Optimal Derivative Investment for Lévy Systems

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OUTLINE

- Spanning with options
- The investment problem
- Solution using Markov Control for Jump processes
- Explicit solutions and examples.

Market Completion using options

- C(S) an arbitrary twice differentiable cash flow contingent on S the price of an underlying asset at maturity T.
- Let F be an arbitrary expansion point.
- Applying the fundamental theorem of calculus twice we have

$$
\begin{aligned}
C(S) &= C(F) + 1_{S>F}\int_F^S C'(u)du - 1_{S<F}\int_S^F C'(u)du \\
&= C(F) + 1_{S>F}\int_F^S \left[C'(F) + \int_F^u C''(v)dv\right] du \\
&\quad -1_{S<F}\int_S^F \left[C'(F) - \int_u^F C''(v)dv\right] du
\end{aligned}
$$

- Integrating and reversing the order of integration of the double integrals we have

$$
\begin{aligned}
C(S) \;=\; & C(F) + C'(F)(S-F) \\
& +1_{S>F} \int\limits_F^S \int\limits_v^S C''(v)\,du\,dv \\
& +1_{S<F} \int\limits_S^F \int\limits_S^v C''(v)\,du\,dv
\end{aligned}
$$

- This may be rewritten as

$$
\begin{aligned}
C(S) \;=\; & C(F) + C'(F)(S-F) \\
& +1_{S>F} \int\limits_F^S C''(v)(S-v)\,dv \\
& +1_{S<F} \int\limits_S^F C''(v)(v-S)\,dv
\end{aligned}
$$

- or equivalently as

$$\begin{aligned}C(S) \;=\; & C(F) + C'(F)(S-F) \\ & +1_{S>F} \int_F^\infty C''(v)(S-v)^+ dv \\ & +1_{S<F} \int_0^F C''(v)(v-S)^+ dv\end{aligned}$$

- **ASSET ALLOCATION IN OPTIONS MARKETS WHEN THE UNDERLYING IS DRIVEN BY A LÉVY SYSTEM**
  a. SUPPOSE THAT THE STOCK PRICE FOLLOWS STATISTICALLY A JUMP PROCESS WITH LÉVY MEASURE $k_P(x)$.
  b. LET THE RISK NEUTRAL PROCESS ALSO BE A LÉVY PROCESS WITH LÉVY MEASURE $k_Q(x)$.
  c. More explicitly we suppose that under P

$$S(t) = S(0) \exp \left(\mu t + X(t) - \int_{-\infty}^{\infty} (e^x - 1) k_P(x) dx \right)$$

  d. While under Q we have that

$$S(t) = S(0) \exp \left(rt + X(t) - \int_{-\infty}^{\infty} (e^x - 1) k_Q(x) dx \right)$$

e. The measure change process is the P martingale given by

$$\Lambda(t) = \exp\left(-\int_{-\infty}^{\infty}(Y(x)-1)\,k_P(x)dx\right)\prod_{s\le t}Y(\Delta X_s)$$

$$Y(x) = \frac{k_Q(x)}{k_P(x)}.$$

f. Let the wealth response function or exposure design function be given by

$$w(x,u)$$

that is the designed response in log wealth at time u if the stock's log price were to jump at this time by x.

g. Consider the problem of finding the optimal response function w. We formulate this problem for infinite time horizons with intermediate consumption and for finite time horizons with no intermediate consumption.

h. The wealth transition equation is

$$W(t) = W(0) + \int_0^t (rW(u) - c(u))du +$$

$$\int_{-\infty}^{\infty} W(u\_) \left(e^{w(x,u)} - 1\right) (k_P(x) - k_Q(x))dxdu$$

$$+ \int_0^t \int_{-\infty}^{\infty} W(u\_) \left(e^{w(x,u)} - 1\right) (\mu(dx, du) - k_P(x))dxdu$$

