Applied Conic Finance

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Conic Finance Explained and Applied

- Bid and ask price modelling introduced via conic trees
- Fundamental theory: Acceptability and coherent risk measures
- Bid and ask prices using distorted expectations
- Applications covered here:
    - Conic MC
    - Implied Liquidity
    - Conic CVA and DVA
- More applications not covered here:
    - Conic Portfolio Theory
    - Conic Hedging
    - Conic Trading

Conic Finance Explained and Applied

In a modern financial economy all risks cannot be eliminated.

Perfect hedging is not possible and some risk exposures must be tolerated.

Hence we need to define the set of acceptable risks as a primitive of the financial economy.

We will later on define the set of acceptable risks and this set will be a cone, hence the name ***Conic Finance***.

Conic Finance Explained and Applied

To make things easy, we will focus on zero cost cash flows.

If we want to a price a derivative, say an European Call option, with payoff $(S_T - K)^+$, the zero cost cash flows we typically look at are of the form :

$(S_T - K)^+ - \exp(rT)\, b$

- Here we essentially agree to pay <u>at time T</u> a cash amount ***exp(rT) b*** (which is equivalent with a cash amount ***b*** at time zero) and receive the payoff.

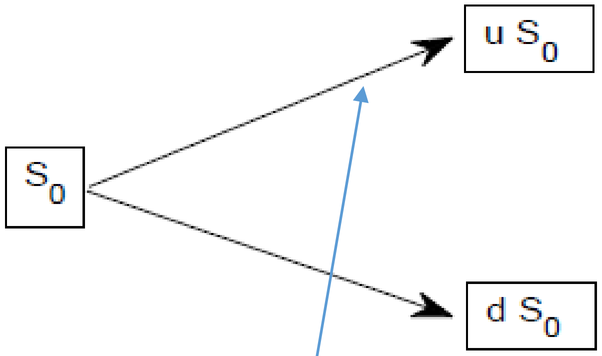
$a \exp(rT) - (S_T - K)^+$

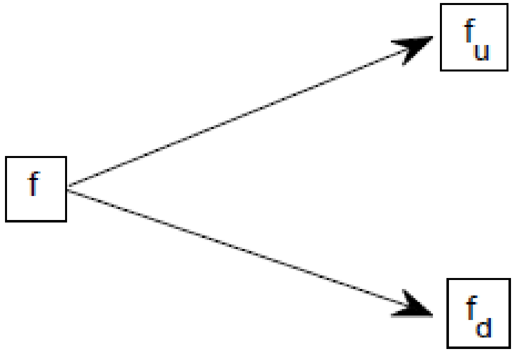
- Here we essentially agree to receive <u>at time T</u> a cash amount ***exp(rT) a*** (which is equivalent with a cash amount ***a*** at time zero) and payout the payoff.

<u>**Essentially:**</u>

- The **bid price** or the price the market wants to pay for a risk X, will be the price b, such that $X - \exp(rT)\, b$ is "acceptable" and any other higher price will make it unacceptable.
- The **ask price** or the price the market wants to receive for a risk X, will be the price a, such that $\exp(rT)\, a - X$ is "acceptable" and any other lower price will make it unacceptable.

Binomial Trees





$$p = \frac{\exp(r\Delta t) - d}{u - d}$$

$$f = \exp(-r\Delta t)\,(p f_u + (1-p) f_d)$$

Conic Binomial Trees

European Call

$$f_d < f_u$$

Increase probability p

$$\text{ask} = \exp(-r\Delta t)\,(\Psi(p) f_u + (1 - \Psi(p)) f_d)$$

Ask > Risk-neutral price

Conic Binomial Trees

European Put

$$f_d > f_u$$

Increase probability *p*

$$\text{ask} = \exp(-r\Delta t)\left(\Psi(p) f_u + (1 - \Psi(p)) f_d\right)$$

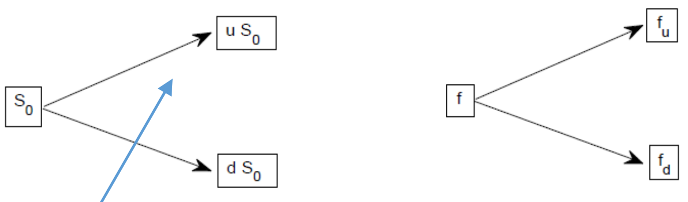
Ask < Risk-neutral price

Increase probability *1-p*

$$\text{ask} = \exp(-r\Delta t)\left((1 - \Psi(1-p)) f_u + \Psi(1-p) f_d\right)$$

Ask > Risk-neutral price

To get an ask price: increase the probability of the state with the highest payoff.



$$p = \frac{\exp(r\Delta t) - d}{u - d}$$

Conic Binomial Trees

Increase probability *1-p*

European Call

$$\text{bid} = \exp(-r\Delta t)\left((1-\Psi(1-p))f_u + \Psi(1-p)f_d\right)$$

$$f_d < f_u$$

bid < Risk-neutral price

Increase probability *p*

European Put

$$\text{bid} = \exp(-r\Delta t)\left(\Psi(p)f_u + (1-\Psi(p))f_d\right)$$

$$f_d > f_u$$

bid < Risk-neutral price

To get a bid price: increase the probability of the state with the lowest payoff.

Bid and ask pricing

	risk-neutral	bid	ask
$f_u > f_d$	p	$1 - \Psi(1-p)$	$\Psi(p)$
$f_d > f_u$	p	$\Psi(p)$	$1 - \Psi(1-p)$

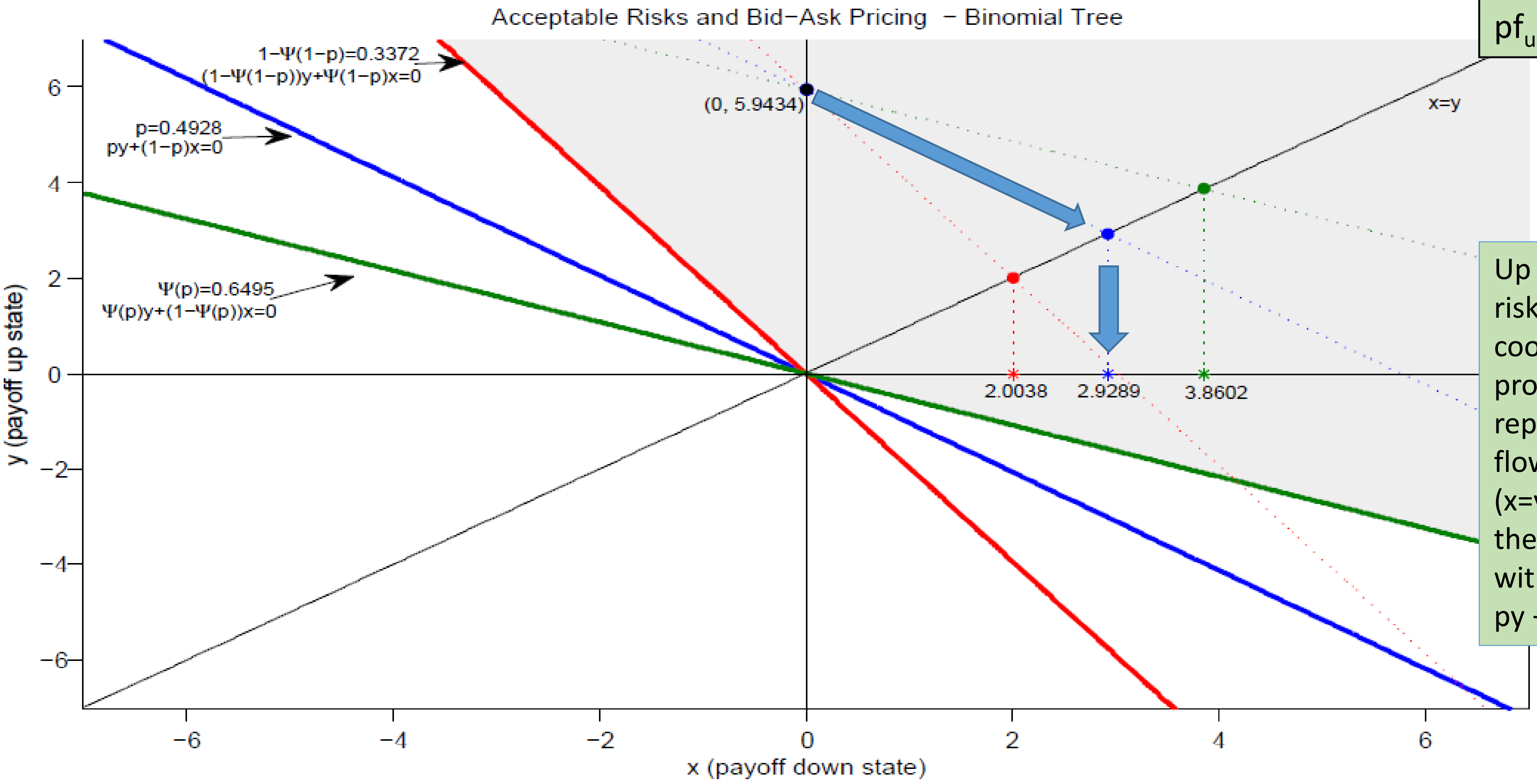
- Bid and ask pricing is payoff dependent !
- **For the ASK: the probability of the higher payoffs is increased and the lower payoff's probabilities are decreased.**
- **For the BID: the probability of the lower payoffs is increased and the higher payoff's probabilities are decreased.**
- The market does this since the BID is the price at which it will go long the risk and hence it values the risk in a prudent fashion by giving the downside more weight.
- The ASK is the price at which it will go short the risk and hence it values the risk in a prudent fashion by giving the upside more weight.

