Volatility-Smile Modeling with Density-Mixture Stochastic Differential Equations

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Overview

- Brief intro to the smile problem
- Brief intro to general no-arbitrage mixture diffusions for single smile
- The no-arbitrage lognormal-mixture dynamics and variants
- Analytical tractability and calibration
- Decorrelation between average volatility and underlying asset, and brief comparison with stochastic volatility
- Examples of calibration to market data
- Example: Good nested structure of the parameterization
- Consistent generalization to multi-asset, basket smile
- Basket of two smiley assets: numerical results
- Conclusions and references to present/future work

Short intro to the smile

Sketchy version of the smile problem.

Financial (risky) asset (**B**lack& **S**choles, stock, FX rate, etc.)

$$dS_t = r(t)S_t dt + \sigma(t)S_t\, dW_t, \quad s_0, \quad t \in [0,T],$$

$$R(t) := \int_0^t r(s)ds,\ V(t)^2 := \int_0^t \sigma(s)^2 ds$$

European call option with maturity T and strike K pays

$(S_T - K)^+$ at time T.

$$E_0^Q[(S_T - K)^+/B(T)] = \text{BSCall}(S_0, K, T, R(T), V(T)).$$

$V(T)/\sqrt{T}$ is the (average) volatility of the option and does not depend on K.

In this formulation, volatility is a characteristic of stock S underlying the contract, and has nothing to do with the nature of the contract itself. In particular, it has nothing to do with K.

Short intro to the smile (cont'd)

Now take two different strikes K_1 and K_2. Suppose that the market provides us with the prices MKTCall(S_0, K_1, T) and MKTCall(S_0, K_2, T). Does there exist a *single* volatility $V(T)$ such that

$$\text{MKTCall}(S_0, K_1, T) = \text{BSCall}(S_0, K_1, T, R(T), V(T)),$$

$$\text{MKTCall}(S_0, K_2, T) = \text{BSCall}(S_0, K_2, T, R(T), V(T))?$$

The answer is a resounding **"NO!!!"**

Market option prices do not behave like this. Instead two *different* **implied volatilities** $V(T, K_1)$ and $V(T, K_2)$ are required to match the observed market prices if one is to use Black & Scholes (BS) formula:

$$\text{MKTCall}(S_0, K_1, T) = \text{BSCall}(S_0, K_1, T, R(T), V(T, K_1)),$$

$$\text{MKTCall}(S_0, K_2, T) = \text{BSCall}(S_0, K_2, T, R(T), V(T, K_2)).$$

In other terms, each market option price requires its own Black and Scholes **implied volatility** $V^{\text{MKT}}(T, K)/\sqrt{T}$ depending on the option strike K.

Short intro to the smile (cont'd)

The market therefore uses BS formula simply as a *metric* to express option prices as volatilities. The curve $K \mapsto V^{\text{MKT}}(T,K)/\sqrt{T}$ is the so called *volatility smile* of the T-maturity option. If BS's model were consistent along different strikes, this curve would be flat, since volatility should not depend on K. Instead, this curve exhibits "smiley" or "skewed" shapes.

BS Geometric Brownian Motion (GBM) assumption for dS is no longer sufficient, need richer dynamics to account for the smile. Alternative diffusion dynamics is just one possible means to model the smile (local volatility models, here). There are many other possibilities (stochastic vol models, jump diffusions, lattices... not here). Example of tractable diffusions: CEV and Shifted GBM (one can also combine the two)

$$dS_t = r\ S_t dt + \sigma S_t^{\gamma}\ dW_t, \quad S_0 = s_0.$$

$$S_t = \alpha e^{\int_0^t r(u)du} + X_t, \quad dX_t = r(t)X_t dt + \sigma(t)X_t\ dW_t.$$

CEV or Shifted-GBM: Just one additional parameter γ or α, not flexible enough. Shifted CEV has two additional parameters but still largely insufficient. Local vol models are too poor or not tractable. Our proposal: tractable and flexible loc vol model.

Smile modeling through alternative diffusions

Alternative dS_t can model a non-flat smile:

1. Set K to a starting value;
2. Compute the model option price

$$\Pi(T,K) = e^{-\int_0^T r(s)ds} E_0^Q[(S_T - K)^+]$$

with S modeled through an alternative dynamics

$$dS_t = r(t)S_t dt + \boxed{\ \sigma(t,S_t)\ } S_t\, dW_t, \quad S_0 = s_0$$

(Harrison and Pliska's Risk Neutral valuation theory still stands)

3. Invert BS formula for this strike, i.e. solve

$$\Pi(K) = \text{BSCall}(S_0, K_1, T, R(T), V(T,K))$$

in $V(T,K)$, thus obtaining the model implied volatility $V(T,K)$.

4. Change K and restart from point 2.

Smile problem: summary

- Traders use the BS "metric" to price plain-vanilla options;
- Options are priced (quoted) with a (implied) volatility that varies with the option strike;
- The term structure of implied volatilities is "skewed" or "smiley"
- The BS model cannot consistently price all options quoted in a market (the real risk-neutral distribution is not lognormal);
- Need for an alternative asset price dynamics (not just terminal distribution) to price exotics or non quoted plain-vanilla options;
- This model should feature, among other qualities:
  - explicit dynamics with known marginal distrib.;
  - analytical formulas for European options ("analytic" and rapid calibration to plain vanilla);
  - good fitting of market data (reasonable number of parameters in the dynamics).

The no-arbitrage mixture diffusion dynamics

$$dS_t = \mu S_t dt + \boxed{\sigma(t, S_t)} S_t \, dW_t, \quad S_0 = s_0 \qquad (1)$$

(assume $r(t) = \mu$ with no loss of generality).

σ is no longer a determinisitic function of time, but depends now on the underlying S itself.

We propose a class of analytically tractable models for an asset-price dynamics that are flexible enough to reproduce a large variety of market volatility structures.

The asset under consideration underlies a given option market (not necessarily a tradeable asset). We can think of an FX rate, a stock index, or a forward LIBOR rate.

The asset dynamics follows from assuming that

- The risk-neutral measure Q exists;
- The dynamics of the asset price S under Q is (1).
- The marginal density of S under Q is a weighted average of the known densities of some given diffusion processes.

More specifically:

The no-arbitrage mixture diffusion dynamics

Let us then consider N instrumental diffusions

$$dS_t^i = \mu S_t^i dt + \boxed{\; v_i(t, S_t^i) \;} S_t^i dW_t, \quad s_0, \quad i = 1, \dots, N,$$

We denote by $p_t^i(\cdot)$ the density function of S_t^i

Problem. Derive the local volatility $\sigma(t, S_t)$ in

$$dS_t = \mu S_t dt + \boxed{\; \sigma(t, S_t) \;} S_t \, dW_t, \quad s_0$$

such that the Q-density of S satisfies

$$p_t(y) = \sum_{i=1}^{N} \lambda_i p_t^i(y), \quad \lambda_i > 0, \quad \sum_{i=1}^{N} \lambda_i = 1.$$

Solution. Back out σ from the Fokker-Planck equation for S. We end up with the following SDE under Q:

$$dS_t \quad = \mu S_t dt + \sqrt{\frac{\sum_{i=1}^{N} \lambda_i v_i^2(t, S_t) p_t^i(S_t)}{\sum_{i=1}^{N} \lambda_i p_t^i(S_t)}} S_t dW_t$$

Option pricing with the mixture dynamics.

Let us give for granted that the previous SDE has a unique strong solution and consider a European Call option with maturity T, strike K. The option value at $t = 0$ is :

$$\text{C}all = P(0,T)E^T\left\{(S_T - K)^+\right\}$$

$$= P(0,T)\int_0^{+\infty}(y-K)^+\sum_{i=1}^N \lambda_i p_T^i(y)dy$$

$$= \sum_{i=1}^N \lambda_i P(0,T)\int(y-K)^+ p_T^i(y)dy = \sum_{i=1}^N \lambda_i \text{C}all_i.$$

Remark [Mixtures without dynamics.] Earlier authors used to postulate a lognormal mixture for the risk neutral density, but *did not provide any consistent arbitrage-free dynamics*.

Remark [Greeks]. Due to linearity of differentiation, the same convex combination applies to all option Greeks (sensitivities).

Remark [Complete market, Hedging]. As for all local volatility models, the mixture diffusion dynamics yields a *complete* market (contrary to stochastic vol) and a delta-hedging strategy.

The no-arbitrage lognormal mixture dynamics

$$v_i(t,y) = \sigma_i(t) y, \quad dS_t^i = \mu S_t^i dt + \boxed{\sigma_i(t)} S_t^i dW_t, \quad s_0,$$

where σ_i's are det. bounded from above and below, continuous and $\exists$ an ε such that we have a common initial value $\sigma_i(t) = \sigma_0$ for each t in $[0, \varepsilon]$. Set $\nu_{\mathsf{mix}}(t,y)^2 = \sum_{i=1}^N \Lambda_i(t,y) \sigma_i(t)^2$

$$\Lambda_i(t,y) = \frac{\lambda_i \; p_{\mathcal{N}(\ln s_0 + \mu t - V_i(t)^2/2, \; V_i(t)^2)}(\ln y)}{\sum_{j=1}^N \lambda_j \; p_{\mathcal{N}(\ln s_0 + \mu t - V_j(t)^2/2, \; V_j(t)^2)}(\ln y)},$$

for $(t,y) > (0,0)$; $\nu_{\mathsf{mix}}(t,y) = \sigma_0$ for $(t,y) = (0, s_0)$.

Then the SDE $\boxed{dS_t = \mu S_t dt + \nu_{\mathsf{mix}}(t, S_t) S_t dW_t}$, has a unique strong solution whose marginal density is the mixture of the lognormal S^i's densities

$$p_{S_t}(y) = \sum_{j=1}^N \lambda_j p_{\mathcal{N}(\ln s_0 + \mu t - V_j(t)^2/2, \; V_j(t)^2)}(\ln y)$$

Why a mixture of lognormals?

- analytically tractable (calibration!) and linked to BS model;
- log-returns $\ln(S_t/S_0)$ are leptokurtic;
- Mixtures of lognormals work well in many practical situations: Ritchey (1990), Melick and Thomas (1997), Bhupinder (1998) and Guo (1998) found a good fitting quality to market options data.