THE INVESTOR'S INVESTMENT MANAGEMENT PROBLEM

- The Infinite Time Horizon Investment Management Problem may be formulated as:

$$\max_{[c(\cdot),w(\cdot)]} U = E^{\mathbf{P}}\left[\int_0^\infty \exp(-\beta s)u(c(s))ds\right]$$

Subject to :

$$W(t) = W(0) + \int_0^t rW(s\_)ds - \int_0^t c(s)ds$$

$$+ \int_0^t \int_{-\infty}^\infty W(s\_)(e^{w(x,s)} - 1)(m(\omega; dx, ds) - k_{\mathbf{Q}}(x)dxds).$$

and $W(\infty) \geq 0$ almost surely.

- The random measure m accounts for accumulating the actual wealth changes experienced as a consequence of jump moves in the underlying, and one's chosen positioning.

- The integration with respect to the Lévy density $k_{\mathbf{Q}}(x)$ accounts for the payment of the cost of the

  positioning purchased.

- In contrast to the static problem, such an infinite horizon investor should be more tolerant of risk taking. However, this investor does have a concern for the immediate future as reflected by his contemporaneous consumption needs and in this regard he is like the static investor with a concern for the morrow.
- Many investors enter the market with long term objectives and are not looking to returns to finance their consumption. For this reason we also study the finite but distant time horizon problem with no interimmediate consumption.

- The finite time horizon problem is formulated as:

$$
\begin{aligned}
&\max_{[w(\cdot)]} U = E^{\mathbf{P}}\left[u(W(\Upsilon))\right] \\
&\text{Subject to :} \\
&W(t) = W(0) + \int_0^t rW(s\_)ds \\
&+ \int_0^t \int_{-\infty}^{\infty} W(s\_)(e^{w(x,s)} - 1) \times \\
&(m(\omega; dx, ds) - k_{\mathbf{Q}}(x)dxds), \\
&\text{and } W(\infty) \geq 0 \text{ almost surely.}
\end{aligned}
$$

- There is only one choice variable here and this is the optimal exposure design.

- For the Solution we follow Rishel (1990) on the solution of Continuous Time Markov Control Problems.
- In the infinite horizon case, the infinitesimal generator of the Markov wealth process is:

$$A^{c,w}[\varphi](W) = \varphi_W \left[\begin{array}{c} rW - c- \\ \int\limits_{-\infty}^{\infty} W(e^{w(x)} - 1)k_{\mathbf{Q}}(x)dx \end{array} \right]$$

$$+ \int\limits_{-\infty}^{\infty} \left[\varphi(We^{w(x)}) - \varphi(W) \right] k_{\mathbf{P}}(x)dx.$$

- The optimal controls satisfy the Hamilton, Jacobi and Bellman (HJB) equation
$$A^{c^*,w^*}[J] - \beta J + u(c^*(\cdot)) = 0$$
- Th controls themselves are given by
$$c^*, w^* = \arg\max_{c,w} \left[A^{c,w}[J] - \beta J + u(c)\right]$$
- The first order condition with respect to consumption yields the familiar equation
$$J_W = u'(c)$$

- The first order condition for the exposure design yields
  $$J_W(We^{w(x)})k_{\mathbf{P}}(x) = J_W(W)k_{\mathbf{Q}}(x)$$
- This condition is comparable to that of the static model, except that we now use the ratios of Lévy densities in place of the probability densities.
- For a solution we conjecture a form for the J function, define c and w by
  $$c^* = (u')^{-1}(J_W)$$

$$w^*(x) = \log\left[(J_W)^{-1}\left(J_W(W)\frac{k_{\mathbf{Q}}(x)}{k_{\mathbf{P}}(x)}\right)\right] - \log(W)$$

- Finally we verify the required HJB equation.
- For the finite horizon case the infinitesimal generator is given by:

$$
\begin{aligned}
&A^w[\varphi](t, W) = \varphi_t + \\
&\varphi_W \left[rW - \int_{-\infty}^{\infty} W(e^{w(x,t)} - 1) k_{\mathbf{Q}}(x) dx \right] \\
&+ \int_{-\infty}^{\infty} \left[\varphi(t, We^{w(x,t)}) - \varphi(t, W) \right] k_{\mathbf{P}}(x) dx.
\end{aligned}
$$

