A new perspective on multiple yield curve models

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- The basic observation in interest rate theory is that 1 EUR today has a different value than 1 EUR at a future timepoints, say in 1 year.

Definition

A **zero-coupon bond** (T-bond) with nominal N and maturity T promises the owner the payment of N units of currency at time T.

- For simplicity we consider always $N = 1$.
- The price of the zero-coupon bond at time $t \le t$ is denoted by $P(t, T)$.

Example (Forward)

For your investment of 1 today you get 1.04 in 2 years. Then the associated (annualized) yield equals $(1.04 - 1)/2$.

Can we fix the yield today for a future period, say from S to T? YES:

- Invest 1 at S, get x at $T > S$

is equivalent to the following trading strategy:

- Sell 1 S-bond at t (and get $P(t, S)$) and
- buy $P(t, S)/P(t, T)$ T-bonds at t.

This trading strategy has zero cost at t, the cash-flow of -1 at S and $P(t, S)/P(t, T)$ at T.

The associated simple rates lead to the following definition.

Definition

The **simple forward rate** for the time interval $[S, T]$ at time $t \leq S$ is given by

$$F(t, S, T) := \frac{1}{T-S}\left(\frac{P(t, S)}{P(t, T)} - 1\right).$$

The **spot forward rate** is

$$F(t, T) := F(t, t, T).$$

Example

In comparision to yearly compounding one could also consider monthly, daily or even finer compounding. In prinicple we observe that letting $n \to \infty$ leads to

$$\left(1 + \frac{R}{n}\right)^{nT} \to e^{RT},$$

which is called **continuous compounding**. ◇

Definition

The **continuously compounded forward rate** for the time interval $[S, T]$ at time $t \leq S$ is given by

$$R(t, S, T) := \frac{1}{T - S} \log \frac{P(t, S)}{P(t, T)}$$

and the **continuously compounded spot rate** is

$$R(t, T) := R(t, t, T).$$

Finally, we define instantaneous rates which make the time variation of the rates best visible by letting $T \downarrow S$. If the following limit exists we define

$$f(t, T) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(\log P(t, T) - \log P(t, T + \varepsilon) \Big)$$

$$\to -\partial_T \log P(t, T).$$

Definition

The (instantaneous) **forward rate** for maturity T at time $t \leq T$, if existing, is given by

$$f(t, T) := -\partial_T \log P(t, T)$$

and the (instantaneous) **spot rate** by

$$r(t) := f(t, t).$$

It is clear in general that a forward rate does not need to exist and neither does a short rate. By intragration we obtain that all bond-prices can be represented by forward rates in the following way

$$P(t, T) = \exp\left(-\int_t^T f(t, s)ds\right).$$

The money market account

Seen in an idealized way, the money market account B is obtained by continuously investing in the short rate,

$$dB(t) = B(t)r(t)dt \qquad B(0) = 1.$$

The solution is of course $B(t) = \exp \int_0^t r(s)ds$. This may be proxied by a roll-over strategy,

$$B^n(t) = \prod_{i=1}^n \frac{1}{P(t_{i-1}^n, t_i^n)} = \exp\left(\int_{t_{i-1}^n}^{t_i^n} f(t_{i-1}^n, s)ds\right)$$
$$\to \exp\left(\int_0^t r(s)ds\right),$$

provided the convergence holds. Also this may fail in general, but some degree of smoothness will be sufficient (think of counterexamples $\to$ Excercise).

In a certain way, the money market account is embedded into the forward-term structure, $\to$ Döberlein and Schweizer (2001); Klein et al. (2016).

What we need from Stochastic Analysis

An excellent reference for the required results is Karatzas and Shreve (1988), see also Jacod and Shiryaev (2003) for the general semimartingale framework.

We consider a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$ satisfies the usual conditions, i.e.

- $\mathscr{F}$ is $\mathbb{P}$-complete, i.e. $A \subset B$ with $B \in \mathscr{F}$ and $\mathbb{P}(B) = 0 \Rightarrow A \in \mathscr{F}$,
- $\mathscr{F}_0$ contains all $\mathbb{P}$-nullsets ,
- $\mathbb{F}$ is right-continuous.