Geometrical interpretation

$py + (1-p)\,x = 0$

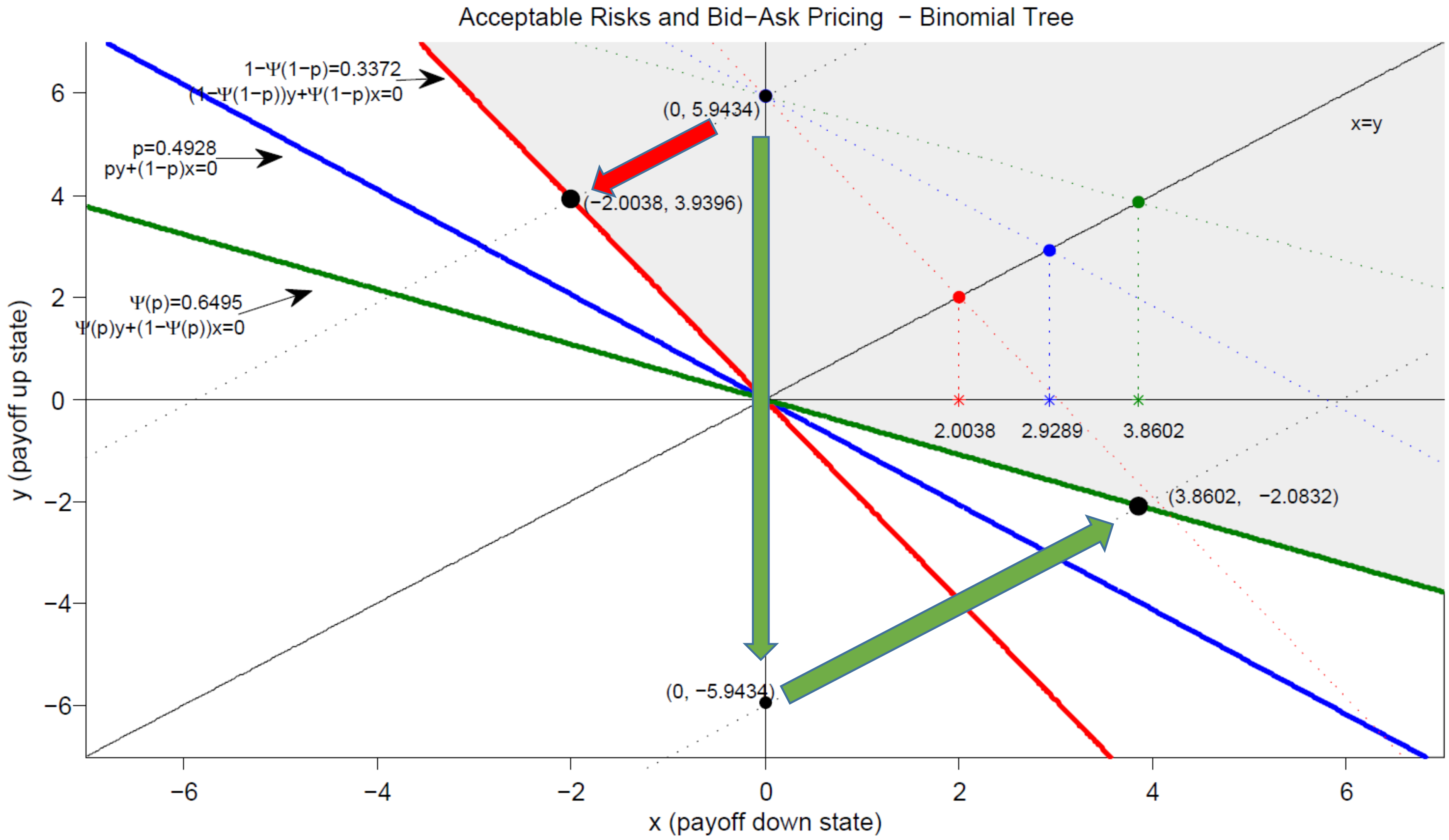
$$\begin{cases} p(y-f_u)+(1-p)(x-f_d)=0 \\ x=y \end{cases}$$

$pf_u-(1-p)\,f_d = x = y$



Up to discounting, the risk-neutral price is the coordinate of the projection of the point representing the cash flow on the diagonal (x=y) along the slope of the "risk-neutral"-line with equation $py + (1-p)\,x = 0$

Geometrical interpretation



Subtracting from the original payoff the undiscounted bid price gives us a payoff with a zero bid price.

$(S_T - K)^+ - \exp(rT)\, b$

Subtracting more will give us a negative price which is unacceptable.

Adding on the negative of the original payoff the undiscounted ask price gives us a payoff with a zero ask price

$a \exp(rT) - (S_T - K)^+$

Adding less will give us a negative price which is unacceptable.

Risks and Risk Measures

- A **risk measure** is nothing but a functional *ρ(.)* that assigns a nonnegative real number to a risk *X*.
- Large values *ρ(X)* will tell us that *X* is very risky. If we think of *X* as a derivative's payoff, then potentially a very large payout needs to be paid out.
- If you promised *X*, you should see *ρ(X)* as the amount of cash (that you'll receive at time *T*) that should be added as a buffer so that the risk to pay out this potential large payoff becomes "**acceptable**".
- The discounted amount of that cash is basically the (ask) price you will charge to take on board that risk.

Coherent Risk Measure

- If a risk-measure satisfies the next four properties we call it a **coherent risk measure**.

1. Translativity

$$\rho(X + c) = \rho(X) + c.$$

You can read this as the cash, $\rho(X + c)$, that needs to added as a buffer so that $X + c$ becomes "acceptable", equals the cash to make X acceptable $\rho(X)$ plus the cash c. As a consequence, one could say that cash is risk-free.

2. Subadditivity

$$\rho(X + Y) \leq \rho(X) + \rho(Y).$$

You can read this as the cash, $\rho(X + Y)$, that needs to added as a buffer so that sum $X + Y$ becomes "acceptable", is lower than the cash $\rho(X)$ to make X acceptable plus the cash $\rho(Y)$ to make Y acceptable. Subadditivity reflects the idea that risk can be reduced by diversification.

3. Positive homogeneity

$$\rho(cX) = c\rho(X),\ c > 0.$$

You can read this as the cash, $\rho(cX)$, that needs to added as a buffer so that cX becomes "acceptable", equals c times the cash $\rho(X)$ to make X acceptable. If you buy a certain number of units c in a risk, then the cash to make that position acceptable equals c times the cash to make one unit of X acceptable; taking positions into the same risk is not bringing in diversification.

4. Monotonicity

$$\text{if } P(X \leq Y) = 1, \text{ then } \rho(X) \leq \rho(Y).$$

You can read this as the cash, $\rho(X)$, that needs to added as a buffer so that X becomes "acceptable", is always lower than the cash $\rho(Y)$ to make Y acceptable, if in all states of the world the risk (or payoff to be paid out) X is smaller or equal than the risk (or payoff to be paid out) Y.

Coherent Risk Measure

- One can prove that

$$\rho(X) = \sup_{Q \in \mathcal{M}} E_Q[X],$$

  where $\mathcal{M}$ is a non-empty set of probability measures satisfies the four above properties and hence **is a coherent risk measure**.

- Furthermore Artzner et al. (1999) showed that **each coherent risk measure** (on a finite set of states of nature) **is of this form**.