- The HJB equation is just
$$A^{w^*}[J] = 0$$
- The optimal exposure design must satisfy
$$w^* = \arg\max_w \left[A^w[J] \right]$$
- The J function must also satisfy the terminal condition
$$J(W, \Upsilon) = u(W)$$
- For this problem we conjecture a form for the J function, define w to satisfy the first order condition as before and then we must show that the ODE for the HJB equation is satisfied.

- The required ODE is

$$0 = J_t + J_W[rW - \int\limits_{-\infty}^{\infty} W(e^{w(x,t)} - 1)k_{\mathbf{Q}}(x)dx] + \int\limits_{-\infty}^{\infty} \left[J(t, We^{w(x,t)}) - J(t,W)\right] k_{\mathbf{P}}(x)dx.$$

HARA INVESTORS IN VG ECONOMIES

- The utility function we employ is
$$u(c) = \frac{\gamma}{1-\gamma}\left(\frac{\alpha}{\gamma}c - A\right)^{1-\gamma}$$
- We have linear risk tolerance and
$$-\frac{u'(c)}{u''(c)} = \frac{c}{\gamma} - \frac{A}{\alpha}$$
with floor consumption of $\gamma A/\alpha$ and cautiousness $1/\gamma$.
- The stock price process is riskneutrally
$$S(t) = S(0)\exp\left(\begin{array}{c} rt + \frac{t}{\nu}\log(1-\theta\nu-\sigma^2\nu/2)+ \\ \theta G(t,\nu) + \sigma W(G(t,\nu)) \end{array}\right).$$
- The Lévy measure for this process is
$$k_{\mathbf{Q}}(x) = \frac{\exp(\theta x/\sigma^2)}{\nu \mid x \mid}\exp\left(-\sqrt{\frac{2}{\nu}+\frac{\theta^2}{\sigma^2}}\frac{\mid x \mid}{\sigma}\right).$$
- For zero θ we have measure that is symmetric about zero.
- The parameter ν gives fatter tails when it is larger.

- The measure goes to infinity near zero and integrates to infinity. So we have an infinite arrival rate of what are necessarily small jumps.
- The statistical price process is given by

$$S(t) = S(0)\exp\left(\mu t + \frac{t}{\kappa}\log(1 - s^2\kappa/2) + sW(G(t,\kappa))\right)$$

- The Lévy density for this process is

$$k_{\mathbf{P}}(x) = \frac{1}{\kappa \mid x \mid}\exp\left(-\sqrt{\frac{2}{\kappa}}\frac{\mid x \mid}{s}\right).$$

- This is a symmetric measure with general properties comparable to $k_{\mathbf{Q}}$.
- For HARA utility and VG price processes we solve the required conditions for both the finite and infinite time horizon problems and observe that in the infinite horizon case

$$\frac{k_{\mathbf{Q}}(x)}{k_{\mathbf{P}}(x)} = Y(x) = \frac{\kappa}{\nu}\exp\left(\zeta x + \lambda \mid x \mid\right)$$

where

$$\zeta = \frac{\theta}{\sigma^2}$$

$$\lambda = \frac{\sqrt{\frac{2}{\kappa}}}{s} - \frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma}$$

- The optimal exposure design has the form

$$\begin{aligned}
w(x) &= \log\left[\Delta_1 + \Delta_2 \exp\left(-\frac{\zeta}{\gamma}x - \frac{\lambda}{\gamma} \mid x \mid\right)\right] \\
\Delta_1 &= \frac{\gamma}{\eta}\frac{B}{W} \\
\Delta_2 &= \left(1 - \frac{\gamma}{\eta}\frac{B}{W}\right)\left(\frac{\kappa}{\nu}\right)^{-\left(\frac{1}{\gamma}\right)}
\end{aligned}$$