A stochastic process $X = (X_t)_{t \geq 0}$ is called

- **adapted** if $\Omega \ni \omega \mapsto X_t(\omega)$ is $\mathscr{F}_t$-measurable for all $t \geq 0$,
- **progressively measurable** (progressive) if $\Omega \times [0, t] \ni (\omega, s) \mapsto X_s(\omega)$ is $\mathscr{F}_t \otimes \mathscr{B}([0, t])$-measurable for all $t \geq 0$.

By Prog_T we denote the σ-algebra generated by all progressive processes on $[0, T]$. We throughout use both X_t and $X(t)$ for referencing the value of X at time t. Consider a d-dimensional Brownian motion W.

The one-dimensional process X is called **Itô**-process if X has the representation

$$X(t) = X(0) + \int_0^t a(s)ds + \int_0^t b(s)dW(s) \tag{1}$$

where a, b are progressive and

- $\mathbb{P}\left(\int_0^t \| a(s) \| \, ds < \infty \right) = 1$, and $\mathbb{P}\left(\int_0^t \| b(s) \| \, ds^2 < \infty \right) = 1$.

For a progressive process b with the second property we abbreviate $b \in \mathscr{L}$. When $b \in \mathscr{L}$, the stochastic integral $(b \cdot W)$ given by

$$(b \cdot W)_t := \int_0^t b(s)dW(s), \qquad t \geq 0$$

is a local martingale. The decomposition in (1) is unique (up to indistinguishability).

The stochastic integral of an Itô-process is again an Itô process, or more precisely: if c is such that $\mathbb{P}(\int_0^t \| c(s)a(s) \| ds < \infty) = 1$ and $cb \in \mathcal{L}$, then we write $c \in \mathcal{L}(X)$ and then

$$Y(t) = Y(0) + \int_0^t c(s)dX(s)$$

is an Itô-process.

Define the covariation of two Itô processes X_1 and X_2 satisfying (1) with coefficients a_i and b_i by

$$<X, Y>_t := \int_0^t b_1(s)b_2^\top(s)ds.$$

Theorem (Itô-formula)

Consider $f \in C^{1,2}$. Then $(f(t, X_t))_{t \geq 0}$ is an Itô-process and

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_t f(s, X_s)ds + \sum_{i=1}^d \int_0^t \partial_{x_i} f(s, X_s)dX_s^i$$

$$+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{x_i} \partial_{x_j} f(s, X_s)d<X^i, X^j>_s .$$

Fubini theorems

In contrast to deterministic analysis Fubini-theorems come at a surprisingly high cost.

Theorem (Fubini)

Consider the d-dimensional process $\phi = (\phi(\omega, s, t))_{0 \leq s, t \leq T}$, such that

- (i) ϕ is $Prog_T \otimes \mathscr{B}([0, T])$-measurable
- (ii) $\sup_{0 \leq s, t \leq T} \| \phi(t, s) \| < \infty$.

Then $\int_0^T \phi(t, s) ds \in \mathscr{L}$ and

$$\int_0^T \left(\int_0^T \phi(t, s) dW_t \right) ds = \int_0^T \left(\int_0^T \phi(t, s) ds \right) dW_t.$$

Alternative Fubini theorems can be found in Protter (2004); Veraar (2012); Fontana and Schmidt (2016).

What we need from No-Arbitrage Theory

Consider a $d+1$-dimensional financial market given by the stochastic process $S = (S^0, \ldots, S^d)^\top$. We assume that

$$dS_t^0 = S_t^0 r_t dt, \qquad S^0(0) = 1$$
$$dS_t^i = S_t^i (\mu_t^i dt + \sigma_t^i dW_t).$$

A **portfolio** (or trading strategy) is a $d+1$-dimensional progressive process $\phi = (\phi_0, \ldots, \phi_d)$. The trading strategy is called **self-financing** if $\phi \in \mathscr{L}(S)$ and

$$dV(t) = \phi(t)dS(t).$$

A self-financing trading strategy is admissible if the value process is bounded from below. Then the fundamental theorem of asset pricing (FTAP) shows that NFLVR (no free lunch with vanishing risk) is equivalent to the existence of an equivalent local martingale measure (ELMM).