Proposition. Consider a Call option with maturity T, strike K and written on the asset. The model yields

$$\text{C}all = P(0,T)\sum_{i=1}^{N}\lambda_i\left[S_0 e^{\mu T}\Phi\left(\frac{\ln\frac{S_0}{K}+\left(\mu+\frac{1}{2}\eta_i^2\right)T}{\eta_i\sqrt{T}}\right)\right.$$

$$\left.-K\Phi\left(\frac{\ln\frac{S_0}{K}+\left(\mu-\frac{1}{2}\eta_i^2\right)T}{\eta_i\sqrt{T}}\right)\right], \qquad \boxed{\eta_i(T) := \frac{V_i(T)}{\sqrt{T}}}$$

This leads to smiles with a minimum at $K = s_0 e^{\mu T}$. Can shift the dynamics to fit asymmetric smiles (skews) by adding a new parameter α.

$$A_t = (A_0 - S_0)\alpha e^{\mu t} + S_t,$$

where α is a real constant. By Ito's formula,

$$dA = \mu A dt + \nu(t, A-(A_0-S_0)\alpha e^{\mu t})(A-(A_0-S_0)\alpha e^{\mu t})dW$$

No arbitrage lognormal-mix dyn: Drifts variants

Consider the instrumental processes

$$dS_t^i = \boxed{\mu_i} S_t^i dt + \sigma_i(t) S_t^i dW_t, \quad s_0$$

and look for a diffusion coefficient $\nu(t, y)$ such that

$$dS_t = \mu S_t dt + \nu_{\mathsf{dmix}}(t, S_t) S_t dW_t, \quad s_0$$

has marginal density $p_{S_t}(y) = \sum_{i=1}^m \lambda_i p_{S_t^i}(y)$.

Call $\nu_{\mathsf{mix}}(t, y)^2$ the solution of the analogous problem when all instrumental processes share the same drift μ (found earlier). It is possible to show that

$$(\nu_{\mathsf{dmix}}(t, y) y)^2 := (\nu_{\mathsf{mix}}(t, y) y)^2 + \frac{2 \sum_{i=1}^N \lambda_i (\mu_i - \mu) \int_y^{+\infty} x p_{S_t^i}(x) dx}{\sum_{j=1}^N \lambda_j p_{S_t^j}(y)}.$$

It is possible to find conditions under which this diffusion coefficient has linear growth and does not explode in finite time (Sartorelli, (2002)). The integral in the numerator is just the Black and Scholes price of an asset or nothing option for the instrumental process S^i, which is readily available in terms of the Gaussian cumulative distribution function.

No arbitrage lognormal-mix dyn: Shifts Variants

Can also shift single basic distributions:

$$A_t^i = \boxed{\beta_i} e^{\mu t} + S_t^i, \quad dS_t^i = \mu S_t^i dt + \sigma_i(t) S_t^i dW_t$$

that leads to instrumental processes

$$dA_t^i = \mu A_t^i dt + \sigma_i(t)(A_t^i - \boxed{\beta_i} e^{\mu t}) dW_t,$$

and look for an SDE for S with $p_{S_t} = \sum_j \lambda_j p_{A_t^j}$, i.e.

$$p_{S_t}(y) = \sum_{j=1}^{N} \lambda_j p_{\mathcal{N}(\ln(s_0 - \beta_j) + \mu t - V_j(t)^2/2,\ V_j(t)^2)} \left(\ln \left(y - \beta_j e^{\mu t} \right) \right)$$

We find (but here no a-priori $\exists$ results available for the SDE)

$$dS_t = \mu S_t dt + v_{\mathsf{smix}}(t, S_t) dW_t, \quad v_{\mathsf{smix}}(t, y)^2 = \sum_{i=1}^{N} \Lambda_i(t, y) v_i(t, y)^2$$

$$v_i(t, y) = \sigma_i(t)(y - \beta_i e^{\mu t}).$$

$$\Lambda_i(t, y) = \frac{\lambda_i \ p_{\mathcal{N}(\ln(s_0 - \beta_i) + \mu t - V_i(t)^2/2,\ V_i(t)^2)} (\ln(y - \beta_i e^{\mu t}))}{\sum_{j=1}^{N} \lambda_j \ p_{\mathcal{N}(\ln(s_0 - \beta_j) + R(t) - V_j(t)^2/2,\ V_j(t)^2)} (\ln(y - \beta_j e^{\mu t}))},$$

Mixture model vs Stochastic volatility: Correlation(average volatility,underlying)

$$dS_t = r(t)S_t dt + \boxed{\gamma(t, S_t)} S_t \, dW_t, \quad S_0 = s_0, \tag{2}$$

in general γ can be either a deterministic or a stochastic function of S_t. In the latter case we have a "stochastic-volatility model" (SVM), for example $\gamma(t, S) = \xi(t)$,

$$d(\xi(t)^2) = b(t, \xi(t)^2)dt + \chi(t, \xi(t)^2)dZ_t,$$

with the important specification $\quad dZ_t dW_t = \rho \, dt.$

It is usually said that SVM are better than local volatility models (LVM), because the *instantaneous correlation* is:

Corr$(dS_t, \, d\gamma^2(t, S_t)) = \rho < 1$ (e.g. $\rho = 0$ in **H**ull-**W**hite SVM)

Corr$(dS_t, \, d\nu^2_{\text{mix}}(t, S_t)) = 1,$ and the same holds for all LVM's
But what about *terminal* correlations?

Mixture model vs Stochastic volatility: Correlation(average volatility,underlying)

$$\boxed{\mathrm{Corr}(dS_t,\ d\gamma^2_{\mathrm{HW}}(t,S_t)) = 0, \quad \mathrm{Corr}(dS_t,\ d\nu^2_{\mathrm{mix}}(t,S_t)) = 1.}$$

$$V_{\mathrm{HW}}(T) := \int_0^T \gamma^2_{\mathrm{HW}}(t,S_t)dt, \quad V_{\mathrm{mix}}(T) := \int_0^T \nu^2_{\mathrm{mix}}(t,S_t)dt$$

the “average variances” of the process S in the Hull-White model and in our mixture model, respectively. Then

$$\boxed{\mathrm{Corr}(S_T, V_{\mathrm{HW}}(T)) = 0, \quad \mathrm{Corr}(S_T, V_{\mathrm{mix}}(T)) = 0 \ \text{ all } \ T}$$

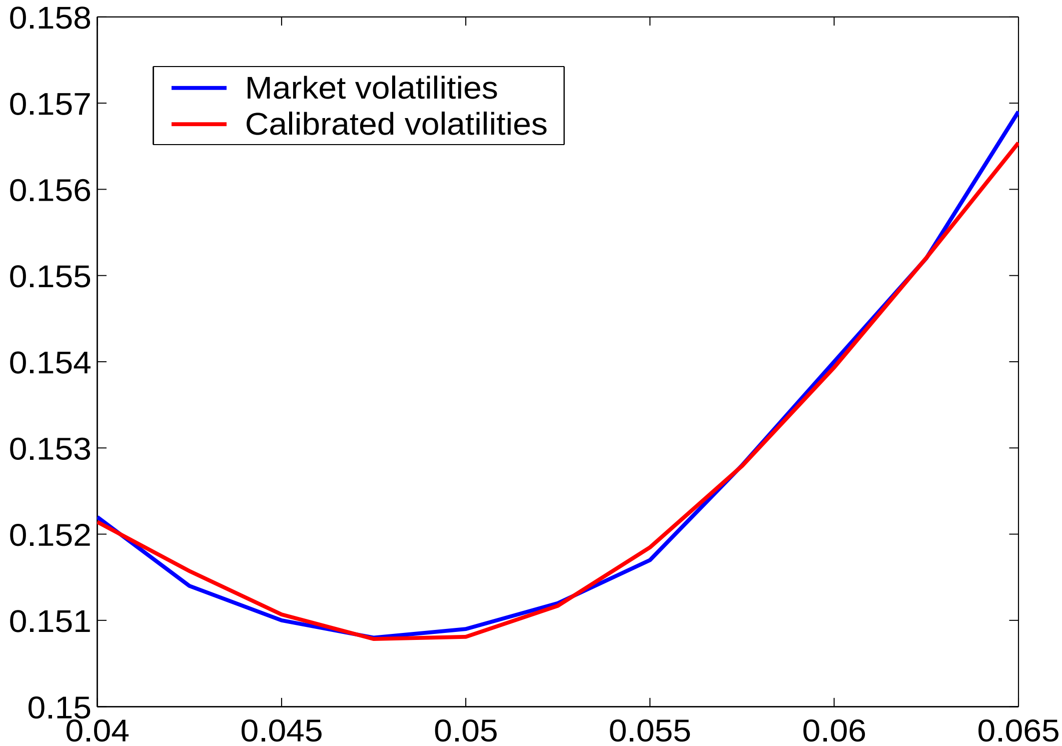
Correlation (Volatility)^2 $\longleftrightarrow$ Asset-Value:

Model	Hull-White SVM	lognormal mix dyn
Intantaneous correlation	0	1
Terminal correlation	0	0

Yet, correlation is not a satisfactory measure of dependence outside the Gaussian world...