- For the finite horizon case we get that

$$\begin{aligned}
w(x,t) &= \log\left[\Delta_1(t) + \Delta_2(t) \exp\left(-\frac{\zeta}{\gamma}x - \frac{\lambda}{\gamma} \mid x \mid\right)\right] \\
\Delta_1(t) &= \frac{\gamma}{\eta(t)}\frac{B(t)}{W} \\
\Delta_2(t) &= \left(1 - \frac{\gamma}{\eta(t)}\frac{B(t)}{W}\right)\left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}}
\end{aligned}$$

- Hence both types of investors prefer products of the same general type, with the finite horizon utility altering the level of investment over time in the risky asset. In fact the investor is getting less risk tolerant as one approaches the horizon and the floor consumption is rising at the interest rate.

THE HARAVG FINANCIAL PRODUCT

- The optimal product is the continuous receipt of a function of the price relative given by
$$f(R) = 1_{R>e^a}\left(\left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}} R^{-\frac{\zeta+\lambda}{\gamma}} - 1\right) + 1_{R<e^{-a}}\left(\left(\frac{\kappa}{\nu}\right)^{-\frac{1}{\gamma}} R^{-\frac{\zeta-\lambda}{\gamma}} - 1\right).$$
- For risk neutral volatilities and kurtosis exceeding their statistical counterparts the investors takes convex positions with respect to market down moves and concave positions with respect to market up moves.
- Relative to holding stock this amounts to buying a droption, that pays a function of the market down move and financing this by selling an uption that pays out on the large up moves.
- It is interesting that there is independent client interest in such structures.

RESULTS II

- **SPOT SLIDE CALIBRATION AND POSITION MEASURES**
  a. WE ALLOW FOR A MEASURE CHANGE FROM OBJECTIVE TO SUBJECTIVE PROBABILITIES, PARAMETERIZING THE RATIO OF OBJECTIVE EXCESS KURTOSIS TO SUBJECTIVE EXCESS KURTOSIS BY β.

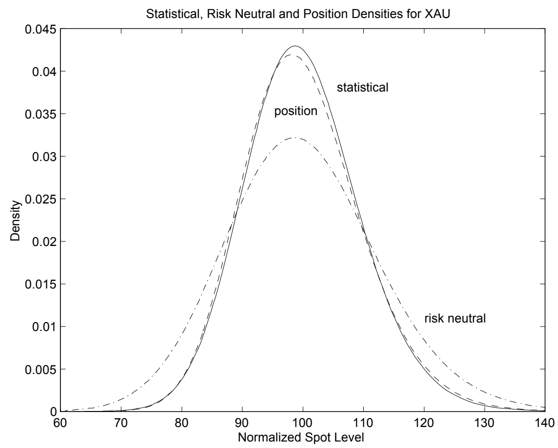
  b. WE ASSUME CONSTANT RELATIVE RISK AVERSION WITH COEFFICIENT OF RELATIVE RISK AVERSION α

  c. WE THEN DETERMINE β AND α SO THAT THE OPTIMAL CRRA VG EXPOSURE DESIGN MATCHES THE SPOT SLIDE OF A PARTICULAR POSITION.

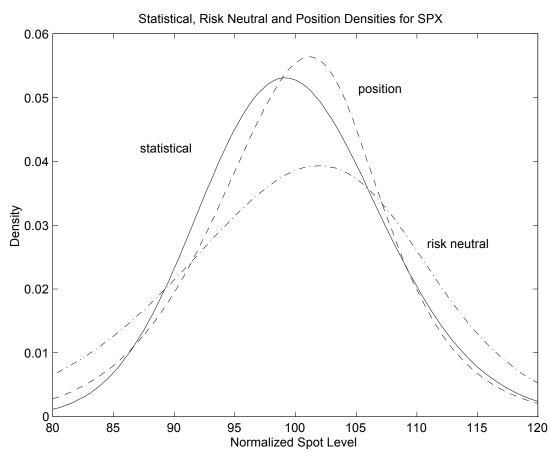
d. THIS IDENTIFIES THE PERSONALIZED MEASURE CONSISTENT WITH THE OBSERVED POSITION AS BEING OPTIMAL. WE TERM THIS THE POSITION MEASURE.

