d\mathbb{A}}{d\mathbb{P}}$ **exogenous**

Extended state price density specification

- Linear state price density specification: market price of risk

$$\lambda_t = -\frac{\sigma(Z_t, U_t)^\top \psi}{\phi + \psi^\top Z_t}.$$

- Alternatively, develop model under auxiliary measure $\mathbb{A}$:
  - State price density: $\zeta_t^{\mathbb{A}} = \mathrm{e}^{-\alpha t}(\phi + \psi^\top Z_t)$
  - Factor process dynamics: $\mathrm{d}Z_t = \kappa(\theta - Z_t)\mathrm{d}t + \mathrm{d}M_t^{\mathbb{A}}$
  - Basic pricing formula: $\Pi(t, T) = \mathbb{E}_t^{\mathbb{A}}\left[\zeta_T^{\mathbb{A}} C_T\right] / \zeta_t^{\mathbb{A}}$

- Extended state price density specification

$$\zeta_t^{\mathbb{P}} = \zeta_t^{\mathbb{A}}\, \mathbb{E}_t^{\mathbb{P}}\left[\mathrm{d}\mathbb{A}/\mathrm{d}\mathbb{P}\right] = \zeta_t^{\mathbb{A}}\, \mathcal{E}\left(-\int_0^t \delta_s^\top \mathrm{d}B_s^{\mathbb{P}}\right)$$

with (Alvarez & Jermann (2005), Hansen & Scheinkman (2009))

  - transitory component $\zeta_t^{\mathbb{A}}$
  - permanent component $\mathbb{E}_t^{\mathbb{P}}\left[\mathrm{d}\mathbb{A}/\mathrm{d}\mathbb{P}\right]$

Extended state price density specification

- Market price of risk now given by

$$\lambda_t^{\mathbb{P}} = -\frac{\sigma(Z_t, U_t)^\top \psi}{\phi + \psi^\top Z_t} + \delta_t$$

- In LRSQ model: no additional unspanned risk premium factors

$$\delta_t = (\delta_1 \sqrt{X_{1t}}, \ldots, \delta_{m+n} \sqrt{X_{m+n,t}})^\top$$

- $\mathbb{A}$ is long forward measure:

$$\frac{\zeta_t^{\mathbb{A}} P(t,T)}{\zeta_0^{\mathbb{A}} P(0,T)} = \frac{\phi + \mathbb{E}_t^{\mathbb{A}}[\psi^\top Z_T]}{\phi + \mathbb{E}^{\mathbb{A}}[\psi^\top Z_T]} \to 1 \quad \text{as } T \to \infty$$

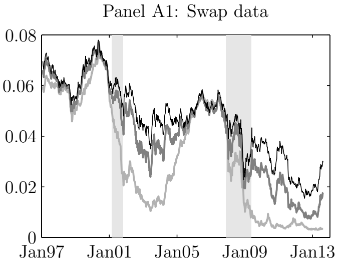
  Hence deflating by $\zeta_t^{\mathbb{A}}/\zeta_0^{\mathbb{A}}$ amounts to discounting by gross return on long-term bond $\lim_{T\to\infty} \frac{P(t,T)}{P(0,T)}$

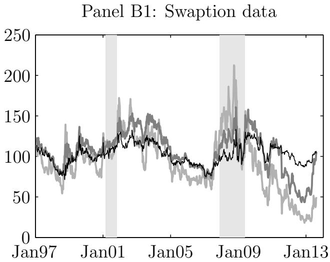
  It also implies that the long-term bond is growth optimal under $\mathbb{A}$ (Qin & Linetsky 2015)

Outline

Data and estimation approach

- Panel data set of swaps and swaptions
- Swap maturities: 1Y, 2Y, 3Y, 5Y, 7Y, 10Y
- Swaptions expiries: 3M, 1Y, 2Y, 5Y
- 866 weekly observations, Jan 29, 1997 – Aug 28, 2013
- Estimation approach: Quasi-maximum likelihood in conjunction with the unscented Kalman Filter





