On Dynamic Models for Portfolio Credit Risk and Credit Contagion

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1. Introduction

Applications of Credit Risk Models:

a) Credit Risk Management

- computation of loss distribution and associated risk measures (such as VaR) for portfolios of defaultable bonds and loans
- determination of risk capital (economic capital or regulatory capital)

b) Pricing of credit risky securities such as

- corporate bonds, swaps and vulnerable securities (eg. options whose writer may default)
- single name credit derivatives such as credit default swaps
- portfolio related products such as collaterized bond obligations (CDO's) or basket credit derivatives (eg. i-th-to-default swap)

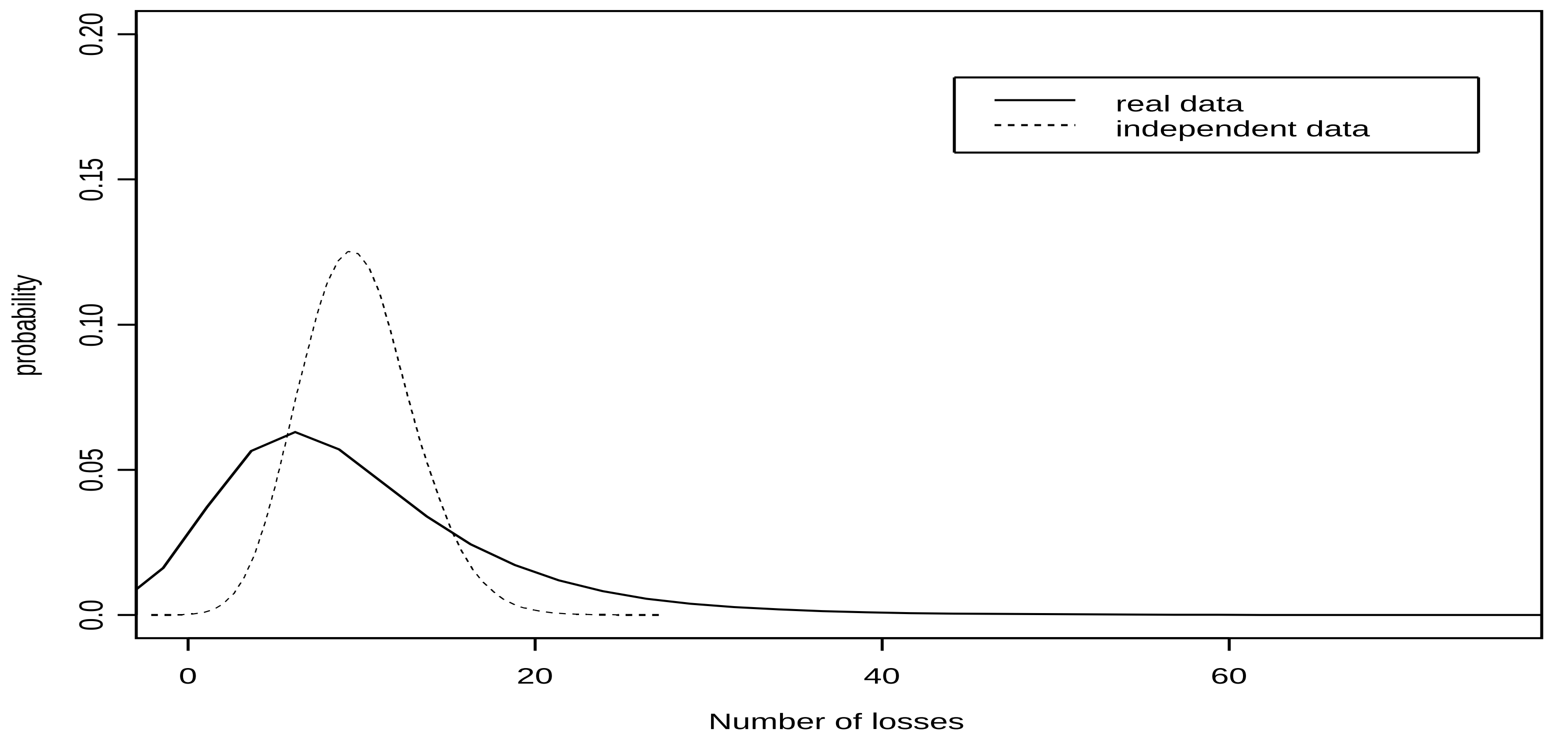
Modelling dependent defaults

Modelling of dependence between defaults critical for performance of credit risk models in both areas