e. THE STOCK PRICE PROCESS UNDER THE POSITION MEASURE IS BY CONSTRUCTION A VG PROCESS AND THIS IS THE MEASURE ONE SHOULD USE TO SIMULATE PROFIT AND LOSS ACCOUNTS TO EVALUATE PROSPECTIVE TRADING POSITIONS.

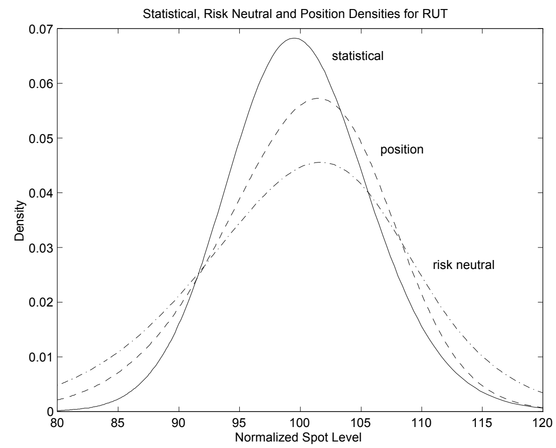
f. DISCOUNTED EXPECTATIONS OF CASH FLOWS UNDER THE POSITION MEASURE ARE PERSONALIZED PRICES CONSISTENT WITH ONE'S POSITIONS. TRADING STRATEGIES ARE DERIVED FROM COMPARISONS OF PERSONALIZED AND MARKET PRICES OF CASH FLOWS.


Statistical, Risk Neutral and Position Densities for XAU
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
Density
statistical
position
risk neutral
60
70
80
90
100
110
120
130
140
Normalized Spot Level

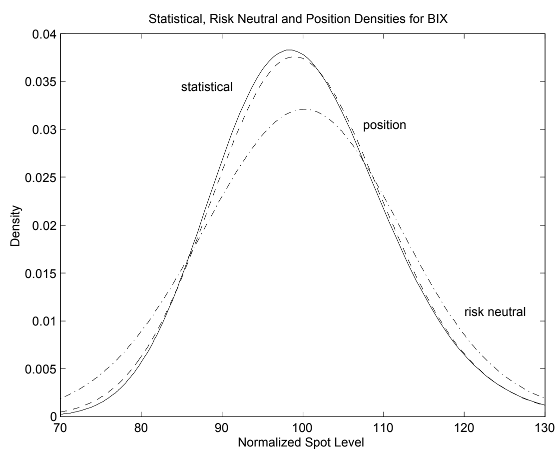
1.


Statistical, Risk Neutral and Position Densities for SPX
0.06
0.05
0.04
0.03
0.02
0.01
0
Density
position
statistical
risk neutral
80
85
90
95
100
105
110
115
120
Normalized Spot Level

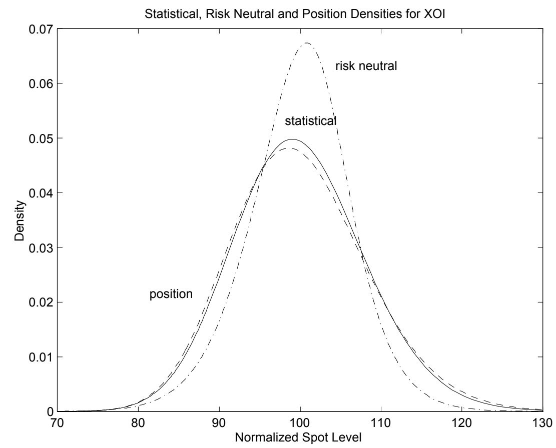
2.