Arbitrages are Acceptable

- Let us define now the set of **acceptable** zero cost cash flows.

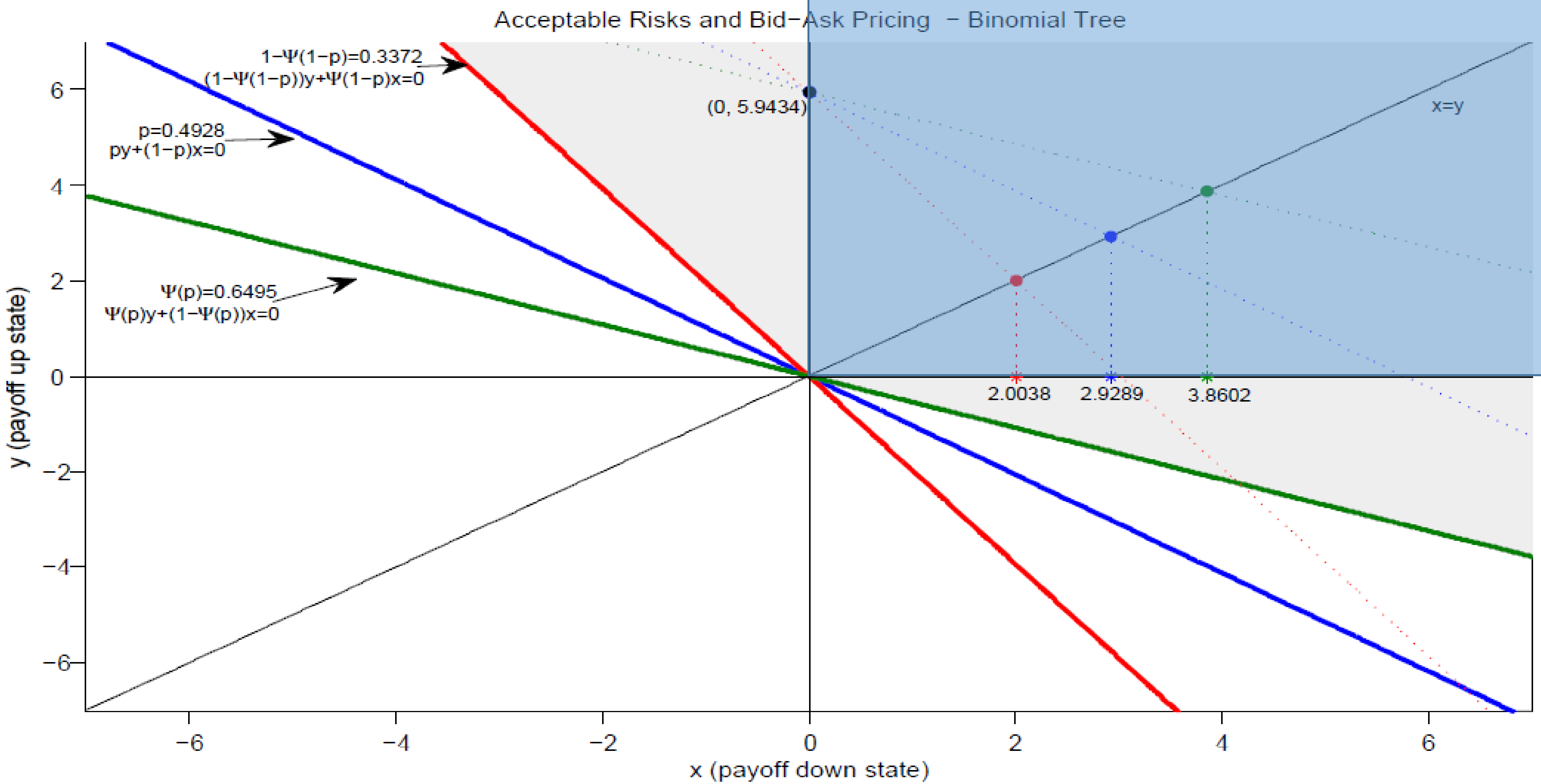
- First, let's first consider the zero cost cash flows with always a nonnegative payout.

Such variables should actually always be acceptable since they are in fact arbitrages.

- However one could say that not only arbitrages are traded and that financial markets accept zero cost cash-flow that are not necessary arbitrages.

Geometrical interpretation

Zero cost arbitrage opportunities have always a positive payout.

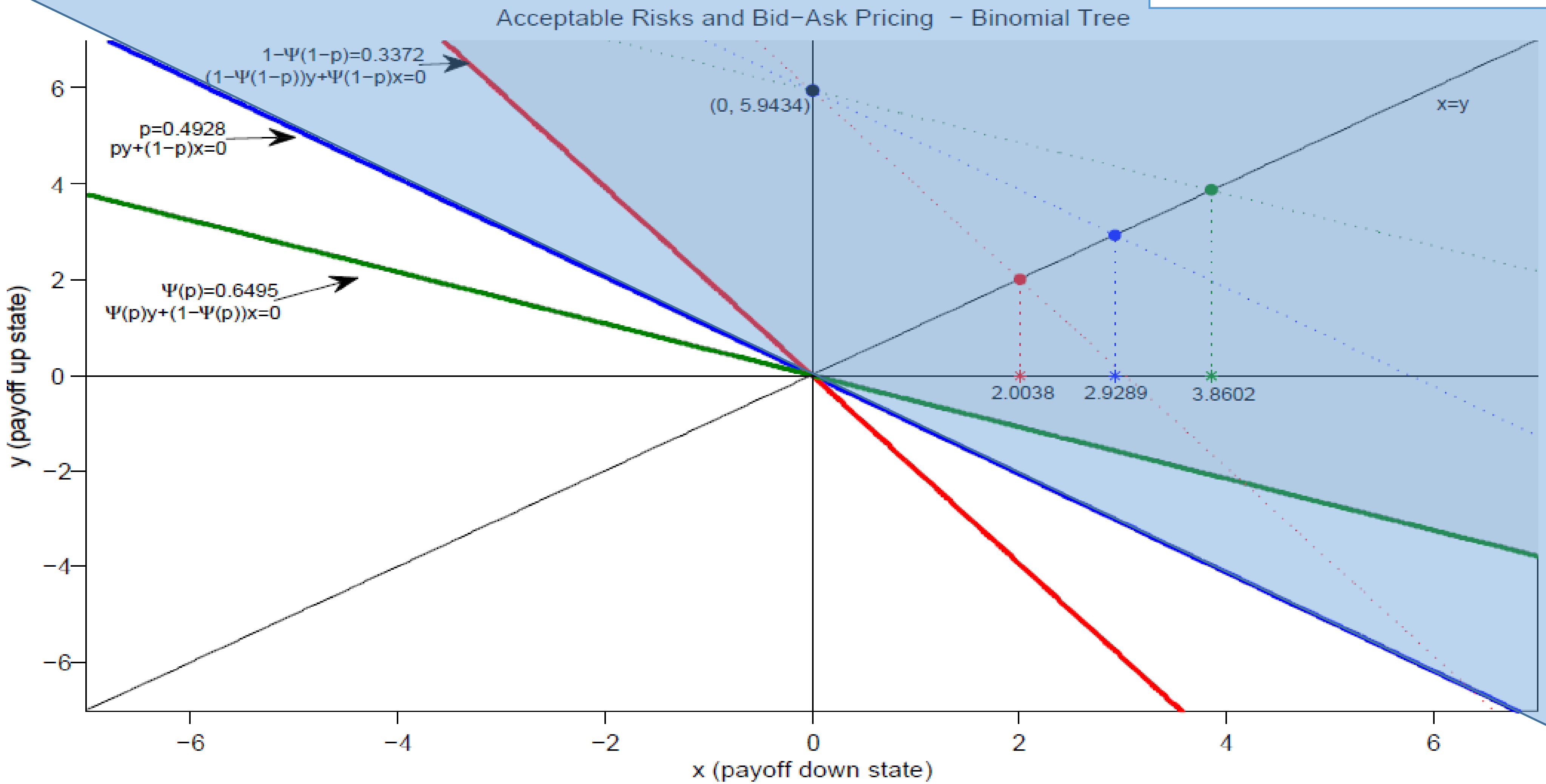


Risk-Neutral Pricing and Acceptability

- Inspired on **risk-neutral pricing**, consider the set of zero cost cash-flows defined as
$$\mathcal{A}^* = \{Z|V(Z) = \exp(-rT)E_Q[Z] \geq 0\}.$$
- Then clearly this set contains the nonnegative random variables.
- The set of acceptable zero cost cash-flows must contain the nonnegative random variables but it is unlikely to be as generous and as large as the half space $\mathcal{A}^*$.
- The set of acceptable zero cost cash-flows of a two price economy will be a proper convex cone containing the nonnegative random variables, but will be not as large as the half space $\mathcal{A}^*$.