Finally, in variants of the basic lognormal mixture model (e.g. mixtures with different drifts) one can impose correlation patterns as part of the calibration procedure

Example 1 of calibration to market data, single smile



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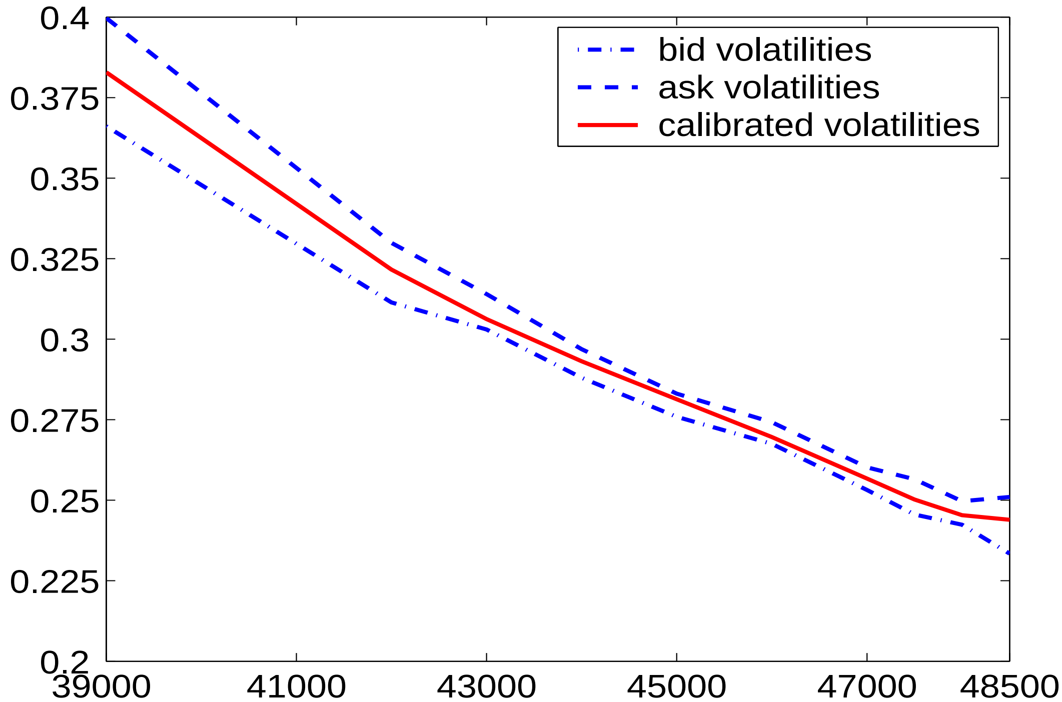
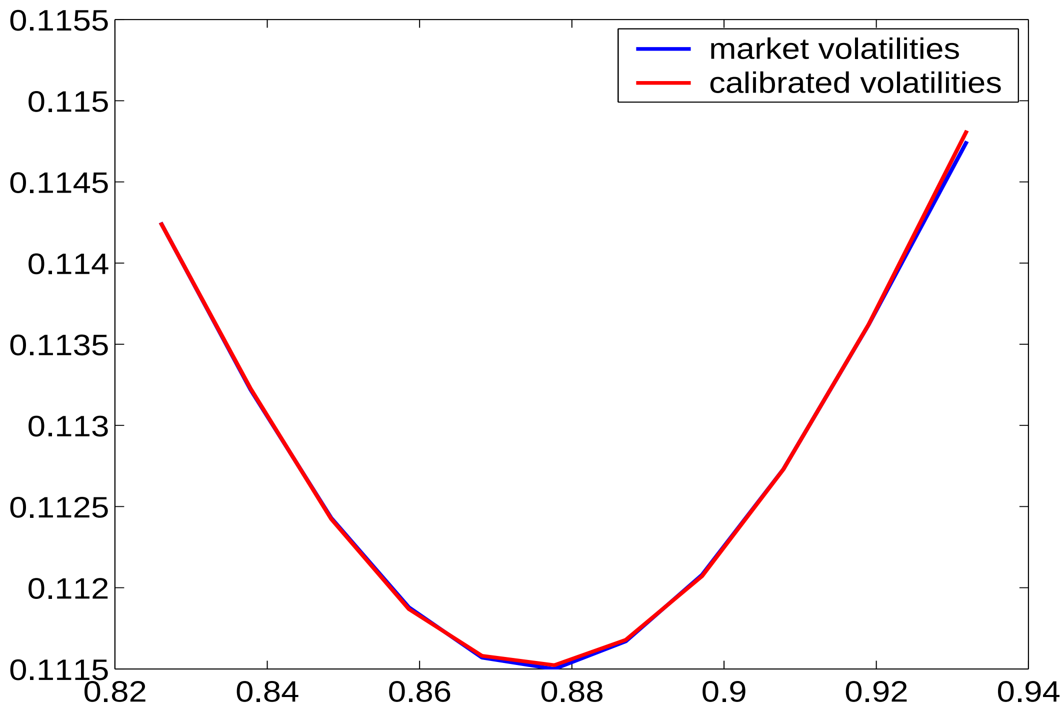
Data: Two-year Euro caplet volatilities as of November 14th, 2000 (Libor resetting at 1.5 years). We set: $N = 2$, $T = 1.5$, $\lambda_2 = 1 - \lambda_1$. We minimize the squared percentage difference between model and market (mid) prices: $\lambda_1 = 0.241$, $\lambda_2 = 0.759$, $\eta_1(T) = 0.125$, $\eta_2(T) = 0.194$, $\alpha = 0.147$.

Example 2 of calibration to market data, single smile



Data: Italian MIB30 equity index on March 29, 2000, at 3,21pm (most liquid puts with the shortest maturity). We set $N = 3$, $T = 0.063014$, $\lambda_3 = 1 - \lambda_1 - \lambda_2$. We minimize the squared percentage difference between model and market mid prices. We get: $\lambda_1 = 0.201$, $\lambda_2 = 0.757$, $\eta_1(T) = 0.019$, $\eta_2(T) = 0.095$, $\eta_3(T) = 0.229$, $\alpha = -1.852$.

Example 3 of calibration to market data, single smile



Data: USD/Euro two-month implied volatilities as of May 21, 2001.

We set $N = 2$, $T = 0.167$, $\lambda_2 = 1 - \lambda_1$. We minimize the squared percentage difference between model and market mid prices. We get: $\lambda_1 = 0.451$, $\eta_1(T) = 0.129$, $\eta_2(T) = 0.114$, $\alpha = 0.076$.

Example 4, calibration of a whole FX vol surface

EUR/USD market volatility surface May 17, 2001.

Vol surface function of $(T-t)$ and δ. FX market: quote volatilities in terms of ATM vol (σ_{ATM}), risk–reversal (r), strangle (s) (e.g. Malz (1997)). Common assumption for a interpolating functional form

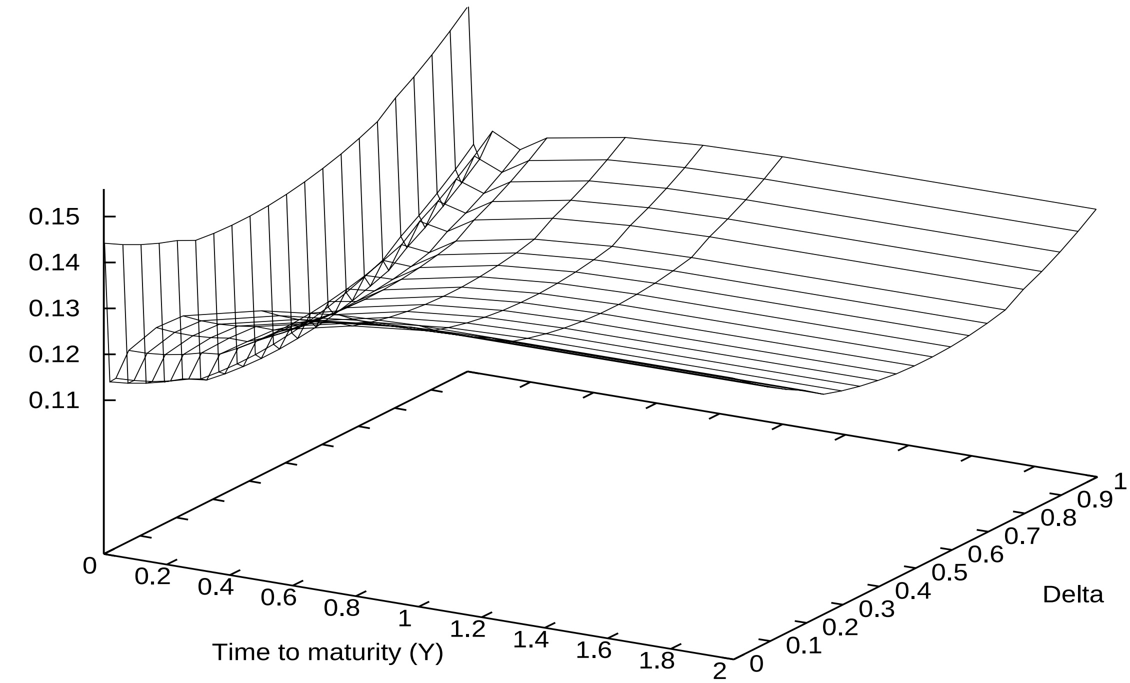
$$\sigma(\delta,T) = \sigma_{ATM}(T) - 2r(T)\left(\delta - \frac{1}{2}\right) + 16s(T)\left(\delta - \frac{1}{2}\right)^2,$$

$$\delta(T) = e^{-r_d T}\Phi\left[\frac{\ln(S_t/X) + (r_d - r_f + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right]$$

being the delta of a call option.

T	σ_{ATM}	r	s	Bid/ask spread
O/N	13.50%	0.60%	0.29 %	2%-2.5%
1W	10.50%	0.60%	0.29%	2%-2.5%
2W	10.40%	0.40%	0.29%	1%-1.5%
1M	11.00%	0.40%	0.30%	0.35%-0.85%
2M	11.15%	-0.05%	0.30%	0.30%-0.80%
3M	11.50%	-0.05%	0.30%	0.30%-0.80%
6M	11.85%	-0.10%	0.30%	0.30%-0.68%
9M	12.00%	-0.14%	0.30%	0.30%-0.55%
1Y	12.05%	-0.15%	0.30%	0.25%-0.45%
2Y	12.05%	-0.15%	0.30%	0.25%-0.45%

Table 1: Market data for ATM implied vols, risk–reversal and strangle prices as of May 17, 2001.