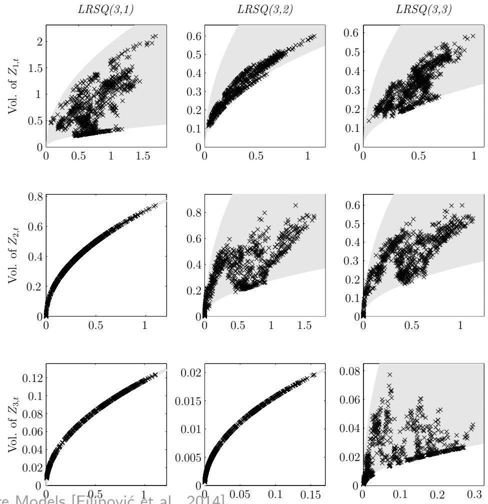
Model specifications

- Model specifications (always 3 term structure factors)
  - **LRSQ(3,1)**: volatility of Z_{1t} containing an unspanned component
  - **LRSQ(3,2)**: volatility of Z_{1t} and Z_{2t} containing unspanned components
  - **LRSQ(3,3)**: volatility of term structure factors containing unspanned components
- $\alpha = \alpha^*$ and range of r_t:

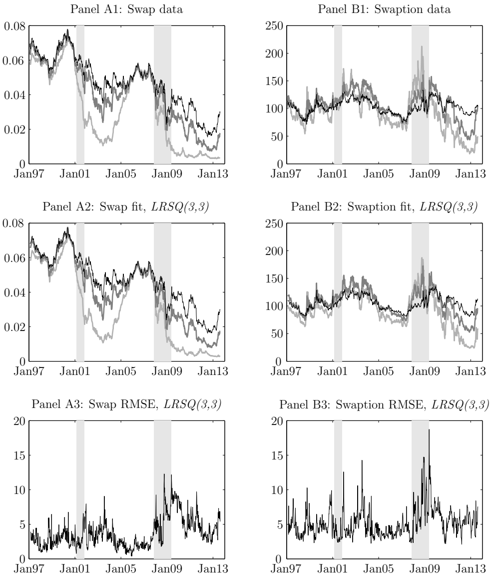
	LRSQ(3,1)	*LRSQ(3,2)*	*LRSQ(3,3)*
Long ZCB yield α	7.46%	6.88%	5.66%
Upper bound on r_t	20%	146%	72%

Level-dependence in factor volatilities

- Volatility of Z_{it} with USV: $\sqrt{\sigma_i^2 Z_{it} + (\sigma_{i+3}^2 - \sigma_i^2) U_{it}}$
- Volatility of Z_{it} without USV: $\sigma_i \sqrt{Z_{it}}$

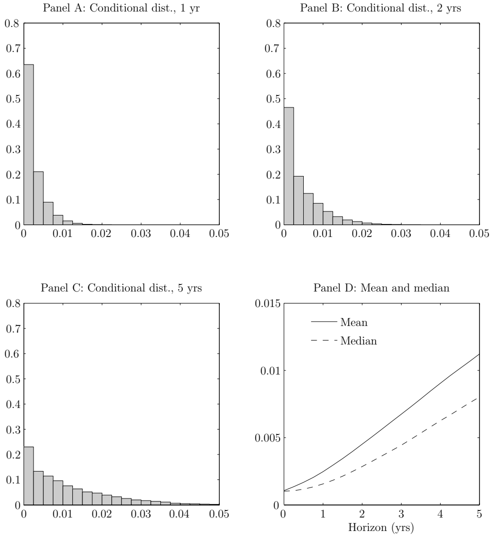


Fit to data, *LRSQ(3,3)*



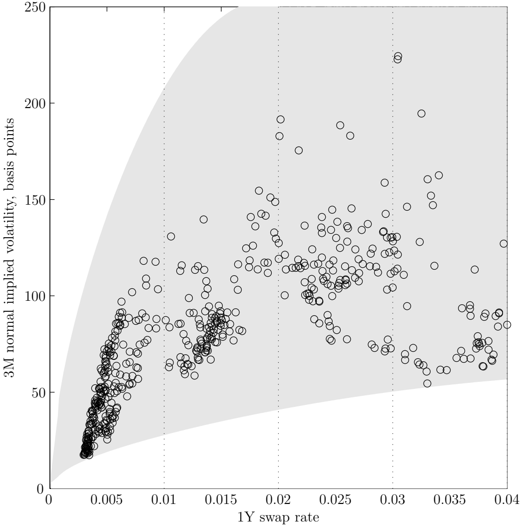
Short-rate dynamics near the ZLB

- Conditional density of r_t given $r_0 \leq 25$bps, *LRSQ(3,3)*



Volatility dynamics near the ZLB

- Level-dependence in volatility, 3M/1Y IV vs. 1Y rate



Level-dependence in volatility

- Regress weekly changes in the 3M swaption IV on weekly changes in the swap rate