Geometrical interpretation

Zero cost acceptable risks from a risk-neutral point of view have a positive risk-neutral expectation.



Acceptable Risks

- Acceptable zero cost cash-flows can be defined by a convex set $\mathcal{M}$ of probability measures whereby

$$Z \in \mathcal{A} \iff \exp(-rT)E_Q[Z] \geq 0 \text{ for all } Q \in \mathcal{M}.$$

- If the market now agrees to buy *X* for the price *b* or it agrees to sell *X* for the price *a* then

$$X - b\exp(rT) \in \mathcal{A} \text{ and } a\exp(rT) - X \in \mathcal{A}.$$

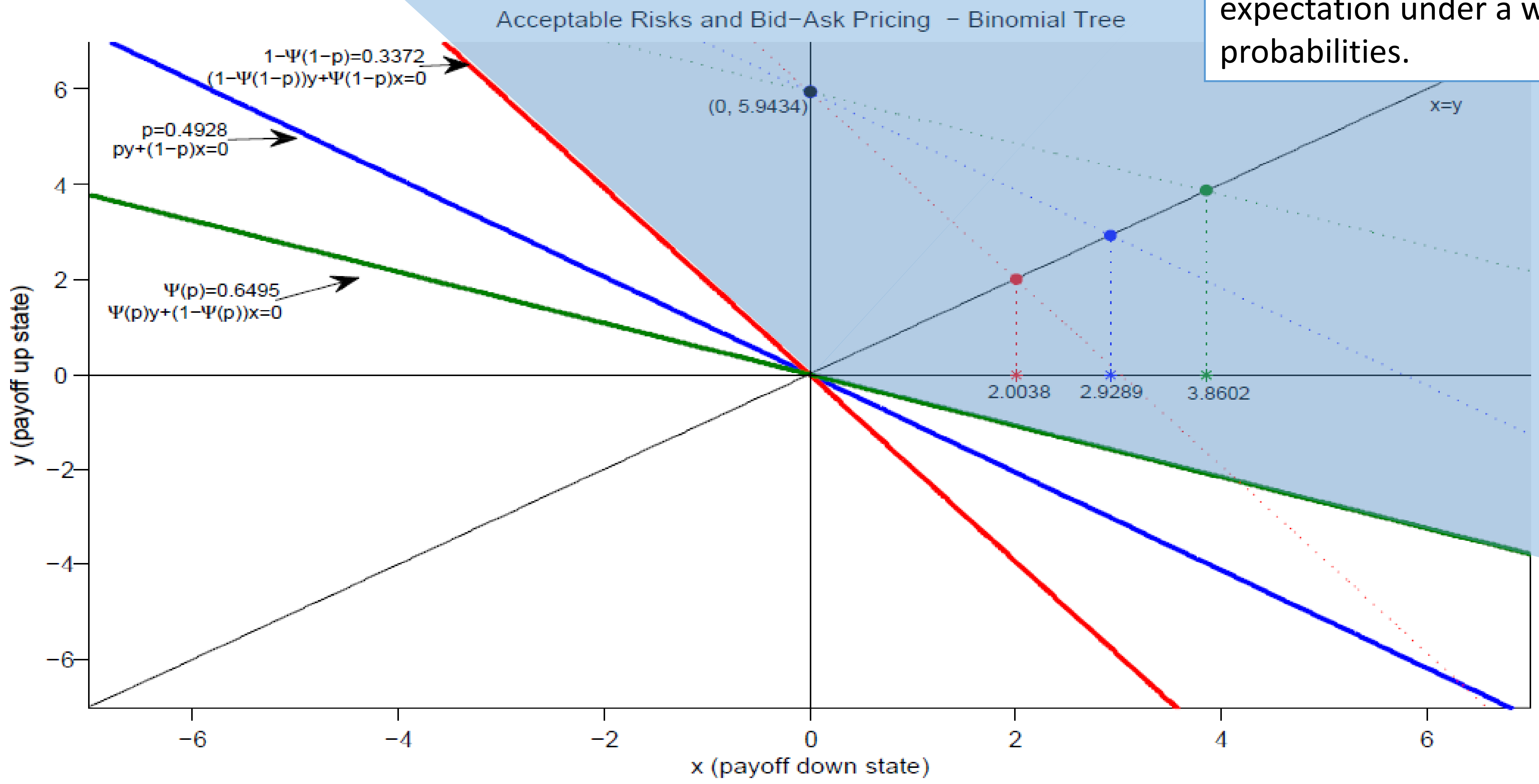
or equivalently for all $Q \in \mathcal{M}$:

$$\exp(-rT)E_Q[X - b\exp(rT)] = \exp(-rT)E_Q[X] - b \geq 0$$

$$\exp(-rT)E_Q[a\exp(rT) - X] = a - \exp(-rT)E_Q[X] \geq 0.$$

Geometrical interpretation

Zero cost acceptable risk from a conic point of view have a positive expectation under a whole set of probabilities.



Bid and Ask Prices

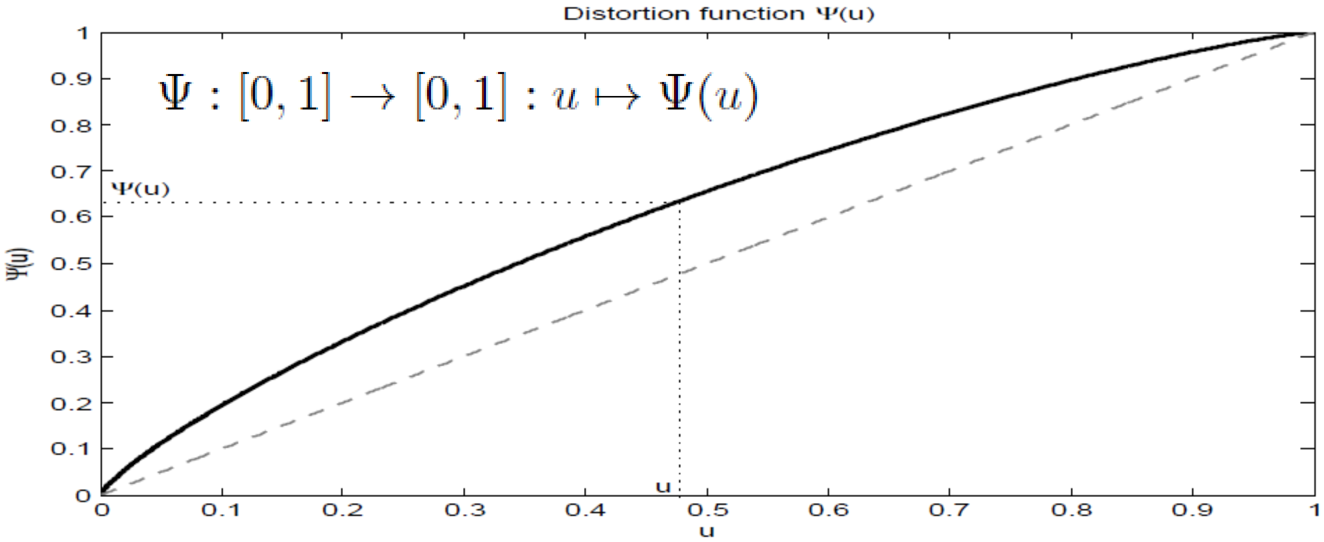
- The best bid and ask prices for *X* provided by the market, denoted *bid(X)* and *ask(X)* respectively, are then given by

$$
\begin{aligned}
\mathrm{bid}(X) &= \exp(-rT) \inf_{Q \in \mathcal{M}} E_Q[X]; \\
\mathrm{ask}(X) &= \exp(-rT) \sup_{Q \in \mathcal{M}} E_Q[X] = \exp(-rT)\rho(X).
\end{aligned}
$$

- Every market is then defined by a convex cone of zero cost cash-flows acceptable to the market, and this cone has associated with it a convex set of probability measures $Q \in \mathcal{M}$ with acceptability equivalently defined as positive expectation under each $Q \in \mathcal{M}$.
- We therefore refer to financial markets for the law of two prices as **conic**, given that they are defined by convex cones of acceptable cash-flows.