As is customary in financial literature we will mainly use the claim that the existence of an equivalent local martingale measure (ELMM) is sufficient for absence of arbitrage. We throughout consider an equivalent measure P and classify when it is a LMM.

Some care is needed

- You may note that $(P(t, T))_{0 \leq t \leq T}$ is, however, a market with an infinite number of assets.

- This requires some special care: one may rely on methodologies from larger financial markets → Klein et al. (2016); Cuchiero et al. (2014).

- In short, NFLVR can be replaced by NAFLVR (no asymptotic free lunch with vanishing risk) and is, in the case considered here (with locally bounded prices of the traded assets), equivalent to the existence of an equivalent local martingale measure.

The classical case - Heath, Jarrow and Morton (1992)

Consider

$$P(t, T) = e^{-\int_t^T f(t,u)du}$$

with numéraire $S^0(t) = e^{\int_0^t r_s ds}$. The classic assumption is that

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)ds + \int_0^t \beta(s, T)dW_s$$

where $f(t, t) = r_t$ and

- α and β are $\text{Prog} \otimes \mathscr{B}$-measurable,
- $\int_0^T \int_0^T |\alpha(s, u)| ds du < \infty$ for all T,
- $\sup_{0 \leq s, t \leq T} \| \beta(s, t) \| < \infty$ for all T.

We denote

$$\bar{\alpha}(t, T) = \int_t^T \alpha(t, u)du, \quad \bar{\beta}(t, T) = \int_t^T \beta(t, u)du.$$

Theorem

P is an LMM if and only if $dt \otimes dP$-almost surely,

$$\bar{\alpha}(t, T) = \| \bar{\beta}(t, T) \|^2. \tag{2}$$

The proof

Lemma

The zero coupon bond is an Itô-process satisfying

$$\frac{dP(t,T)}{P(t,T)} = \left(f(t,t) - \bar{\alpha}(t,T) + \frac{1}{2} \parallel \bar{\beta}(t,T) \parallel^2\right) dt - \bar{\beta}(t,T) dW_t$$

Proof. We first consider

$$\log P(t,T) = -\int_t^T f(t,u) du = -\int_t^T \left[f(0,u) + \int_0^t \alpha(s,u) ds + \int_0^t \beta(s,u) dW_s \right] du$$

$$= -\int_t^T f(0,u) du - \int_0^t \int_t^T \alpha(s,u) du\, ds - \int_0^t \int_t^T \beta(s,u) du\, dW_s$$

$$= -\int_0^T f(0,u) du - \int_0^t \int_s^T \alpha(s,u) du\, ds - \int_0^t \int_s^T \beta(s,u) du\, dW_s$$

$$+ \int_0^t f(0,u) du + \int_0^t \int_s^t \alpha(s,u) du\, ds + \int_0^t \int_s^t \beta(s,u) du\, dW_s.$$

Moreover note that, using Fubini,

$$\int_0^t \int_s^t \alpha(s,u)du\,ds = \int_0^t \int_0^t \mathbb{1}_{\{s<u\}}\alpha(s,u)du\,ds = \int_0^t \int_0^u \alpha(s,u)ds\,du$$

and similarly for the other integrals in the last line. Summarizing, the last line equals

$$\int_0^t f(0,u)du + \int_0^t \int_s^t \alpha(s,u)du\,ds + \int_0^t \int_s^t \beta(s,u)du\,dW_s$$

$$= \int_0^t f(0,u)du + \int_0^t \int_0^u \alpha(s,u)ds\,du + \int_0^t \int_0^u \beta(s,u)dW_s\,du = \int_0^t f(u,u)du.$$