$$\Delta\sigma_{N,t} = \beta_0 + \beta_1 \Delta S_t + \epsilon_t$$

	1 yr	2 yrs	3 yrs	5 yrs	7 yrs	10 yrs	Mean
Panel A: $\hat{\beta}_1$							
All	0.18^{**} (2.38)	0.16^{***} (2.88)	0.16^{***} (3.31)	0.16^{***} (4.12)	0.16^{***} (4.59)	0.16^{***} (4.97)	0.16
0%-1%	1.20^{***} (8.03)	0.74^{***} (8.79)	0.62^{***} (8.19)	0.48^{***} (7.83)			0.76
1%-2%	0.54^{***} (2.70)	0.64^{***} (6.21)	0.46^{***} (6.77)	0.52^{***} (5.02)	0.45^{***} (5.23)	0.26^{***} (8.24)	0.48
2%-3%	0.28^{***} (3.10)	0.11^{**} (1.97)	0.30^{***} (3.77)	0.36^{***} (5.08)	0.40^{***} (5.62)	0.40^{***} (4.93)	0.31
3%-4%	-0.02 (-0.22)	0.11 (1.21)	0.06 (0.92)	0.05 (0.80)	0.11^{*} (1.82)	0.17^{*} (1.96)	0.08
4%-5%	0.04 (0.31)	-0.07 (-0.82)	0.01 (0.08)	0.08 (1.59)	0.07^{*} (1.76)	0.07^{*} (1.65)	0.03
Panel B: R^2							
All	0.05	0.06	0.08	0.10	0.11	0.10	0.08
0%-1%	0.52	0.54	0.54	0.44			0.51
1%-2%	0.25	0.49	0.45	0.55	0.55	0.27	0.43
2%-3%	0.16	0.06	0.28	0.37	0.44	0.45	0.29
3%-4%	0.00	0.03	0.01	0.01	0.07	0.12	0.04
4%-5%	0.00	0.01	0.00	0.03	0.03	0.03	0.02

Level-dependence in volatility

- Regress weekly changes in the 3M swaption IV on weekly changes in the swap rate

$$\Delta\sigma_{N,t} = \beta_0 + \beta_1 \Delta S_t + \epsilon_t$$

	1 yr	2 yrs	3 yrs	5 yrs	7 yrs	10 yrs	Mean
Panel A: $\hat{\beta}_1$							
All	0.18^{**} (2.38)	0.16^{***} (2.88)	0.16^{***} (3.31)	0.16^{***} (4.12)	0.16^{***} (4.59)	0.16^{***} (4.97)	0.16
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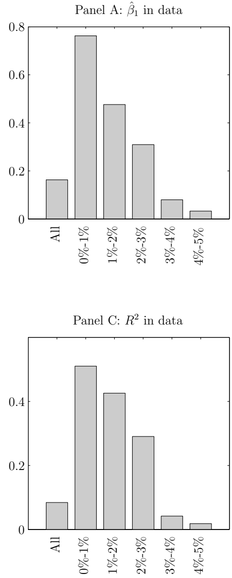
Panel B: R^2							
All	0.05	0.06	0.08	0.10	0.11	0.10	0.08
0%-1%	0.52	0.54	0.54	0.44			0.51
1%-2%	0.25	0.49	0.45	0.55	0.55	0.27	0.43
2%-3%	0.16	0.06	0.28	0.37	0.44	0.45	0.29
3%-4%	0.00	0.03	0.01	0.01	0.07	0.12	0.04
4%-5%	0.00	0.01	0.00	0.03	0.03	0.03	0.02