Operational cones

- A market model may be constructed by specifying a set of supporting measures and bid and ask prices are then calculated by infimum and supremum of expectations over this set of test measures.
- Operational cones were defined by Cherny and Madan (2009) and employ concave distortion functions.
- A concave distortion function is nothing else than a concave distribution function from the unit interval to itself:



Bid and ask prices as distorted expectations

- Under the hypotheses of co-monotone additivity and a dependence on just the distribution function results of Kusuoka (2001) imply that the bid and ask price must be an expectation under a concave distortion.
- More specially, there must exist a concave distribution (i.e. a distortion) from the unit interval to itself such that for any risk *X* with distribution function $F_X(x)$ we have

$$\mathrm{bid}(X) = \exp(-rT) \int_{-\infty}^{+\infty} x \mathrm{d}\Psi(F_X(x))$$

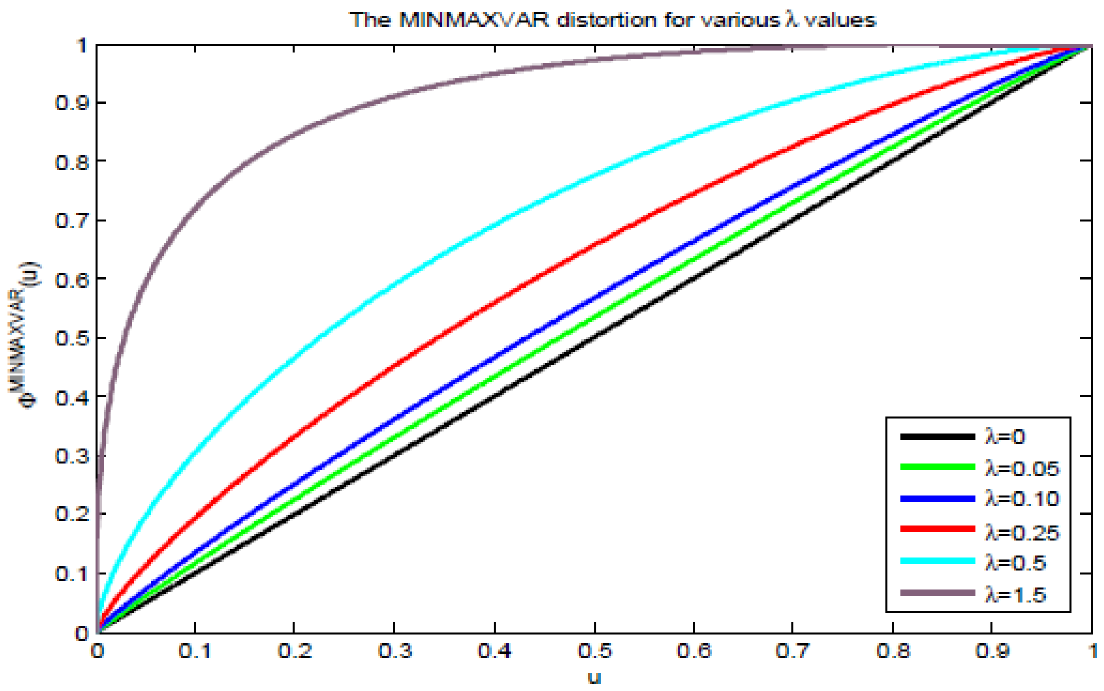
$$\text{risk neutral price}(X) = \exp(-rT) \int_{-\infty}^{+\infty} x \mathrm{d}F_X(x)$$

$$\mathrm{ask}(X) = -\exp(-rT) \int_{-\infty}^{+\infty} x \mathrm{d}\Psi(F_{-X}(x))$$

Some examples of distortion functions

$$\Psi_\lambda^{MINMAXVAR}(u) = 1 - \left(1 - u^{\frac{1}{\lambda+1}}\right)^{1+\lambda}, \qquad \lambda \geq 0.$$

$$\Psi_\lambda^{WANG}(u) = \mathrm{N}\left(\mathrm{N}^{[-1]}(u) + \lambda\right) \lambda \geq 0.$$



Conic vanillas

$$
\begin{aligned}
bidEC &= \exp(-rT)\int_{K}^{+\infty}(x-K)\mathrm{d}\Psi(F_S(x)) \\
askEC &= \exp(-rT)\int_{K}^{+\infty}(K-x)\mathrm{d}\Psi(1-F_S(x)) \\
bidEP &= \exp(-rT)\int_{0}^{K}(x-K)\mathrm{d}\Psi(1-F_S(x)) \\
askEP &= \exp(-rT)\int_{0}^{K}(K-x)\mathrm{d}\Psi(F_S(x))
\end{aligned}
$$

Conic Black-Scholes with Wang

$$F_{(S_T-K)^+}(x) = P((S_T - K)^+ \leq x) \;=\; 1 - P(S_T > K + x) \;=\; 1 - \mathrm{N}\left(\frac{\log(S_0/(K+x)) + (r - q - \sigma^2/2)}{\sigma\sqrt{T}}\right), \quad x \geq 0$$

$$\mathrm{bidEC}(K,T) = \exp(-rT)\int_0^{+\infty} x \mathrm{d}\Psi_\lambda(F_{(S_T-K)^+}(x)).$$

$$\Psi_\lambda^{WANG}(F_S(x)) = N\left(\frac{\log(x) - (\log S_0 + (r - q - \sigma^2/2)T - \lambda\sigma\sqrt{T})}{\sigma\sqrt{T}}\right)$$

$$bidEC(K,T;S_0,r,q,\sigma\;) = EC(K,T;S_0,r,q+\lambda\sigma/\sqrt{T},\sigma)$$

Application: Conic Monte-Carlo

$S_{T,1}$	$S_{T,2}$	$S_{T,3}$	$S_{T,4}$	$S_{T,5}$	$S_{T,6}$	$S_{T,7}$	$S_{T,8}$	$S_{T,9}$	$S_{T,10}$
85.2	103.5	69.4	124.2	108.5	98.5	106.2	125.3	98.4	106.9
payoff_1	payoff_2	payoff_3	payoff_4	payoff_5	payoff_6	payoff_7	payoff_8	payoff_9	payoff_{10}
0	3.5	0	24.2	8.5	0	6.2	25.3	0	6.9
$\text{payoff}_{(1)}$	$\text{payoff}_{(2)}$	$\text{payoff}_{(3)}$	$\text{payoff}_{(4)}$	$\text{payoff}_{(5)}$	$\text{payoff}_{(6)}$	$\text{payoff}_{(7)}$	$\text{payoff}_{(8)}$	$\text{payoff}_{(9)}$	$\text{payoff}_{(10)}$
0	0	0	0	3.5	6.2	6.9	8.5	24.2	25.3

Instead of assigning a weight 1/N to each payoff we will assign a distorted weight to it.
The distorted cdf is approximated by distorting the empirical cdf.

$$\text{payoff}_{(1)} \leq \text{payoff}_{(2)} \leq \cdots \leq \text{payoff}_{(N)},$$

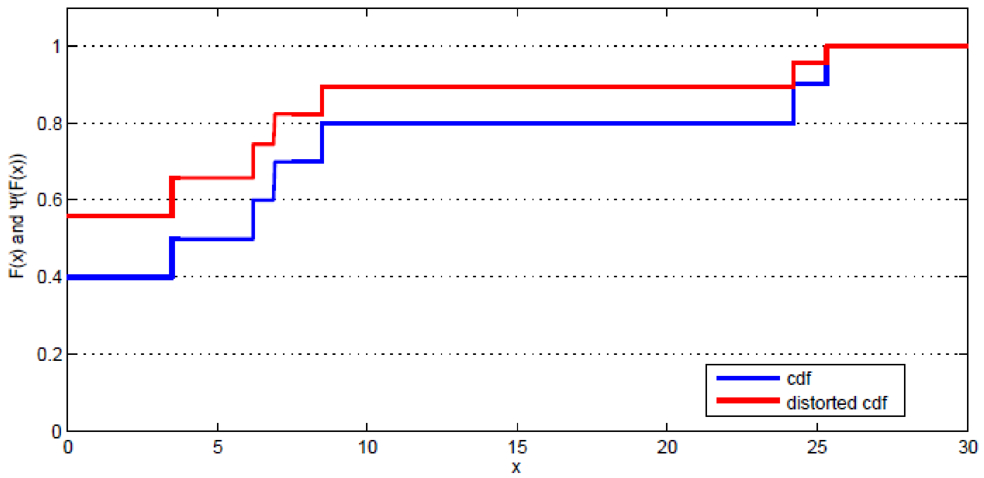
$$\Psi\left(\bar{F}_{\text{payoff}}(\text{payoff}_{(i)})\right) = \Psi\left(\frac{i}{N}\right)$$