Statistical, Risk Neutral and Position Densities for RUT
statistical
position
risk neutral
Density
Normalized Spot Level
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
80
85
90
95
100
105
110
115
120

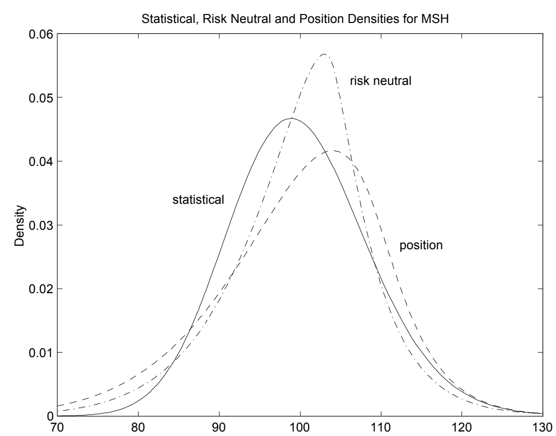
3.


Statistical, Risk Neutral and Position Densities for BIX
statistical
position
risk neutral
Density
Normalized Spot Level
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
70
80
90
100
110
120
130

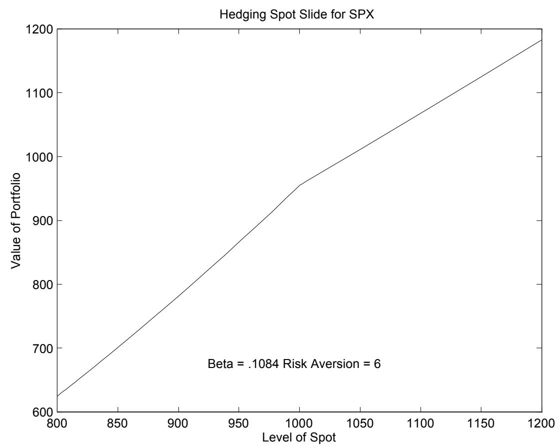
4.


Statistical, Risk Neutral and Position Densities for XOI
risk neutral
statistical
position
Density
Normalized Spot Level

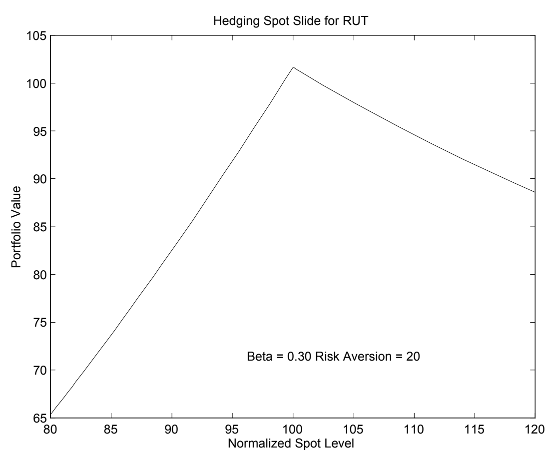
5.


Statistical, Risk Neutral and Position Densities for MSH
risk neutral
statistical
position
Density
Normalized Spot Level

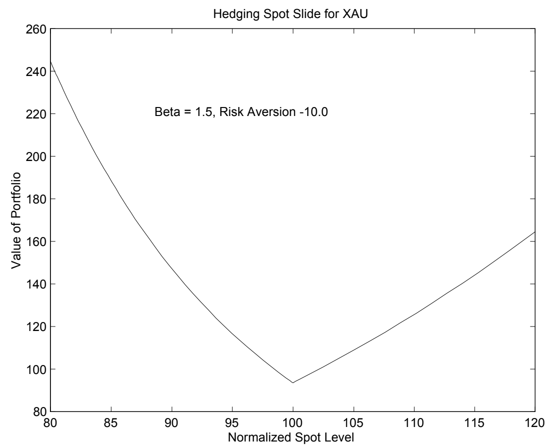
6.