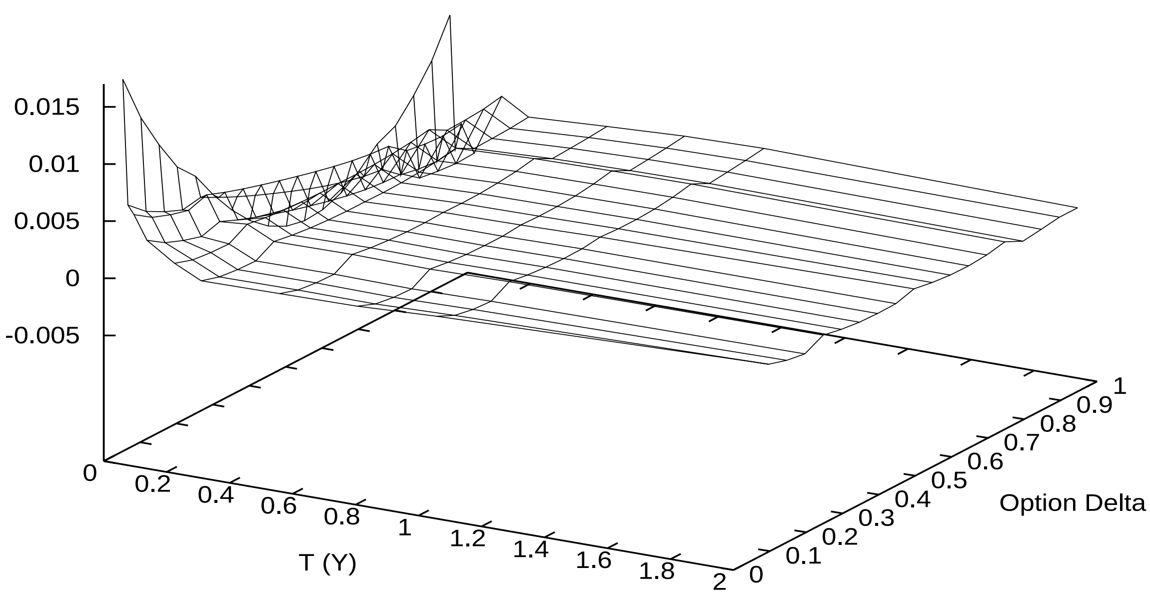


Figure 1: The market implied volatility surface (above) and absolute difference in implied volatility after calibration of the model with $N = 3$ (below) for the May 17, 2001 market data.

Example 4 (cont'd)

Use Shifts variant of the lognormal mixture dynamics

explore the effect of varying the number of basis densities n

$\eta_i(T) = \sqrt{\frac{1}{T}\int_0^T \sigma_i^2(s)ds}$ are taken as

$$\eta^i(T) = a_i + b_i \left[1 - \exp\left(-\frac{T}{\tau_i}\right)\right]\frac{\tau_i}{T} + c_i \exp\left(-\frac{T}{\tau_i}\right),$$

(Nelson and Siegel (1987) for yield curves)

Model parameters to calibrate:

$\mathbf{x} = (\lambda_{1:N}, \beta_{1:N}, a_{1:N}, b_{1:N}, c_{1:N}, \tau_{1:N})$ has dimensionality $4N + N + (N-1) = 6N - 1$.

minimize the sum of the relative squared discrepancies The resulting root–mean–square error is 3×10^{-4} and 7×10^{-5} for calibrations with $N = 2$ and $N = 4$ respectively.

The maximum error for any maturity is well below the corresponding bid–ask spread already with $N = 3$.

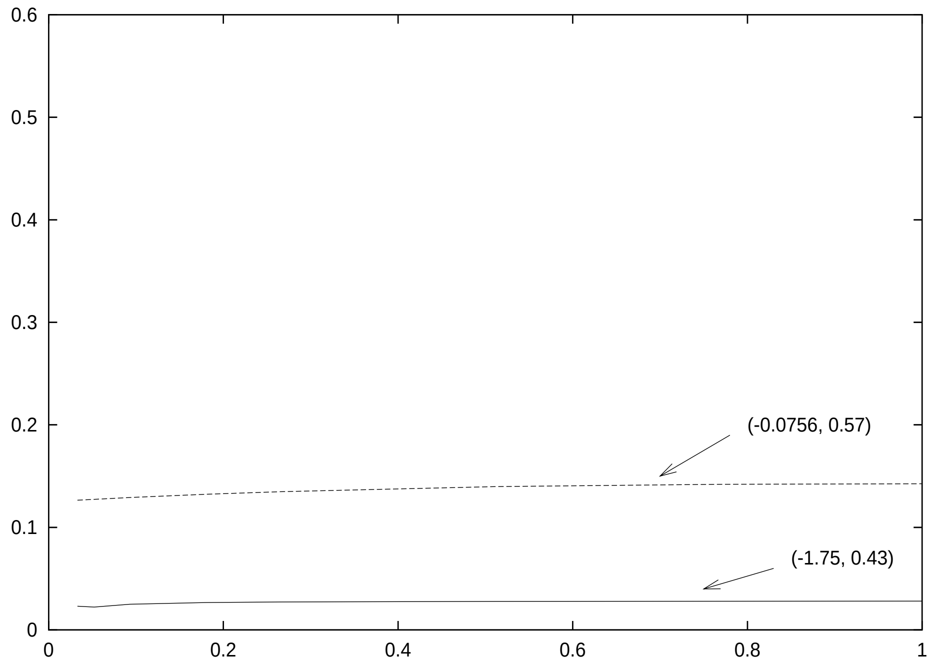


Figure 2: $T \mapsto \eta_i(T; a, b, c, \tau)$ after calibration of the model with $N = 2$; We show (β_i, λ_i) for each component.

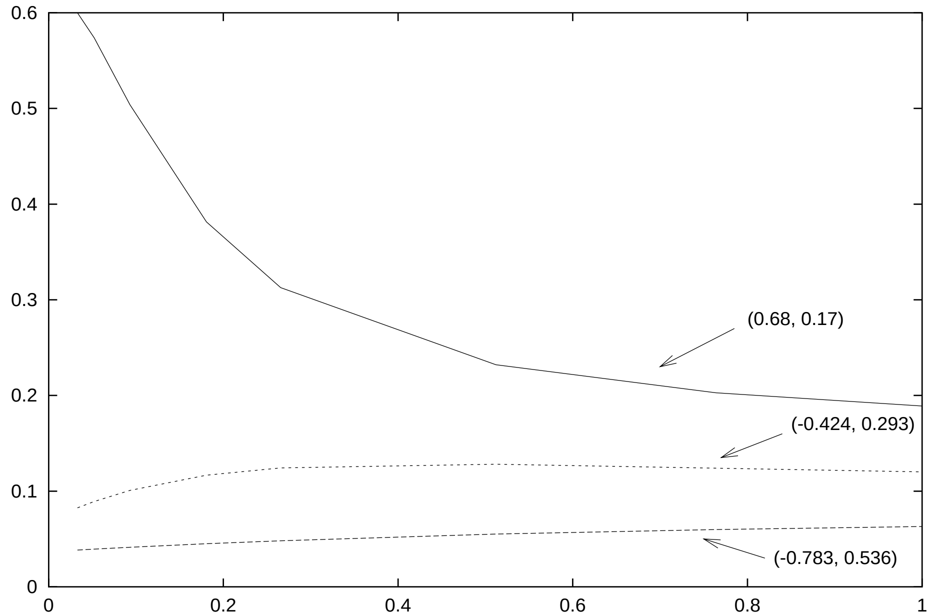


Figure 3: $N = 3$

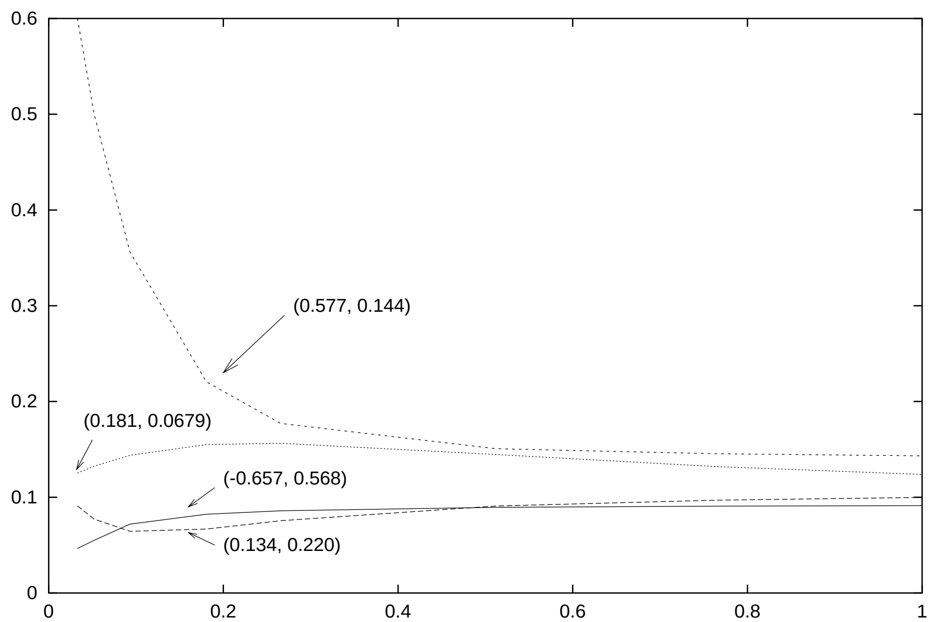


Figure 4: $N = 4$. Notice the "nested" structure

Baskets: Consistent multivariate extension

Single assets: $\underline{S}(t) = [S_1(t), S_2(t), \ldots, S_\alpha(t), \ldots, S_n(t)]'$.