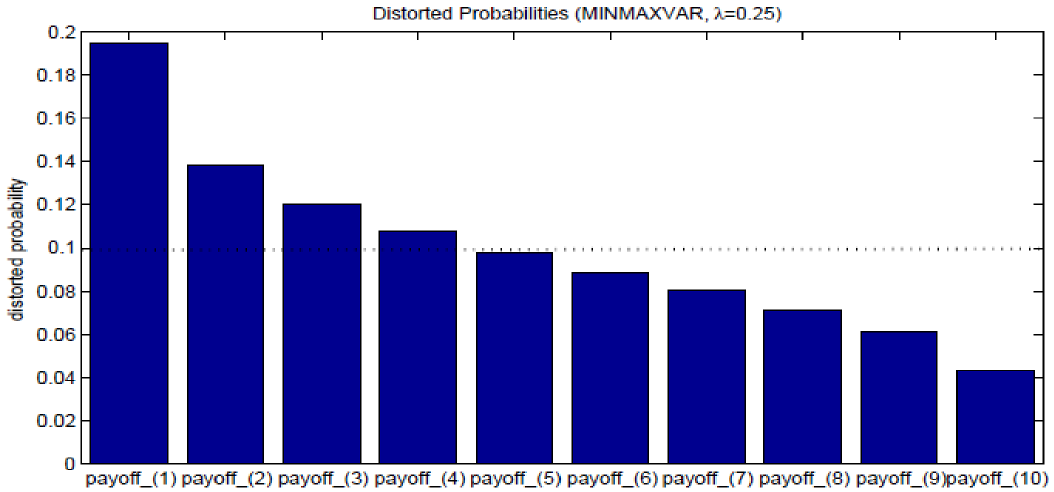
Bid: $\tilde{p}_i = \Psi\left(\frac{i}{N}\right) - \Psi\left(\frac{i-1}{N}\right)$

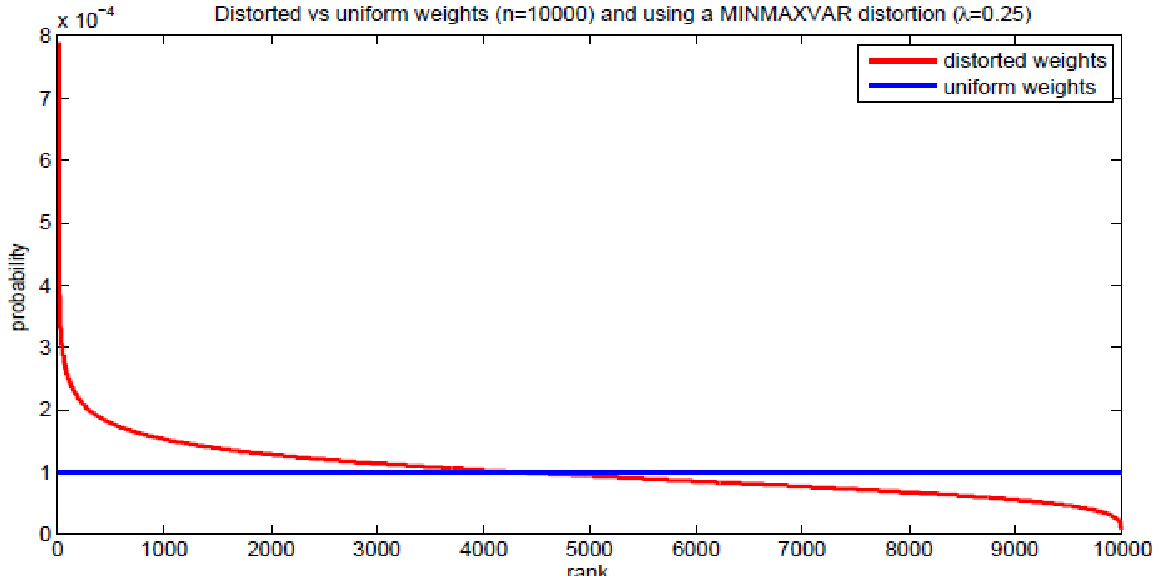
Ask: $\bar{p}_i = \Psi\left(\frac{N-i+1}{N}\right) - \Psi\left(\frac{N-i}{N}\right)$

x	$\text{payoff}_{(1)}$	$\text{payoff}_{(2)}$	$\text{payoff}_{(3)}$	$\text{payoff}_{(4)}$	$\text{payoff}_{(5)}$	$\text{payoff}_{(6)}$	$\text{payoff}_{(7)}$	$\text{payoff}_{(8)}$	$\text{payoff}_{(9)}$	$\text{payoff}_{(10)}$
$F(x)$	0.4	0.4	0.4	0.4	0.5	0.6	0.7	0.8	0.9	1
$\Psi(F(x))$	0.5589	0.5589	0.5589	0.5589	0.6562	0.7447	0.8248	0.8960	0.9569	1.0000
p	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
$\tilde{p}_i$	0.1940	0.1381	0.1196	0.1072	0.0973	0.0885	0.0801	0.0713	0.0609	0.0431
$\bar{p}_i$	0.0431	0.0609	0.0713	0.0801	0.0885	0.0973	0.1072	0.1196	0.1381	0.1940

Application: Conic Monte Carlo

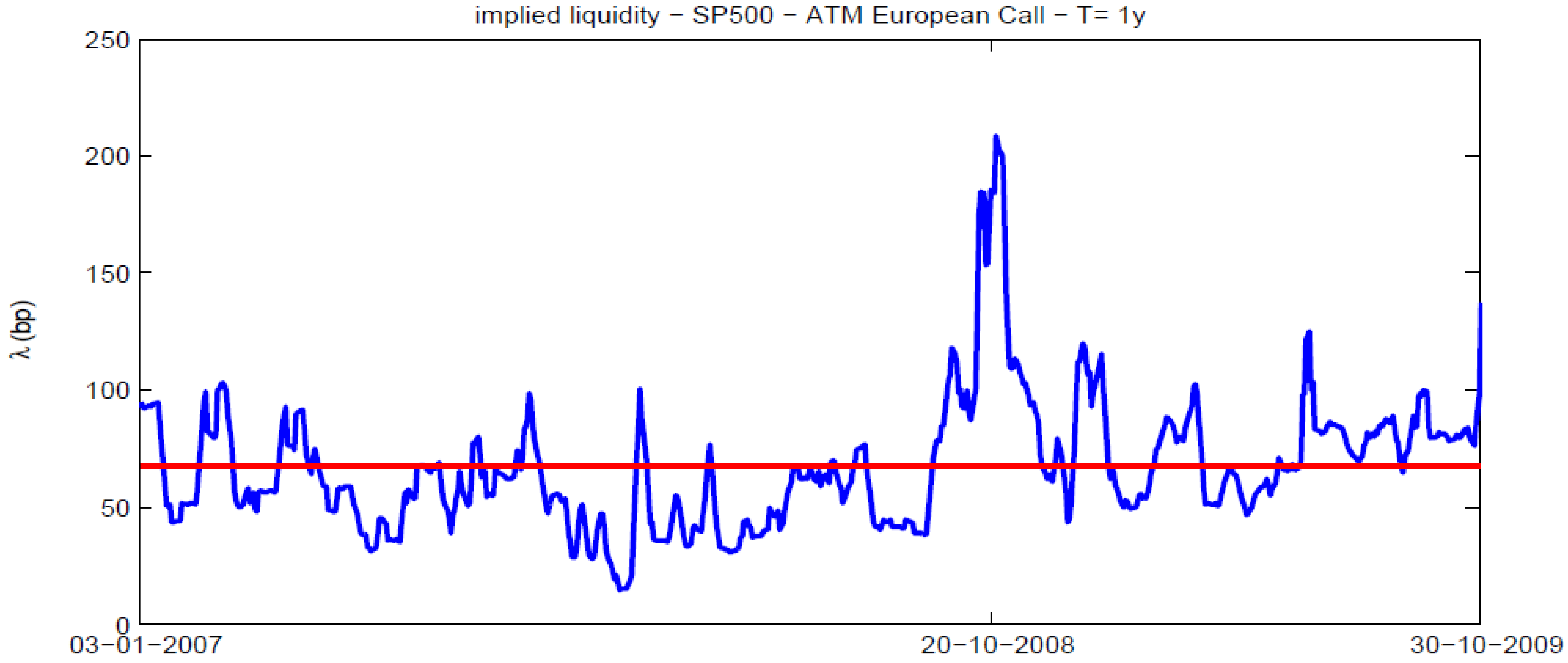






Application: Implied Liquidity

Matching the distorting parameter with a given market spread



Application: Conic DVA and CVA

- Let us consider a very simple zero-coupon defaultable bond.
- In the one-price framework, the value of this bond then equals:

$$ZCB = \exp(-rT)\left(pR + (1-p)\right),$$

- In a two-price world, the default and no-default probabilities are distorted by a distortion function:

$$bidZCB = \exp(-rT)\left(\Psi(p)R + (1-\Psi(p))\right) \quad askZCB = \exp(-rT)\left((1-\Psi(1-p))R + \Psi(1-p)\right).$$

- Example: *T = 1, r = 1%, R = 20%* and *p = 2%, MINMAXVAR2, λ= 0.25; γ=2*

ZCB = 0.9742; bidZCB = 0.8906; askZCB = 0.9900

Traditional CVA and DVA

- CVA equals the difference between the non-defaultable value (*exp(-rT)* = 0.99) and the defaultable value (0.9742)

  → CVA = 0.0158.