Hence,

$$\log P(t,T) = -\int_0^T f(0,u)du - \int_0^t \bar{\alpha}(s,u)ds - \int_0^t \bar{\beta}(s,T)dW_s + \int_0^t f(s,s)ds. \quad (3)$$

Applying the Itô-formula and using $f(s,s) = r(s)$ yields that

$$dP(t,T) = P(t,T)\Big[\Big(f(t,t) - \int_t^T \bar{\alpha}(t,u)du + \frac{1}{2}\parallel \bar{\beta}(t,T)\parallel^2\Big)dt - \bar{\beta}(t,T)dW_t\Big]$$

and we conclude. ∎

The theorem immediately follows as

$$dP(t,T) = P(t,T)\Big[\Big(f(t,t) - \int_t^T \bar{\alpha}(t,u)du + \frac{1}{2}\parallel \bar{\beta}(t,T) \parallel^2 \Big)dt - \bar{\beta}(t,T)dW_t\Big]$$

is (because $\bar{\beta} \in \mathscr{L}$) a local martingale (after discounting), if and only if

$$\int_0^s \Big(f(t,t) - \int_t^T \bar{\alpha}(t,u)du + \frac{1}{2}\parallel \bar{\beta}(t,T) \parallel^2 \Big)dt = \int_0^s r_t dt, \qquad (4)$$

for all $0 \le s \le T$. For $T = s$ we obtain $r_t = f(t,t)$ and thereafter the remaining drift condition (2).

Multiple yield curve markets

- Central instruments are **forward rate agreements** (FRA): the fixation of a rate on the future interval $[T, S]$. If bond prices are sufficiently liquid (and not risky), one obtains the "classical" FRA rate[1]

$$F(t, T, S) = \frac{1}{S - T} \left(\frac{P(t, T)}{P(t, S)} - 1 \right).$$ (5)

- If the bonds carry credit risk, this is no longer possible and one considers **multiple yield curves**.

- The literature on multiple yield curve models is huge:**Heath-Jarrow-Morton (HJM)**-like approaches have been considered in Crépey, Grbac, and Nguyen (2012), Crépey, Grbac, Ngor, and Skovmand (2014), Moreni and Pallavicini (2014) as well as in Cuchiero, Gnoatto and Fontana (2016).

- A nice survey: $\rightarrow$ Grbac and Runggaldier (2015).

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[1]See the first example.

NA starting points: the central instruments are FRAs

- In a FRA, a discretely compounded rate is exchanged with payments based on a fixed rate K. Denote its price by $\Pi^{\mathrm{FRA}}(t, T, \delta, K)$.
- Denote the (spot) Libor rate (at T) for $[T, T + \delta]$ by $L(T, T, \delta)$.
- The **forward Libor rate** $L(t, T, \delta)$ is the unique K, such that

$$\Pi^{\mathrm{FRA}}(t, T, \delta, K) = 0. \tag{6}$$

- At maturity T,

$$\Pi^{\mathrm{FRA}}(T, T, \delta, K) = (1 + \delta L(T, T, \delta)) - (1 + \delta K),$$

- Discounting, we arrive at the fundamental equation No 1:

$$\Pi^{\mathrm{FRA}}(t, T, \delta, K) = (1 + \delta L(t, T, \delta)) P(t, T + \delta) - \bar{K}(\delta) P(t, T + \delta)$$
$$:= S_t(\delta) P(t, T, \delta) - \bar{K}(\delta) P(t, T + \delta).$$

Dynamic multiple term-structures

- We assume
$$P(t, T, \delta) = \exp \left(- \int_{(t,T]} f(t, u, \delta) \mu(du) \right),$$
with the convention that $(t, t] = \emptyset$ and $\delta \in \{\delta_1, \ldots, \delta_N\} =: \mathscr{D}$.