Mixture diffusion model for each asset S_α: Find ν^α_{mix} with

$$dS_\alpha = S_\alpha[\mu_\alpha dt + \nu^\alpha_{\text{mix}}(t, S_\alpha) dW_\alpha] \Rightarrow p_{S_\alpha(t)} = \sum_{k=1}^N \lambda^k_\alpha p_{S^k_\alpha(t)}$$

where $S^1_\alpha, \ldots, S^k_\alpha, \ldots, S^N_\alpha$ are instrumental procs for S_α:

$$dS^k_\alpha(t) = \mu^k_\alpha S^k_\alpha dt + \sigma^k_\alpha S^k_\alpha dW_\alpha, \quad dW_\alpha dW_\beta = \rho_{\alpha,\beta}\, dt$$

Multivariate extension: def $n \times n$ matrix C by $C(t, \underline{x})\ \rho\ C'(t, \underline{x})$

$$= \frac{\sum_{k_1,\ldots,k_n=1}^N \lambda_1^{k_1} \cdots \lambda_n^{k_n} \ \ V^{k_1,..,k_n}(t) \ \ p_{[S_1^{k_1}(t),\ldots,S_n^{k_n}(t)]'}(\underline{x})}{\sum_{k_1,\ldots,k_n=1}^N \lambda_1^{k_1} \cdots \lambda_n^{k_n} \ \ p_{[S_1^{k_1}(t),\ldots,S_n^{k_n}(t)]'}(\underline{x})}$$

$$V^{k_1,..,k_n}(t) = \left(\sigma_\alpha^{k_\alpha}(t)\ \rho_{\alpha,\beta}\ \sigma_\beta^{k_\beta}(t)\right)_{\alpha,\beta=1,\ldots,n}$$

Our extension is (Rapisarda (2001), "**M**ulti **V**ariate **M**ixture **D**ynamics")

$$d\underline{S}(t) = \text{diag}(\mu)\underline{S}(t)dt + \text{diag}(\underline{S}(t))C(t, \underline{S}(t))d[W_1, .., W_n]'$$

and satisfies $p_{\underline{S}(t)} = \sum_{k_1,\ldots,k_n=1}^N \lambda_1^{k_1} \cdots \lambda_n^{k_n} \ \ p_{[S_1^{k_1}(t),\ldots,S_n^{k_n}(t)]'}$

Multivariate Mixture Dynamics (MVMD)

$$d\underline{S}(t) = \mathrm{diag}(\mu)\underline{S}(t)dt + \mathrm{diag}(\underline{S}(t))C(t, \underline{S}(t))d[W_1, .., W_n]'$$

$$(C\rho C')(t, \underline{x}) = \frac{\sum_{k_1,\ldots,k_n=1}^{N} \lambda_1^{k_1} \cdots \lambda_n^{k_n} \; V^{k_1,..,k_n}(t) \; p_{[S_1^{k_1},\ldots,S_n^{k_n}]'}(\underline{x})}{\sum_{k_1,\ldots,k_n=1}^{N} \lambda_1^{k_1} \cdots \lambda_n^{k_n} \; p_{[S_1^{k_1}(t),\ldots,S_n^{k_n}(t)]'}(\underline{x})}$$

$$V^{k_1,..,k_n}(t) = \left(\sigma_\alpha^{k_\alpha}(t) \; \rho_{\alpha,\beta} \; \sigma_\beta^{k_\beta}(t)\right)_{\alpha,\beta=1,\ldots,n}$$

$$dW_i dW_j = \rho_{i,j} dt, \quad p_{\underline{S}(t)} = \sum_{k_1,\ldots,k_n=1}^{N} \lambda_1^{k_1} \cdots \lambda_n^{k_n} \quad p_{[S_1^{k_1}(t),\ldots,S_n^{k_n}(t)]'}$$

Can evaluate simple claims (e.g. call option) on basket $A(t) = \sum_{i=1}^{n} a_i S_i(t) = \underline{a}'\underline{S}$ "one shot", since we know $p_{\underline{S}}$.

Compare with naive numerical Euler or Milstein scheme for Monte Carlo, consisting of a time-discretization of the "**S**imply-**C**orrelated **M**ixture **D**ynamics", obtained by single mixture sde's by instantaneously correlated Brownian motions

$$d\underline{S} = \mathrm{diag}(\mu)\underline{S}dt + \mathrm{diag}((\nu_{\mathrm{mix}}^{\alpha}(t, S_\alpha))_\alpha)\mathrm{diag}(\underline{S})d[W_1, .., W_n]'$$

$p_{\underline{S}(t)}$ =?!? Both are consistent with single mixture smiles

$$dS_\alpha = S_\alpha[\mu_\alpha dt + \nu_{\mathrm{mix}}^{\alpha}(t, S_\alpha)dW_\alpha] \Rightarrow p_{S_\alpha} = \sum_{k=1}^{N} \lambda_\alpha^k p_{S_\alpha^k}$$

MVMD vs SCMD

$$d\underline{S}(t) = \text{diag}(\mu)\underline{S}(t)dt + \text{diag}(\underline{S}(t))C(t, \underline{S}(t))d[W_1, .., W_n]'$$

$$p_{\underline{S}(t)} = \sum_{k_1,\ldots,k_n=1}^{N} \lambda_1^{k_1} \cdots \lambda_n^{k_n} \quad p_{[S_1^{k_1}(t),\ldots,S_n^{k_n}(t)]'} \qquad \textbf{VS}$$

$$d\underline{S} = \text{diag}(\mu)\underline{S}dt + \text{diag}((\nu_{\text{mix}}^{\alpha}(t, S_\alpha))_\alpha)\text{diag}(\underline{S})d[W_1, .., W_n]'$$

$$dW_i dW_j = \rho_{i,j}dt, \qquad p_{\underline{S}(t)} = ?!?,$$

Pro's MVMD: 1) Terminal distribution of $\underline{S}_T$ (and of basket $A_T = \underline{a}'\underline{S}_T$) can be simulated one-shot, no time discretization. 2) Explicit multivariate distribution that is the most natural non-trivial generalization of the scalar case, i.e. a multivariate mixture

Con's MVMD: 1) "Combinatorial explosion": A possibly large number of densities to mix in the multivariate mixture (typically N^n, e.g. $3^{10} = 59049$). BUT... Typically $N = 2, 3$, and there is a hierarchy in the base densities: $\lambda_\alpha^1 > \lambda_\alpha^2 >> \lambda_\alpha^3$, so that with weights given by $\lambda_\alpha^1\lambda_\beta^2\lambda_\gamma^3\cdots$ only few multivariate densities have appreciable weights, thus easing the simulations.

2) No immediate statistical interpretation of ρ if not as cross-sectional fitting parameters...

Pro's SCMD: 1) A clear interpretation for ρ as instantaneous correlation among the single names. 2) Number of densities to mix does not increase with n but remains equal to N.

Con's SCMD: 1) Need time discretization to price simple claims on $A_T = \underline{a}'\underline{S}_T$. As T increases, we need more time steps.

Defining the basket smile

$$B(t) = \sum_{i=1}^{b} w_i S_i(t); \;\; dS_i(t) = (r(t) - q_i) S_i(t) dt + (\ldots) dW_i$$

Moment-matching paradigm. Call q_i the continuous dividends of S_i, so that its risk-neutral drift is $\mu_i = r - q_i$. Find q in

$$d\bar{B}_t = (r(t) - q)\bar{B}_t dt + (\ldots.)dW_t, \quad \bar{B}_0 = B(0)$$

such that

$$E\bar{B}_T = B(0)e^{R(T)-qT} = \sum_{i=1}^{b} S_i(0)e^{R(T)-q_iT} = EB(T).$$

Now that we have q, we may decide to quote basket implied volatilities by inverting BS's formula, by solving the following equation in $V(T,K)$:

$$\text{BSCall}(\text{Basket}_0, K, T, R(T), \mathbf{q}, V(T,K)) = \text{Model-Basket-Call}(\text{Basket}_0, T, K).$$

From the multivariate model prices of basket European options on the right hand side, back out the "basket implied volatilities" $V(K,T)/\sqrt{T}$ such that Black–Scholes formulas with the synthetic dividend q reproduce such prices.

Basket of 2 smiley assets, each modeled with a 2-mixture

We now give some examples with a basket of 2 assets.

Asset S_1, mixture with $(\lambda_1^1, \sigma_1^1(t)), (\lambda_1^2, \sigma_1^2(t))$

Asset S_2, mixture with $(\lambda_2^1, \sigma_2^1(t)), (\lambda_2^2, \sigma_2^2(t))$

Define

$$V_\alpha^j(t) := \left(\frac{1}{t}\int_0^t \sigma_\alpha^j(u)^2 du\right)^{1/2}$$

$$V_\alpha^j(t) = A_\alpha^j + B_\alpha^j \left(1 - \exp\left(\frac{-t}{\sqrt{D_\alpha^j}}\right)\right)\frac{\sqrt{D_\alpha^j}}{t} + C_\alpha^j \exp\left(\frac{-t}{\sqrt{D_\alpha^j}}\right)$$

In all examples we will take:

First Asset $S_1 : S_1(0) = 1, \mu_1 = 5\%$

$A_1^1 = 0.3, B_1^1 = 0.01, C_1^1 = 0.01, D_1^1 = 5, \lambda_1^1 = 0.6$

$A_1^2 = .2, B_1^2 = .001, C_1^2 = -.001, D_1^2 = 3, \lambda_1^2 = 0.4$

Second Asset $S_2 : S_2(0) = 1, \mu_2 = 3\%$

$A_2^1 = 0.25, B_2^1 = 0.008, C_2^1 = 0.008, D_2^1 = 4.8, \lambda_2^1 = 0.7$

$A_2^2 = 0.35, B_2^2 = 0.0008, C_2^2 = -0.008, D_2^2 = 2.8, \lambda_2^2 = 0.3$

Average volatilities in the mixtures components

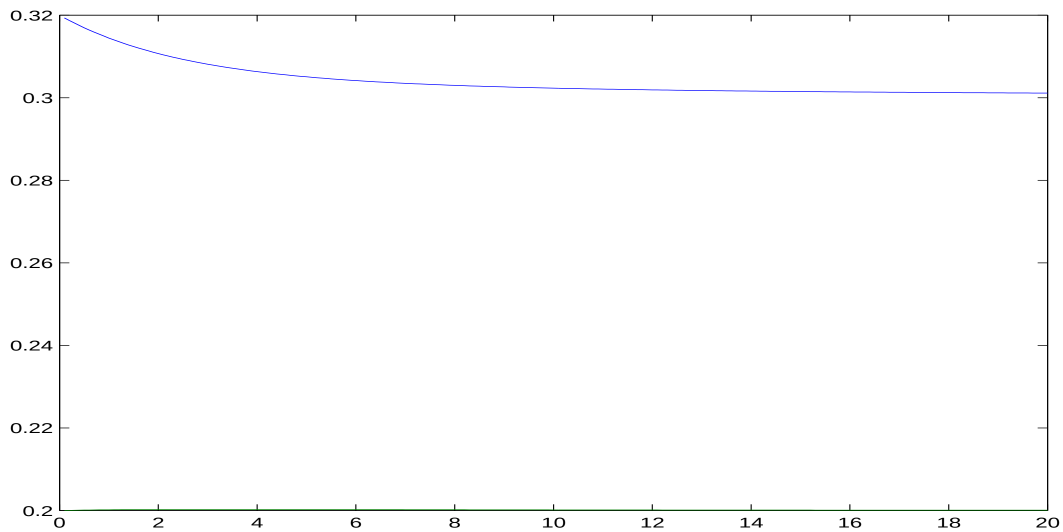


Figure 5: Average vols V_1^1 (below) and V_1^2 (above) for the mixture concurring to S_1