Sources for dependence between defaults

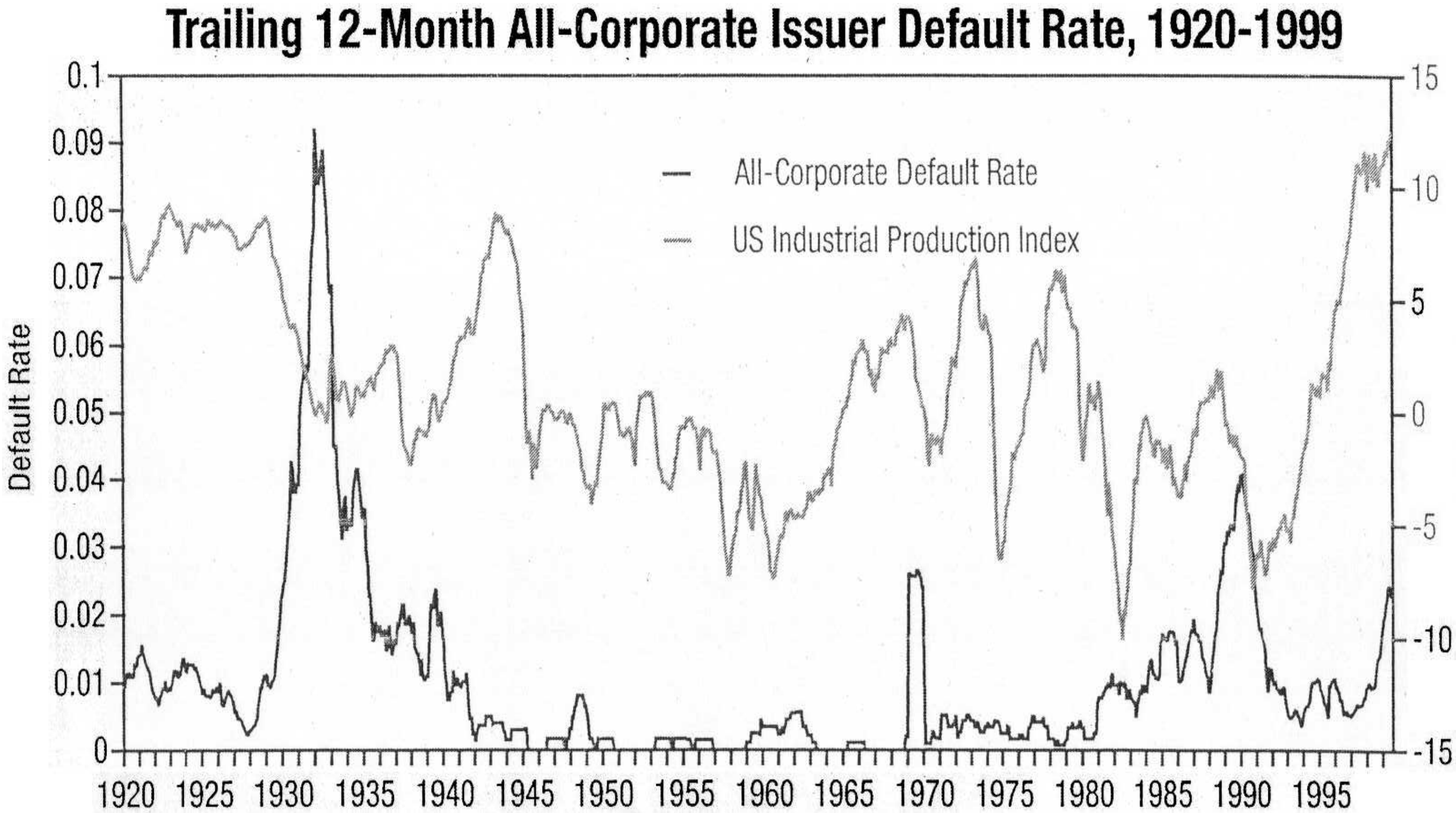
- Dependence between defaults is caused by common factors (eg. interest rates and changes in economic growth) affecting all obligors
- Direct interaction: default of company A may affect the default probability of company B and vice versa because of direct business relations. This type of interaction between defaults is termed counterparty risk
- Default of one firm may contain information about financial health/default probability of other firms