Hedging Spot Slide for SPX
Value of Portfolio
Level of Spot
Beta = .1084 Risk Aversion = 6
1200
1100
1000
900
800
700
600
800
850
900
950
1000
1050
1100
1150
1200

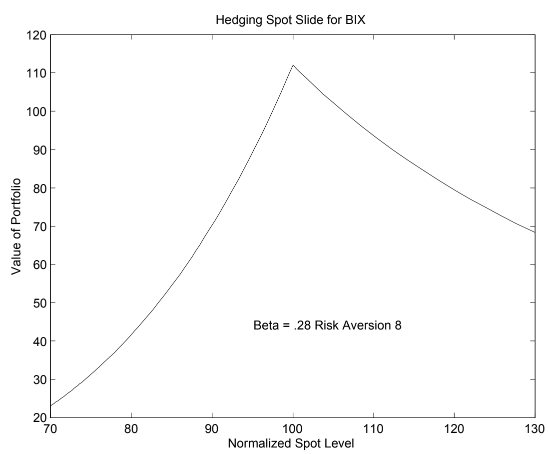
7.


Hedging Spot Slide for RUT
Portfolio Value
Normalized Spot Level
Beta = 0.30 Risk Aversion = 20
105
100
95
90
85
80
75
70
65
80
85
90
95
100
105
110
115
120

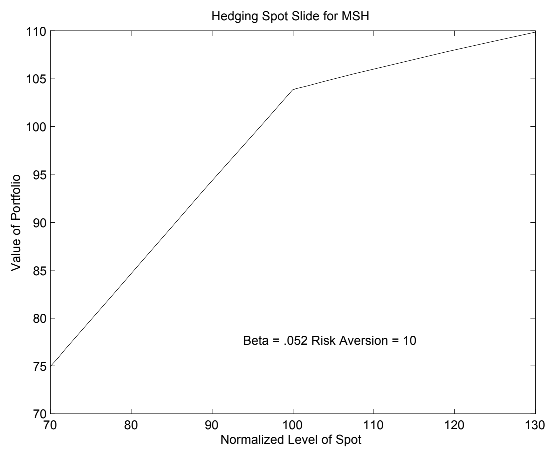
8.


Hedging Spot Slide for XAU
Beta = 1.5, Risk Aversion -10.0
Value of Portfolio
Normalized Spot Level
260
240
220
200
180
160
140
120
100
80
80
85
90
95
100
105
110
115
120

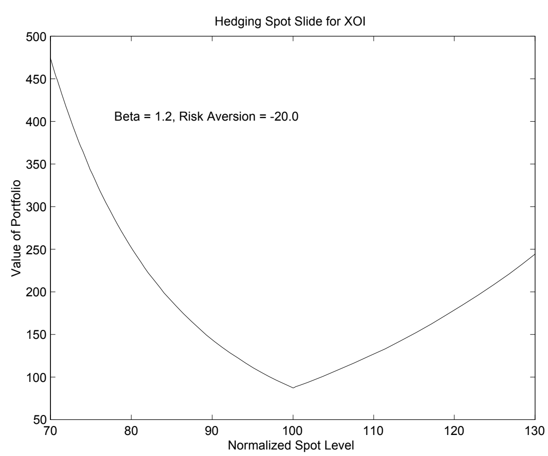
9.


Hedging Spot Slide for BIX
Beta = .28 Risk Aversion 8
Value of Portfolio
Normalized Spot Level
120
110
100
90
80
70
60
50
40
30
20
70
80
90
100
110
120
130

10.


Hedging Spot Slide for MSH
Value of Portfolio
Normalized Level of Spot
Beta = .052 Risk Aversion = 10
110
105
100
95
90
85
80
75
70
70 80 90 100 110 120 130

11.


Hedging Spot Slide for XOI
Value of Portfolio
Normalized Spot Level
Beta = 1.2, Risk Aversion = -20.0
500
450
400
350
300
250
200
150
100
50
70 80 90 100 110 120 130

12.