- The buyer of the zero coupon bond would make a CVA and will book the bond, which is an asset for the buyer, at the default-free value minus the CVA.
- For the seller, the bond is booked as a liability, at the default free value minus the DVA.
- The seller books the bond as a liability at 0.9742 and hence uses a

  → DVA = 0.0158.

Conic CVA and DVA

- The situation is different in a two price setting. Prudence accounting would mean that **assets are booked at the bid value**, because it is this value the investor would get if he immediately would like to exit its position.
- Similarly, **liabilities are booked at the ask value**, because if one needs to close the position, one needs to buy back the asset and this can be done at the ask value.
- Therefore, the buyer now would book the bond as asset at 0.8906.

  CVA = 0.1094.

- The seller books the bond as a liability at 0.9900.

  DVA = 0

Conic CVA and DVA

- DVA has been criticized a lot because of the counter-intuitive effect that firms can book profits due to their own credit deterioration.
- Indeed, let's recalculate the above prices, but now for a default probability equal to the double of the original value, i.e. now p = 4%.
- We get: ZCB = 0.9584; bidZCB = 0.8225; askZCB = 0.9900.
- For the seller the liability is in the one-price-setting now valued at 0.9584.

$\Rightarrow$ Traditional DVA P&L = 0.0158 *(0.9742 - 0.9584)*

- In the two price world, the "profit" for the seller due to its own credit should be calculated using the ask price

$\Rightarrow$ Conic DVA P&L = 0

Conic Hedging

- We design hedging strategies that maximize the concave bid price for positions held or minimize the convex ask price for positions promised.
- For any set of chosen hedging instruments such optimization problems are control problems related to non-linear valuation functionals.
- The concept of (delta and/or gamma) conic hedging introduced here differs fundamentally from risk management approaches aimed at delta and/or gamma neutrality. The latter seek to zero out certain derivatives of the current value function. We seek to add positions to future value functions that enhance current market values.

Conic Delta Hedging

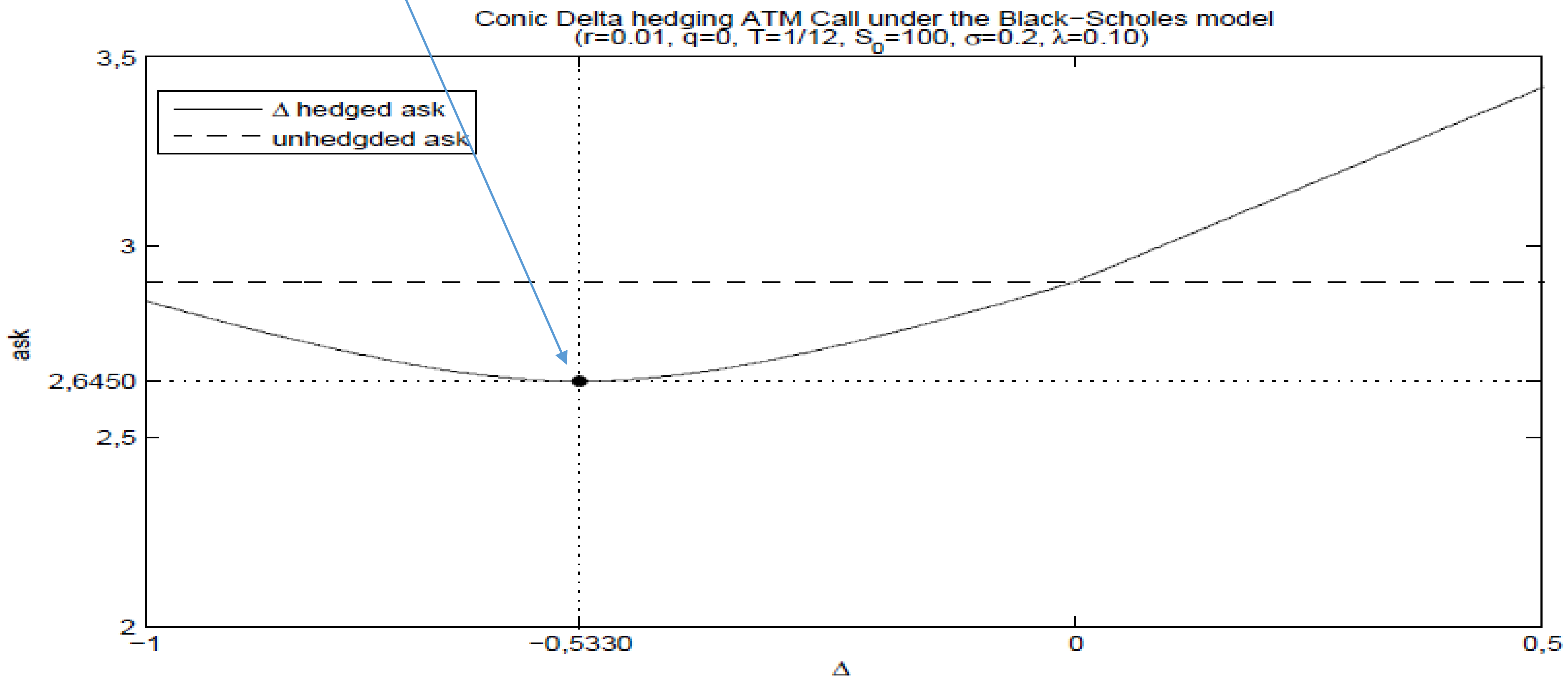
- The key idea of conic hedging is that we wrap the derivative in a package, which has the same risk-neutral price but a more competitive bid or ask price.
- We consider the portfolio of one derivative and a position of *Δ* forward contracts each paying out at maturity *T* the amount

$$(S_T - \exp(rT)S_0)$$

- Note that the risk-neutral upfront price to be paid for such a forward contract is zero and hence the risk-neutral price of the portfolio is unchanged.