- It is time to improve notation: we assume (as general as the previous approach) that forward rates are given by
$$f(t, T, \delta) = f(0, T, \delta) + \int_0^t \beta(s, T, \delta) dX_s. \tag{7}$$

where X is an Itô process of the form
$$X_t = X_0 + \int_0^t a_s^X ds + \int_0^t b_s^X dW_s. \tag{8}$$

We denote the local exponent of X by
$$\Psi_t(u) = u a_t^X + \frac{1}{2} \parallel u b_t^X \parallel^2 .$$

We assume that the multiplicative spread process satisfies

$$S_t(\delta) = S_0(\delta) \exp\left(\int_0^t a_s(\delta)ds - \frac{1}{2}\int_0^t \|b_s(\delta)\|^2 ds + \int_0^t b_s(\delta)dW_s\right),$$

for all $0 \leq t \leq T^*$. Finally we set for all $0 \leq t \leq T \leq T^*$ and $\delta \in \{0, \delta_1, \ldots, \delta_N\}$:

$$\bar{\beta}(t, T, \delta) := \int_{(t,T]} \beta(t, u, \delta)du$$

where $\beta(t, u, 0) = \beta(t, u)$ corresponds to the risk-free case.

Summarizing, we consider all FRAs as traded assets as well as the risk-free bonds (those can be thougt of being the term-structure associated with OIS-rates. It is no difficulty to include additionally credit risk here).

Theorem

P is an LMM if and only if:

1. $r_t = f(t, t, 0)$, *for a.e.* $0 \le t \le T^*$
2. *for every* $T \in [0, T^*]$ *and a.e.* $0 \le t \le T$ *the* **drift condition** *for the risk-free curve,*

$$0 = \Psi_t(-\bar{\beta}(t, T, 0)) \tag{9}$$

holds

3. $f(t, t, \delta) = f(t, t, 0) - a_t(\delta)$, *for a.e.* $0 \le t \le T^*$
4. *for every* $T \in [0, T^*]$ *and a.e.* $0 \le t \le T$ *the* **drift condition** *for the tenor* δ,

$$0 = \Psi_t(-\bar{\beta}(t, T, \delta)) - \langle \bar{\beta}(t, T, \delta) b_t^X, b_t(\delta) \rangle, \tag{10}$$

holds.

The drift condition in the semimartingale exactly looks like (9) for $\delta = 0$. For other δ's the case is more complicated $\rightarrow$ Fontana and Schmidt (2016).

The proof only requires in addition to the previous results to consider

$$S_t(\delta)P(t,T,\delta)$$

instead of simply $P(t,T,\delta)$. This again can be achieved by the Itô-formula.
It will prove useful to utilize the following results on stochastic exponentials.
The stochastic exponential $Z = \mathscr{E}(X)$ of a semimartingale solves the equation

$$dZ_t = Z_{t-}dX_t$$

with $Z_0 = 1$. Examples are

$$Z_t = \exp\left(\int_0^t b_s dW_s - \frac{1}{2} \int_0^t \| b_s \|^2 \right)$$

if $X = W$ and $b \in \mathscr{L}$ (Proof: Itô-formula). Z is in general only a local
martingale (if X is).

We have the following: If X is the Itô-process from (8), with $X_0 = 0$ then[2]

$$e^X = \mathscr{E}(\bar{X})$$

with

$$\bar{X} = X + \frac{1}{2} \int_0^{\cdot} \langle X \rangle_s^2 ds.$$

Moreover, Yor's formula (see JS, II.8.19):

$$\mathscr{E}(X)\mathscr{E}(Y) = \mathscr{E}(X + Y + \langle X, Y \rangle)$$

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[2]Jacod and Shiryaev (2003) will be abbreviated JS. This follows from JS, II.8.10.