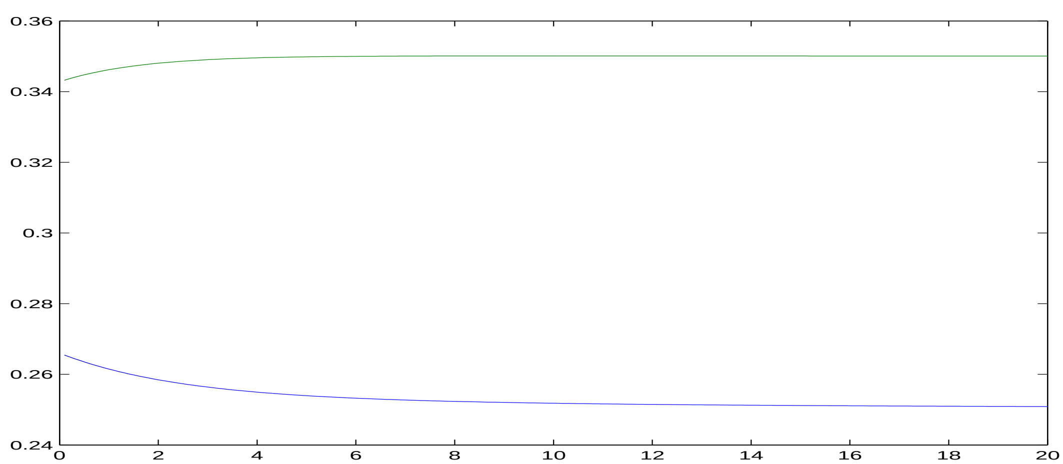


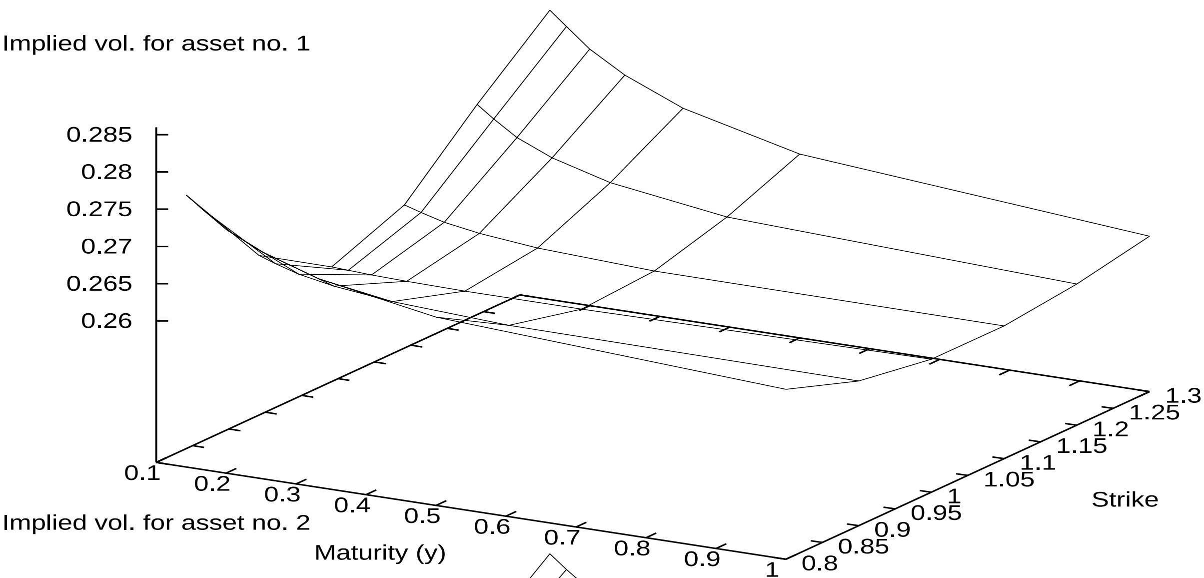
Figure 6: Average vols V_2^1 (below) and V_2^2 (above) for the mixture concurring to S_2

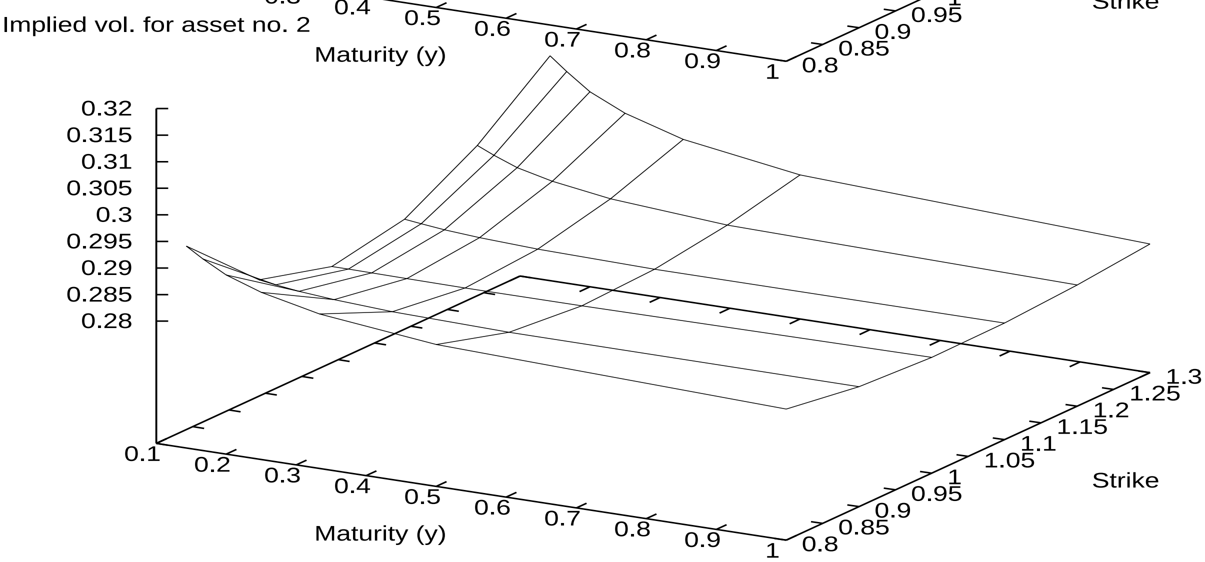
Example: 2×2 basket. Single smiles

$S_1(0) = 1$, $\mu_1 = 5\%$, $\sigma_1^1(t)$, $\sigma_1^2(t)$, $\lambda_1^1 = 0.6$, $\lambda_1^2 = 1 - \lambda_1^1$

$S_2(0) = 1$, $\mu_2 = 3\%$, $\sigma_2^1(t)$, $\sigma_2^2(t)$, $\lambda_2^1 = 0.7$, $\lambda_2^2 = 1 - \lambda_2^1$

$A_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, T =1Y.





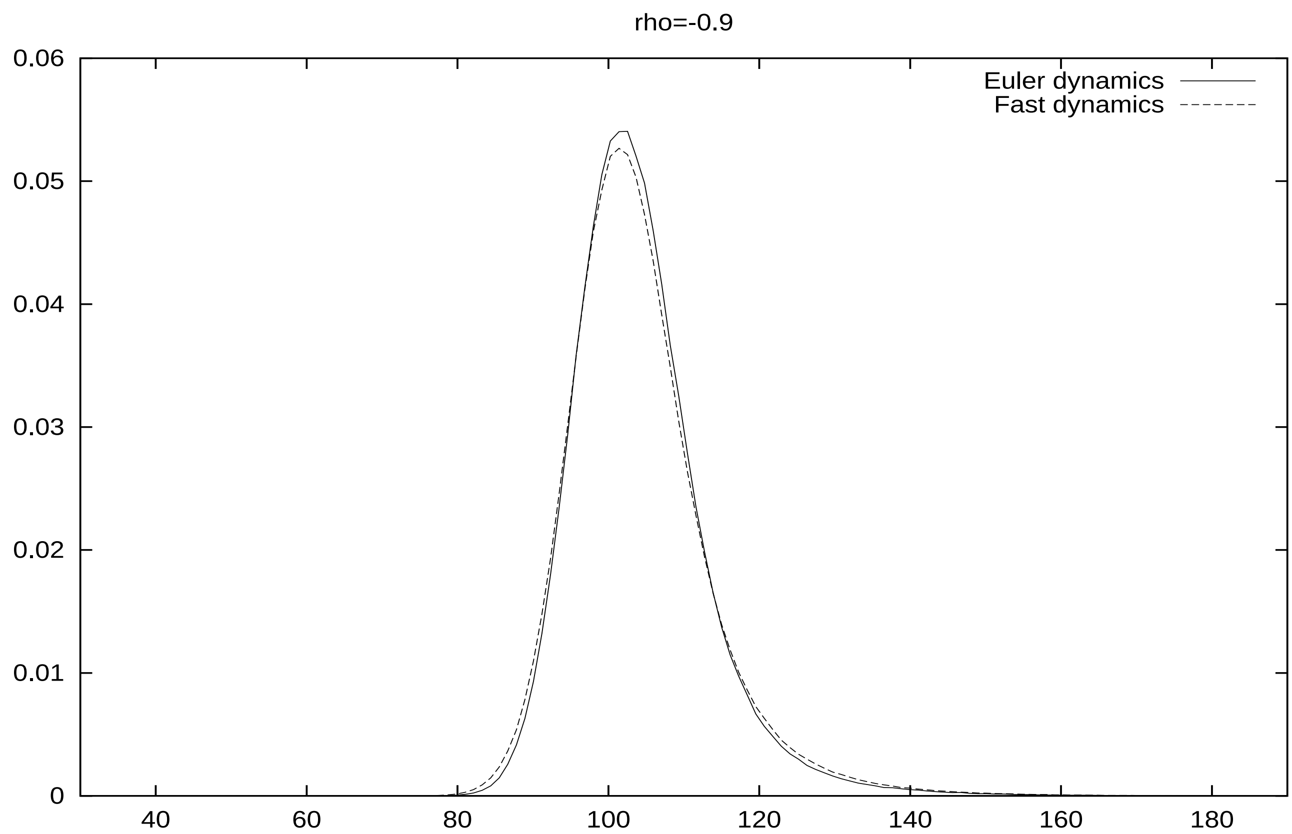
2×2 MVMD vs SCMD basket densit, $\rho = -0.9$

$S_1(0) = 1$, $\mu_1 = 5\%$, $\sigma_1^1(t)$, $\sigma_1^2(t)$, $\lambda_1^1 = 0.6$, $\lambda_1^2 = 1 - \lambda_1^1$

$S_2(0) = 1$, $\mu_2 = 3\%$, $\sigma_2^1(t)$, $\sigma_2^2(t)$, $\lambda_2^1 = 0.7$, $\lambda_2^2 = 1 - \lambda_2^1$

$A_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

$\rho = -90\%$; the basket density $p_{A(T)}(\cdot;\ \rho)$ under the SCMD-EMCscheme, continuous line; $p_{A(T)}(\cdot;\ \rho)$ under the MVMD scheme, dashed line.