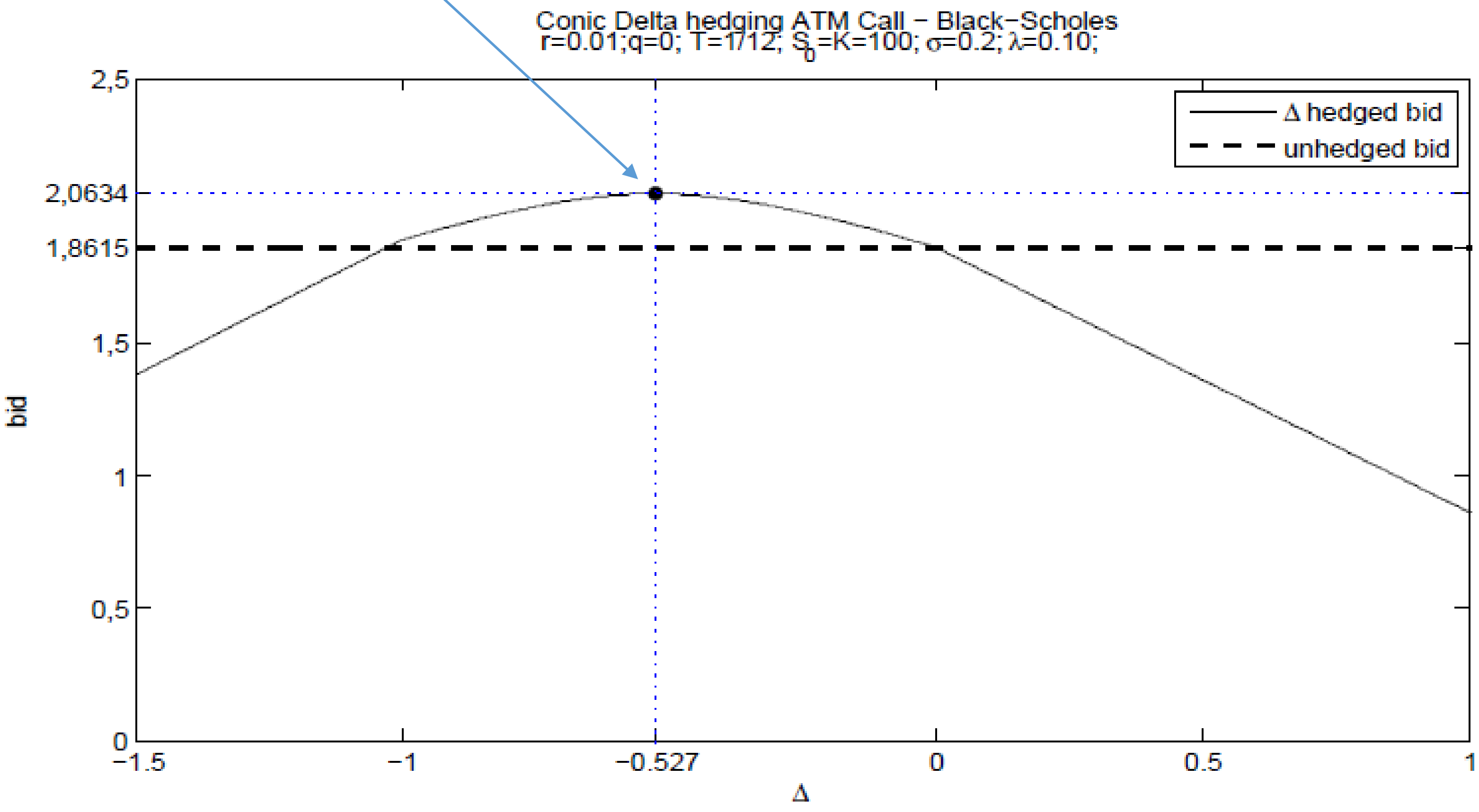
Conic Delta Hedging

Example (ask) : Δ is the value that minimize the ask price of the package consisting out of the option and Δ forward contracts



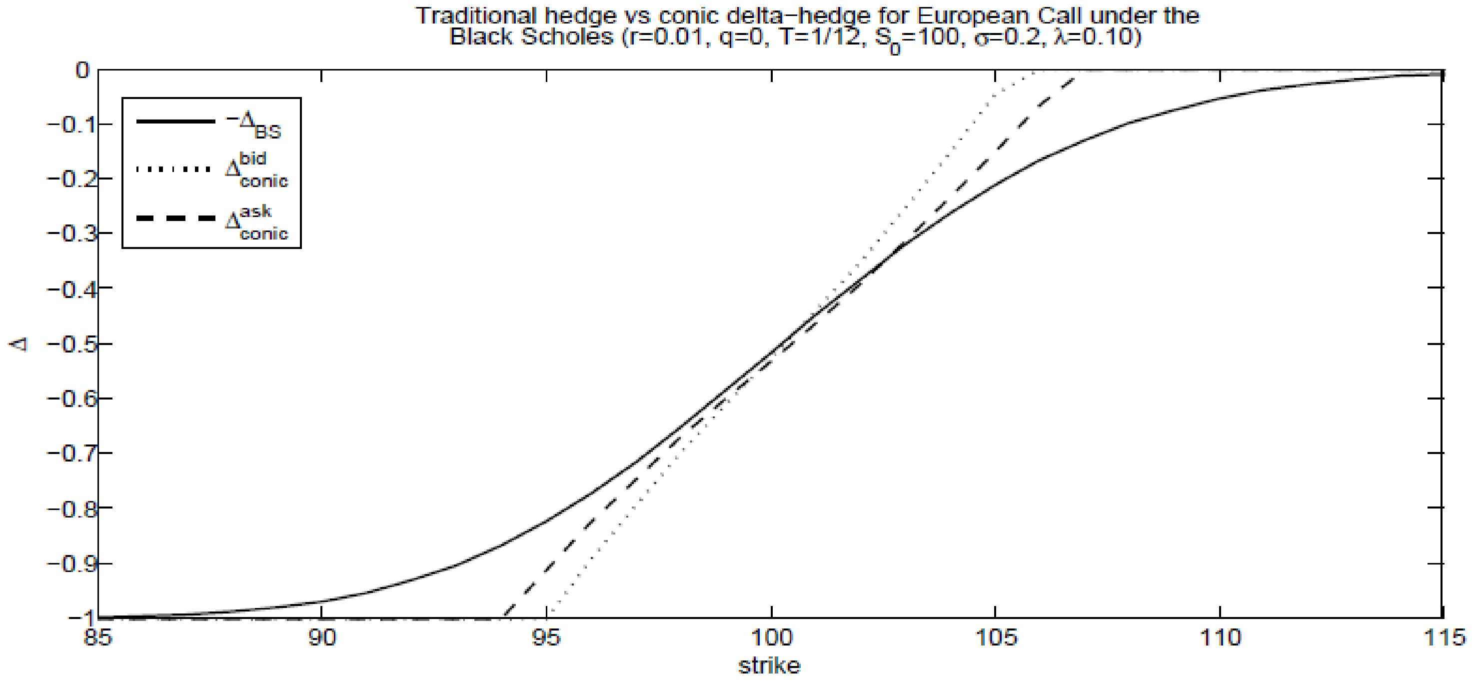
Conic Delta Hedging

Example (bid) : Δ is the value that maximizes the bid price of the package consisting out of the option and *Δ* forward contracts



Conic Delta Hedging

- The traditional delta and the conic delta are completely different concept, with the former operating very locally and on a small time-scale or even instantaneously and the latter operating globally on the final maturity.
- The traditional delta hedging seeks to zero out the first derivative of the current value function. Conic hedging seeks to add positions to future value functions that enhance current market values.

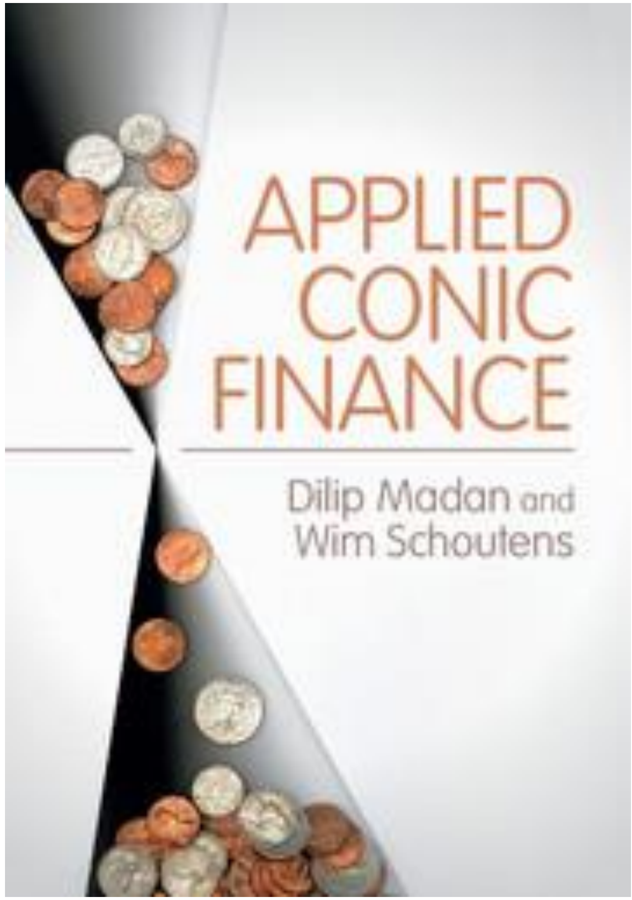


Conclusion

- In traditional financial mathematics the focus of derivative pricing is often solely on the so-called risk-neutral price (cfr. the law of one price), or the (equilibrium) price at which we supposedly can buy and sell.
- However in real markets, one is observing continuously two prices, namely the price at which the market is willing to buy (bid) and a price at which the market is willing to sell (ask).
- Conic Finance is delivering a two-price-theory that is about determining such bid and ask prices in a consistent and fundamentally motivated manner.
- Under the conic finance theory many traditional finance chapters can be rewritten and extended to a two-price setting.

Thank you

- More info: wim@schoutens.be



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