Now we are ready for the Proof:

First, recall from (3) that

$$I_t := -\int_t^T f(t,u,\delta)du = -\int_0^T f(0,u,\delta)du - \int_0^t \bar{\beta}(s,T,\delta)dX_s + \int_0^t f(s,s,\delta)ds$$

and we obtain $P(.,T,\delta) = P(0,T,\delta)\mathscr{E}(\bar{I})$ with

$$\bar{I}_t = I_t - I_0 + \frac{1}{2}\int_0^t \| \bar{\beta}(s,T,\delta)b_s^X \|^2 ds.$$

Similarly, $S(\delta) = S_0(\delta)\mathscr{E}(\bar{J})$ with

$$\bar{J}_t = \int_0^t a_s(\delta)ds + \int_0^t b_s(\delta)dW_s.$$

Hence,

$$\frac{S(\delta)P(.,T,\delta)}{S_0(\delta)P(0,T,\delta)} = \mathscr{E}(\bar{I}+\bar{J}+<\bar{I},\bar{J}>)$$

with

$$<\bar{I},\bar{J}>=<I,J>=-\int_0^{\cdot} \langle\bar{\beta}(s,T,\delta)b_s^X,b_s(\delta)\rangle ds.$$

Summarizing, after discounting SP is a local martingale if and only if the drifts vanish, i.e.

$$0 = \int_0^t \left(f(s, s, \delta) - r_s + a_s(\delta) + \Psi_s(-\bar{\beta}(s, T, \delta)) - \langle \bar{\beta}(s, T, \delta) b_s^X, b_s(\delta) \rangle \right) ds$$

and we obtain (beause $r_t = f(t, t, 0)$ - the risk-free market is also arbitrage free)

$$f(t, t, \delta) = f(t, t, 0) - a_t(\delta)$$
$$0 = \Psi_t(-\bar{\beta}(t, T, \delta)) - \langle \bar{\beta}(t, T, \delta) b_t^X, b_t(\delta) \rangle$$

such that we can conclude. ∎

This machinery can also be applied to credit risk: here τ is a stopping time and the credit risky bond is modeled as

$$P(t,T) = \mathbb{1}_{\{\tau > t\}} e^{-\int_t^T f(t,u)du}.$$

However, jumps need to be taken into account as the default indicator is discontinuous. Also, it is natural to introduce jumps in the default compensator, see $\rightarrow$ Gehmlich and Schmidt (2016).

Before we come back to this we consider market models under multiple yield curves.

Market Models in Multiple Yield Curve Markets

- Starting from a slightly different representation we are able to study market models in general. Consider fundamental equation No 2:

$$\Pi^{\mathrm{FRA}}(t, T, T + \delta, K) = \delta\left(L(t, T, \delta) - K\right)P(t, T + \delta);$$

here $0 \leq t \leq T \leq T^*$, $\delta \in \mathscr{D}$ and, in contrast to the HJM-approach, we only consider maturities $T \in \mathscr{T} = \{T_1, \ldots, T_N\}$.

- We assume that Libor rate is the Itô-process

$$L(t, T, \delta) = L(0, T, \delta) + \int_0^t a^L(s, T, \delta)ds + \int_0^t b^L(s, T, \delta)dW_s, \tag{11}$$

Theorem

P is a LMM if and only if the following conditions hold for all $T \in \mathscr{T}$ and $\delta \in \mathscr{D}$:

1 *for $dt \otimes dP$-almost all $t \leq T$*

$$a^L(t, T, \delta) = \bar{\beta}(t, T + \delta, 0)^\top b^L(t, T, \delta). \tag{12}$$

The drift condition suggests a change of measure and we indeed obtain the following local-martingale condition. Define the density

$$Z_t^{T+\delta} := \frac{1}{X_t^0} \frac{P(t, T+\delta, 0)}{P(0, T+\delta, 0)}, \quad 0 \leq t \leq T + \delta.$$

If this is a true martingale we define $dP^{T+\delta} := Z_{T+\delta}^{T+\delta} dP$ - the so-called $T + \delta$-forward measure.

Proposition

Assume that for each $(\delta, T) \in \mathscr{D} \times \mathscr{T}$ the processes $(Z_t^{T+\delta})_{0 \leq t \leq T+\delta}$ are true martingales. Then P is an LMM if and only if for each $(\delta, T) \in \mathscr{D} \times \mathscr{T}$, the process $(L(t, T, \delta))_{0 \leq t \leq T}$ is a $P^{T+\delta}$-local martingale.