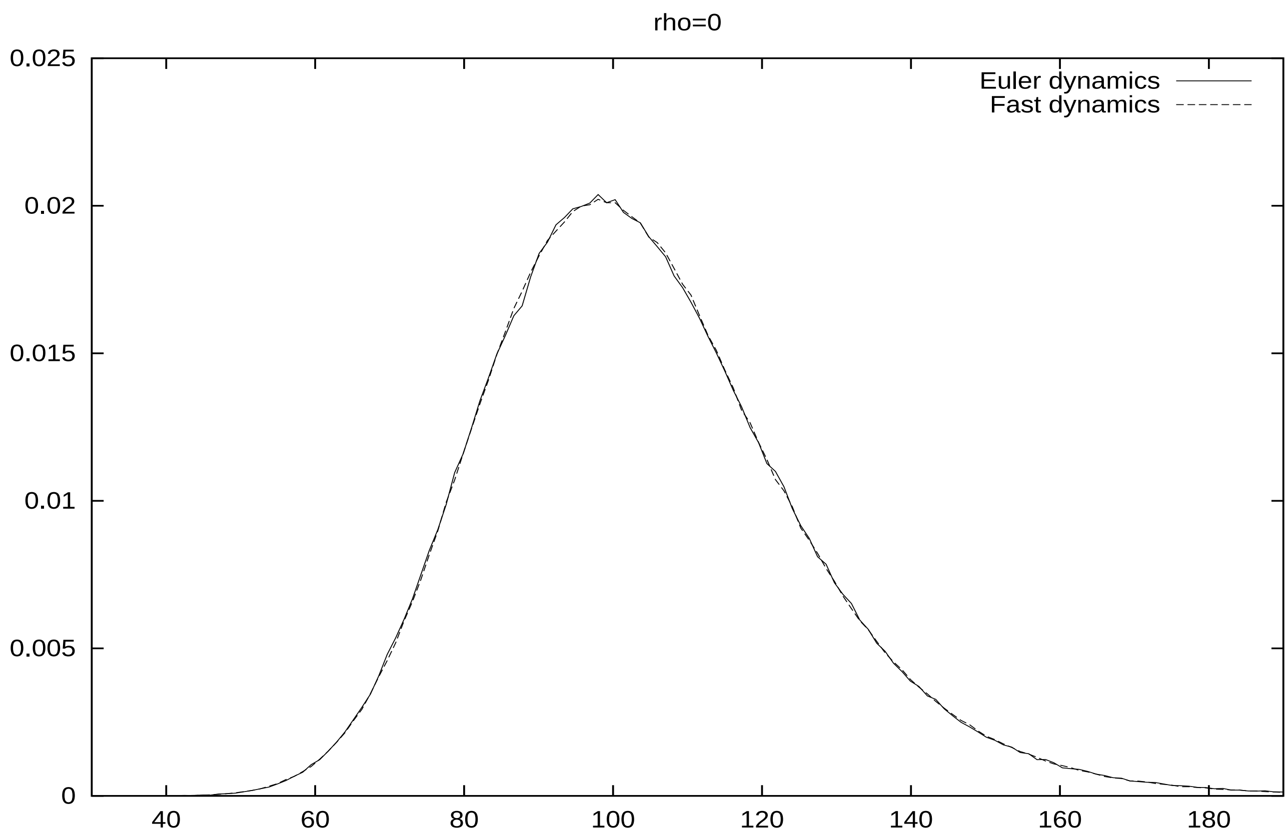
2×2 MVMD vs SCMD basket densities, $\rho = 0$

$S_1(0) = 1$, $\mu_1 = 5\%$, $\sigma_1^1(t)$, $\sigma_1^2(t)$, $\lambda_1^1 = 0.6$, $\lambda_1^2 = 1 - \lambda_1^1$

$S_2(0) = 1$, $\mu_2 = 3\%$, $\sigma_2^1(t)$, $\sigma_2^2(t)$, $\lambda_2^1 = 0.7$, $\lambda_2^2 = 1 - \lambda_2^1$

$A_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

$\rho = 0$; the basket density $p_{A(T)}(\cdot;\ \rho)$ under the SCMD-EMCscheme, continuous line; $p_{A(T)}(\cdot;\ \rho)$ under the MVMD scheme, dashed line.



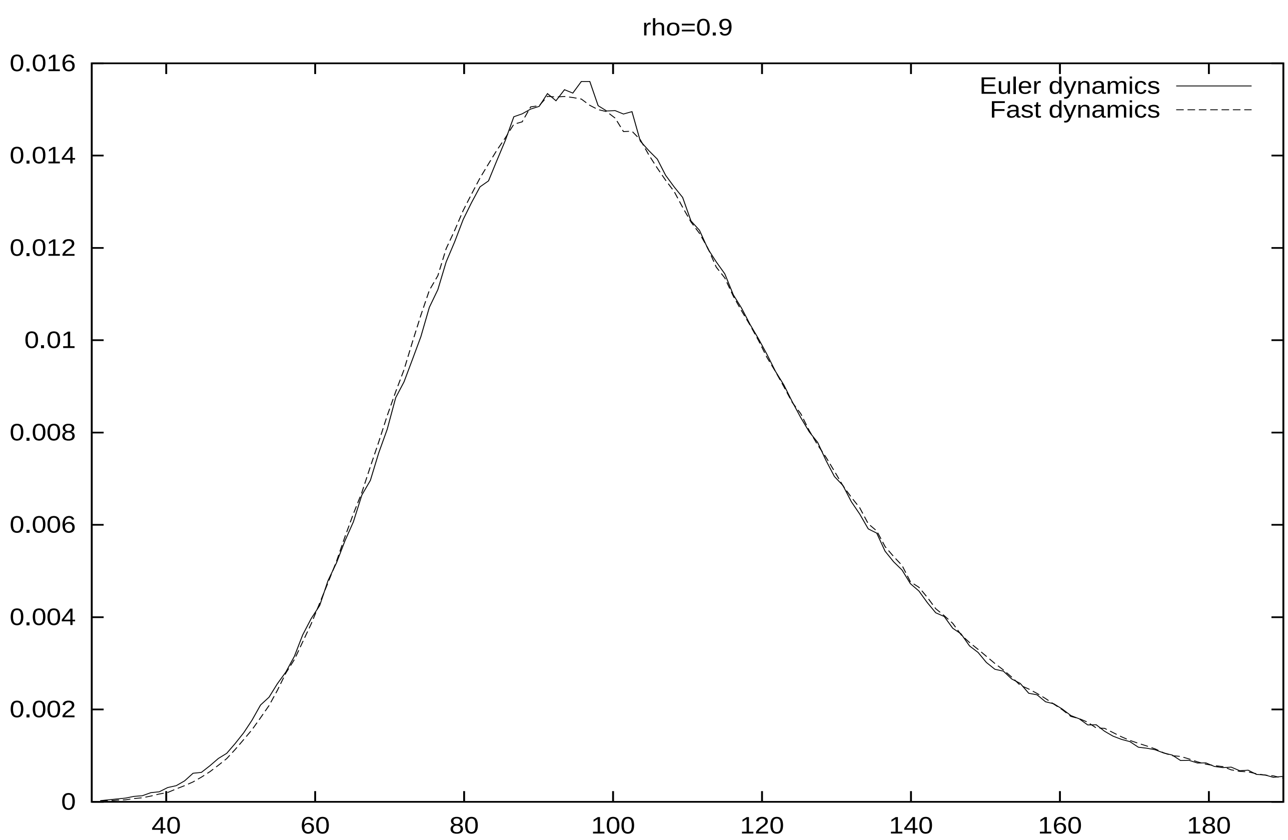
2×2 MVMD vs SCMD basket densities, $\rho = 0.9$

$S_1(0) = 1$, $\mu_1 = 5\%$, $\sigma_1^1(t)$, $\sigma_1^2(t)$, $\lambda_1^1 = 0.6$, $\lambda_1^2 = 1 - \lambda_1^1$

$S_2(0) = 1$, $\mu_2 = 3\%$, $\sigma_2^1(t)$, $\sigma_2^2(t)$, $\lambda_2^1 = 0.7$, $\lambda_2^2 = 1 - \lambda_2^1$

$A_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

$\rho = 90\%$; the basket density $p_{A(T)}(\cdot;\ \rho)$ under the SCMD-EMCscheme, continuous line; $p_{A(T)}(\cdot;\ \rho)$ under the MVMD scheme, dashed line.