Level-dependence in volatility, *LRSQ(3,3)*



Unconditional excess returns

- Unconditional 1M excess ZCB returns, % annualized

		1 yr	2 yrs	3 yrs	5 yrs	7 yrs	10 yrs
Data	Mean	0.58	1.56	2.39	3.61	4.46	5.43
	Vol	0.71	1.72	2.82	4.96	6.96	9.86
	SR	0.82	0.91	0.85	0.73	0.64	0.55
LRSQ(3,1)	Mean	0.37	0.74	1.10	1.77	2.39	3.21
	Vol	0.57	1.28	2.14	4.02	5.83	8.19
	SR	0.64	0.58	0.51	0.44	0.41	0.39
LRSQ(3,2)	Mean	0.37	0.70	1.01	1.60	2.14	2.83
	Vol	0.53	1.21	1.97	3.54	5.04	7.08
	SR	0.69	0.58	0.51	0.45	0.42	0.40
LRSQ(3,3)	Mean	0.25	0.58	0.91	1.53	2.04	2.63
	Vol	0.57	1.19	1.92	3.51	5.06	7.21
	SR	0.43	0.48	0.47	0.44	0.40	0.36
LRSQ(3,3), $\delta_t = 0$	Mean	-0.03	0.01	0.10	0.34	0.60	0.97
	Vol	1.01	1.71	2.35	3.75	5.23	7.31
	SR	-0.03	0.01	0.04	0.09	0.11	0.13

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Conditional expected excess returns

- Regress $R^e_{t+1} = \beta_0 + \beta_{Slp} Slp_t + \beta_{Vol} Vol_t + \epsilon_{t+1}$
- Slp_t: slope of swap term structure (standardized)
- Vol_t: 1M swaption IV (standardized)

		1 yr	2 yrs	3 yrs	5 yrs	7 yrs	10 yrs
Data	$\hat{\beta}_{Slp}$	-0.025 (-1.548)	-0.009 (-0.215)	0.027 (0.403)	0.092 (0.838)	0.121 (0.845)	0.166 (0.832)
	$\hat{\beta}_{Vol}$	0.058*** (4.459)	0.114*** (3.409)	0.144** (2.506)	0.169 (1.546)	0.206 (1.395)	0.210 (0.963)
	R^2	0.102	0.051	0.037	0.025	0.020	0.013
LRSQ(3,1)	$\hat{\beta}_{Slp}$	0.004	0.003	-0.004	-0.032	-0.065	-0.102
	$\hat{\beta}_{Vol}$	0.012	0.017	0.026	0.058	0.096	0.148
	R^2	0.007	0.003	0.002	0.002	0.003	0.004
LRSQ(3,2)	$\hat{\beta}_{Slp}$	0.000	0.002	0.008	0.018	0.021	0.014
	$\hat{\beta}_{Vol}$	0.016	0.033	0.049	0.072	0.088	0.112
	R^2	0.011	0.009	0.008	0.005	0.004	0.003
LRSQ(3,3)	$\hat{\beta}_{Slp}$	0.025	0.038	0.046	0.055	0.059	0.059
	$\hat{\beta}_{Vol}$	0.031	0.054	0.074	0.112	0.143	0.182
	R^2	0.082	0.054	0.035	0.020	0.014	0.010
LRSQ(3,3), $\delta_t = 0$	$\hat{\beta}_{Slp}$	-0.002	-0.001	0.001	0.006	0.010	0.015
	$\hat{\beta}_{Vol}$	-0.004	-0.002	0.005	0.026	0.049	0.080
	R^2	0.000	0.000	0.000	0.001	0.001	0.001

Conditional expected excess returns

- Regress $R^e_{t+1} = \beta_0 + \beta_{Slp} Slp_t + \beta_{Vol} Vol_t + \epsilon_{t+1}$
- Slp_t: slope of swap term structure (standardized)
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	$\hat{\beta}_{Vol}$	0.058***	0.114***	0.144**	0.169	0.206	0.210
		(4.459)	(3.409)	(2.506)	(1.546)	(1.395)	(0.963)
	R^2	0.102	0.051	0.037	0.025	0.020	0.013
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	R^2	0.000	0.000	0.000	0.001	0.001	0.001

Conclusion

- Key features of framework:
  - Respects ZLB on interest rates
  - Easily accommodates unspanned factors affecting volatility and risk premia
  - Admits semi-analytical solutions to swaptions
- Extensive empirical analysis:
  - Parsimonious model specification has very good fit to interest rate swaps and swaptions since 1997
  - Captures many features of term structure, volatility, and risk premia dynamics.

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