We are able to include all existing market models into our framework.

The proof is quite simple: Recall (with $\tilde{P} = (S^0)^{-1}P$) by NA for $\delta = 0$,

$$d\tilde{P}(t,T,0) = \tilde{P}(t,T,0)\left(-\bar{\beta}(t,T,0)b_t^X dW_t \right).$$

Hence, by integration by parts,

$$dL(t,T,\delta)\tilde{P}(t,T+\delta) = L(t,T,\delta)\,d\tilde{P}(t,T+\delta) + \tilde{P}(t,T+\delta)\,dL(t,T,\delta)$$
$$+ d < L(.,T,\delta), \tilde{P}(.,T+\delta) >_t$$

and we obtain

$$\frac{dL(t,T,\delta)\tilde{P}(t,T+\delta)}{L(t,T,\delta)\tilde{P}(t,T+\delta)} = -\bar{\beta}(t,T+\delta,0)b_t^X dW_t + a^L(t,T,\delta)dt + b^L(t,T,\delta)dW_t$$
$$- \bar{\beta}(t,T+\delta,0)b_t^X b^L(t,T,\delta)dt$$

which is a local martingale if and only if $dt \otimes dP$-a.s.

$$a^L(t,T,\delta) = \bar{\beta}(t,T+\delta,0)b_t^X b^L(t,T,\delta).$$

Towards a general approach in credit risk

- Here we follow Gehmlich and Schmidt (2016). The idea is to incorporate all existing model on the one side and to follow economical facts on the other side.
- Credit risk is, in a first step, described by a stopping time τ. (Default is public information!)
- Consider the default indicator $H_t = \mathbb{1}_{\{\tau \leq t\}}$, $t \geq 0$.
- Then, H is an increasing process (a submartingale) and hence by the Doob-Meyer decomposition there exists the compensator H^p, wich itself is predictable and increasing.
- Then by the Lebesgue decomposition[3]

$$H_t^p = \int_0^t h_s ds + v_t + \sum_{0 < s \leq t} \Delta H_s,$$

here v is singular continuous. We consider $v = 0$ in the following.

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[3]See Lemma 2.1 in Fontana and Schmidt (2016).

A short story on discontinuities

The nature of 'discontinuities' in financial markets may be quite different:

- **Type I discontinuities:** Events with substantial impact that come as a complete surprise (at 'unpredictable' times).
- **Type II discontinuities:** Events with substantial impact that occur at predictable or deterministic times (but with random outcome).

With the exception of firm-value-based credit risk models, most modeling frameworks have focused on type I events: e.g. intensity-based credit risk models; Lévy-based models of asset returns; ...

- However, the economic literature long acknowledge jumps at predictable times, e.g. Piazzesi (2001, 2010) (here deterministic times were considered).
- We could include[4] this in multiple yield curve markets by considering the numéraire

$$S^0 = \exp\left(\int_0^\cdot r_t \mu(dt) \right)$$

where μ is a deterministic and finite measure on $(\mathbb{R}, \mathcal{B})$.

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[4]See Fontana et al. (2017).

For credit risk, we consider

$$P(t,T) = \mathbb{1}_{\{\tau > t\}} \exp\left(- \int_{(t,T]} f(t,u)\mu(du) \right)$$

where - for simplicity - $\mu(du) = du + \sum_{i=1}^{n} \delta_{T_i}(du)$.