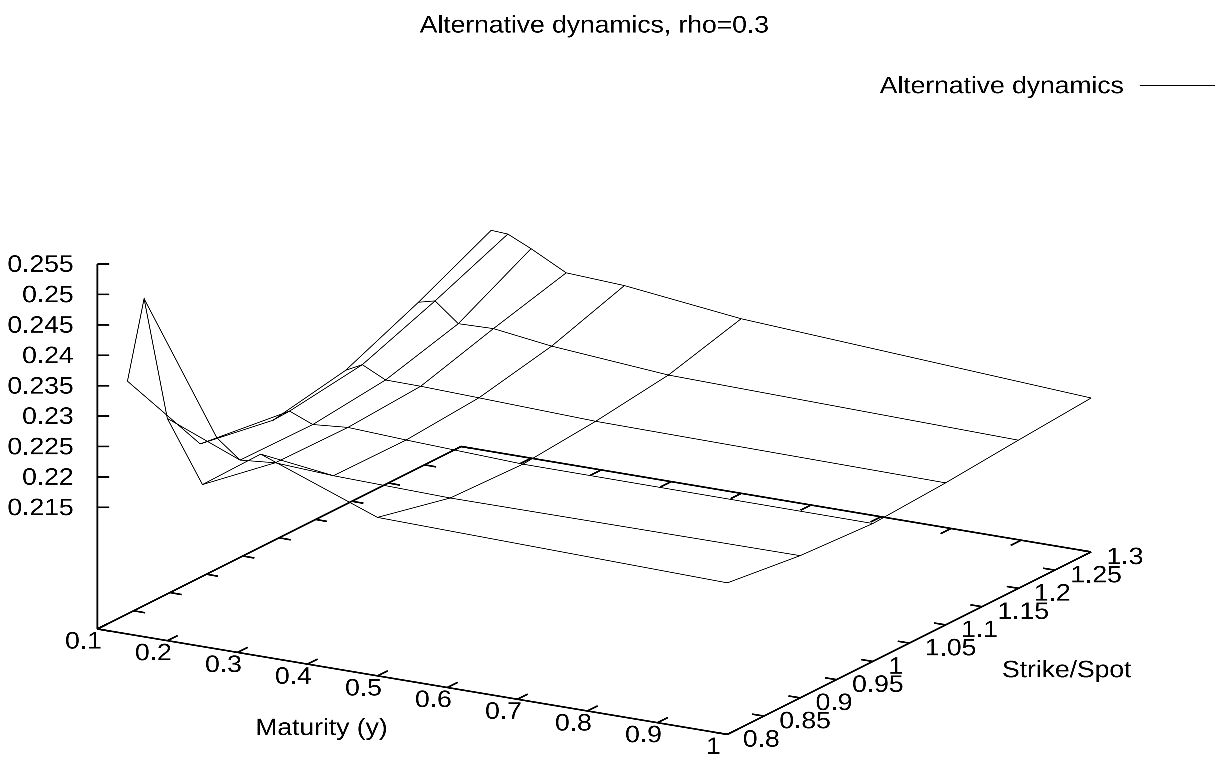
2×2 calibrated MVMD basket smile, $\rho = 0.3$

$S_1(0) = 1$, $\mu_1 = 5\%$, $\sigma_1^1(t)$, $\sigma_1^2(t)$, $\lambda_1^1 = 0.6$, $\lambda_1^2 = 1 - \lambda_1^1$

$S_2(0) = 1$, $\mu_2 = 3\%$, $\sigma_2^1(t)$, $\sigma_2^2(t)$, $\lambda_2^1 = 0.7$, $\lambda_2^2 = 1 - \lambda_2^1$

$A_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

The basket smile for $\rho = 30\%$.



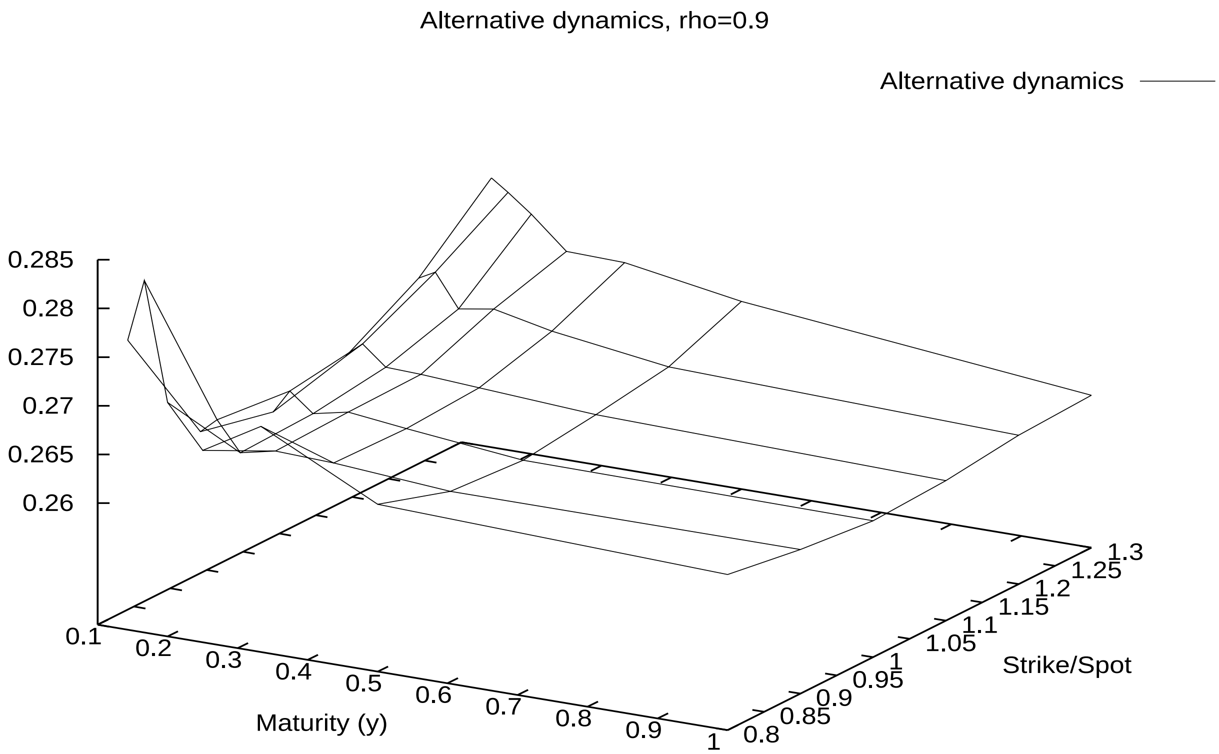
2×2 calibrated MVMD basket smile, $\rho = 0.9$

$S_1(0) = 1$, $\mu_1 = 5\%$, $\sigma_1^1(t)$, $\sigma_1^2(t)$, $\lambda_1^1 = 0.6$, $\lambda_1^2 = 1 - \lambda_1^1$

$S_2(0) = 1$, $\mu_2 = 3\%$, $\sigma_2^1(t)$, $\sigma_2^2(t)$, $\lambda_2^1 = 0.7$, $\lambda_2^2 = 1 - \lambda_2^1$

$A_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

The basket smile for $\rho = 90\%$.



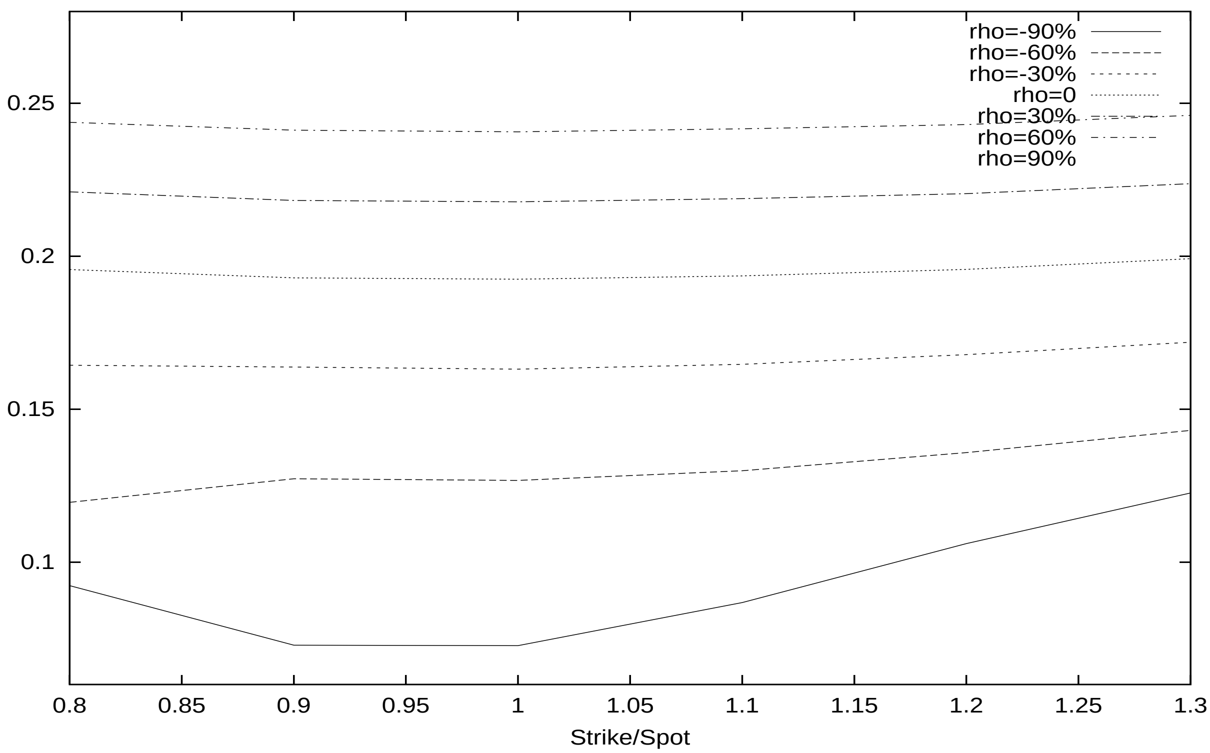
2×2 MVMD basket smile at $T = 6m$ against ρ

$S_1(0) = 1$, $\mu_1 = 5\%$, $\sigma_1^1(t)$, $\sigma_1^2(t)$, $\lambda_1^1 = 0.6$, $\lambda_1^2 = 1 - \lambda_1^1$

$S_2(0) = 1$, $\mu_2 = 3\%$, $\sigma_2^1(t)$, $\sigma_2^2(t)$, $\lambda_2^1 = 0.7$, $\lambda_2^2 = 1 - \lambda_2^1$

$A_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, T =6m.

The basket smile for T=6m for different ρ's.



Conclusions and perspectives

Mixture diffusion local volatility model: lognormal mixture and variants

- Good and analytic calibration to market data
- Explicit dynamics, $\exists$ results, complete market, delta hedging
- Known marginal densities, unknown transitions
- Possible Monte Carlo (Euler scheme)
- Decorrelation between average volatility and underlying asset
- Nested structure for parameterization;
- Extension to multivariate diffusion
- A model to coherently link a basket smile to the smiles of its single names

Current/Future work:

- Stochastic volatility versions of the mixture dynamics lose market completeness but have known transition densities, more tractable for exotic options and diagnostics
- Develop analytical approximations for exotic options with smile (e.g. barrier);
- Diagnostic tests on future volatility structures following calibration;
- More tests on baskets and the role of correlation;

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