Theorem

P is a LMM if and only if on $\{\tau > t\}$,

1. $f(t,t) = r_t + h_t$, $dP \otimes dt$-a.s. for $t \in [0, T^*]$;
2. $\Delta H_t^p = 0$ for any $t \in \{T_1, \ldots, T_n\}^c$ and for $i = 1, \ldots, n$

$$f(T_i, T_i) = -\log(1 - \Delta H_{T_i}^p)$$

3. *for all $0 \le t \le T \le T^*$,*

$$0 = \Psi_t(-\bar{\beta}(t,T))$$

Let's have a look at the proof: First, we define $\bar{H} = 1 - H$ and have

$$d\bar{H} = dM - \int_0^{\cdot} h_s ds - \Delta H^p$$
$$= \bar{H}_- \left(dM - \int_0^{\cdot} h_s ds - \Delta H^p \right).$$

As previously, for $I_t := -\int_{(t,T]} f(t,u)\mu(du)$ with $(t,t] = \emptyset$ we obtain

$$dI_t = -\bar{\beta}(t,T)dX_t + f(t,t)\mu(dt) = -\bar{\beta}(t,T)dX + f(t,t)dt + \sum_{i=1}^n f(T_i,T_i)\delta_{T_i}(dt)$$

By the Itô-formula

$$de_t^I = e^{I_{t-}} \left(dI_t^c + \frac{1}{2}d < I^c >_t + (e^{\Delta I_t} - 1) \right)$$
$$= e^{I_{t-}} \left(\left[f(t,t) + \psi_t(-\bar{\beta}(t,T)) \right]dt + dM_t + (e^{\Delta I_t} - 1) \right)$$

Now we are ready to consider the bond prices themselves, i.e.

$$d(\bar{H}e^{l_t}) = \bar{H}_{t-}de^{l_t} + e^{l_{t-}}d\bar{H}_t + d[\bar{H}, e^l]_t$$

$$= \bar{H}_{t-}e^{l_{t-}} \left(\left[f(t,t) + \psi_t(-\bar{\beta}(t,T)) \right] dt + (e^{\Delta l_t} - 1) \right.$$

$$\left. - h_t dt - \Delta H_t^p - \mathbb{1}_{\{\tau=t\}}(e^{\Delta l_t} - 1) + dM_t' \right)$$

$$= \bar{H}_{t-}e^{l_{t-}} \left(\left[f(t,t) - h_t + \psi_t(-\bar{\beta}(t,T)) \right] dt + \mathbb{1}_{\{\tau>t\}}(e^{\Delta l_t} - 1) - \Delta H_t^p + dM_t' \right)$$

Hence, we have three kind of equations (after discounting):

- $T = t$ implies $f(t,t) = r_t + h_t$
- jumps occur at T_i, and they must vanish, i.e. $e^{\Delta l_t} - 1 = \Delta H_t^p$
- the classical drift condition needs to hold, i.e. $0 = \Psi_t(-\bar{\beta}(t,T))$

and the result is prooved. ∎

Example

Consider $\lambda > 0$, $0 < T_1 < \cdots < T_N$, and positive random variables $\lambda_1', \ldots, \lambda_N'$. Set

$$\Lambda_t = \lambda t + \sum_{T_i \leq t} \lambda_i'.$$

Let E be a standard exponential random variable, independent from Λ, and set

$$\tau = \inf\{t \geq 0 : \Lambda_t \geq E\}.$$

Then $\Delta H_{T_i}^p > 0$ because T_i is a possible default date:

$$P(\tau = T_i) = P(\lambda_i' \geq E) = \mathbb{E}[1 - \exp(-\lambda_i')].$$

If Λ is deterministic and $r = 0$, we obtain

$$P(t, T) = P(\tau > T \mid \tau > t) = \mathbb{1}_{\{\tau > t\}} \exp\left(-\lambda(T - t) - \sum_{T_i \in (t, T]} \lambda_i'\right)$$

Note that

$$H_t^p = \lambda(t \wedge \tau) + \sum_{i : T_i \leq (t \wedge \tau)} (1 - e^{-\lambda_i'})$$

Applications

There are many interesting implications of the HJM-approach:

1. The drift condition is the starting point for affine and polynomial models of the term structure
2. Positivity of the forward rates could be analyzed as well as
3. finite-dimensional realizatons,
4. applications to energy markets, foreign exchange, ...
5. consistency and many more,

Many thanks for your attention !

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