Impact of dependence on loss distribution



Number of defaults for homogeneous portfolio of 1000 BB loans (default probability $\approx 1\%$ and default correlation $\approx 0.22\%$) and independent loans with identical default probability

Number of annual defaults (Source: Moodys)



2. Existing Models: Overview

For pricing credit risky securities we need dynamic (continuous time) models. Basically there exist two classes.

- Firm value models: Default occurs if the asset value of the firm (typically modelled as a diffusion) falls below some default threshold (interpreted as liability).

- Reduced form models: default occurs at the first jump of some point process, typically with stochastic intensity.

This distinction is closely related to information available to investors. As shown eg. by [Duffie and Lando, 2001], from the viewpoint of investors who have incomplete information about firm value or default thresholds a firm value model may become a reduced form model.

Reduced-form Models: Overview

Existing reduced form portfolio models can be divided in following model classes

- Standard models with conditionally independent defaults such as [Duffie and Singleton, 1999] [Lando, 1998] . . .
- Copula models such as [Li, 2001], [Schönbucher and Schubert, 2001], [Laurent and Gregory, 2003]
- Explicit models for counterparty risk (focus of part 3 and 4).
  In this class default intensity of one firm is modelled as function of default state of other firms.

General Model Structure and Notation:

- Portfolio of m counterparties. Default-indicator process $\mathbf{Y}_t = \big(Y_t(1), \dots, Y_t(m)\big)' \in \{0,1\}^m$. $Y_t(i) = 1$ if firm i default in t, $Y_t(i) = 0$ else.
- Default time: $\tau_i = \inf\{t \geq 0 : Y_t(i) = 1\} \Leftrightarrow Y_t(i) = 1_{\{\tau_i \leq t\}}$.
- $S := \{0,1\}^m$ state space of $\mathbf{Y}$; note $\mathrm{card}\, S = 2^m$;
- d-dimensional state variable process $\mathbf{\Psi} = (\mathbf{\Psi}_t)_{t \in [0,\infty)}$ modelling the evolution of macroeconomic variables; typically $\mathbf{\Psi}$ is modelled as autonomous Markov process.
- Filtrations: Define $\mathcal{F}_t^1 := \sigma(\mathbf{\Psi}_s : s \leq t)$, $\mathcal{F}_t^2 := \sigma(\mathbf{Y}_s : s \leq t)$, $\mathcal{F}_t := \mathcal{F}_t^1 \times \mathcal{F}_t^2$; Investors have access to $\{\mathcal{F}_t\}$.

Models with conditionally independent defaults

Definition: Given functions $\lambda_i : \mathbb{R}^d \to \mathbb{R}_+$, $1 \le i \le m$ with $\int_0^t \lambda_i(\boldsymbol{\Psi}_s)ds < \infty$ a.s. for all t. $\mathbf{Y}$ resp. the random times $(\tau_i)_{1\le i\le m}$ follow a model with conditionally independent defaults and default intensities $\lambda_i(\boldsymbol{\Psi}_s)$ if

- $P(\tau_i > t \mid \mathcal{F}^1_\infty) = P(\tau_i > t \mid \mathcal{F}^1_t) = \exp\left(-\int_0^t \lambda_i(\boldsymbol{\Psi_s})ds\right)$
- The rvs $\tau_1, \ldots, \tau_m$ are conditionally independent given $\mathcal{F}^1_\infty$, i.e. given information about the economic factors

Construction: Given a vector $\boldsymbol{\Theta} = (\Theta_1, \ldots, \Theta_m)'$ of independent standard exponentially distributed rvs indep. of $\mathcal{F}^1_\infty$. Then $\tau_i := \inf\{t \ge 0 : \int_0^t \lambda_i(\boldsymbol{\Psi_s})ds \ge \Theta_i\}$ has the desired properties.

Relation to martingales: $Y_t(i) - \int_0^{t\wedge\tau_i} \lambda_i(\boldsymbol{\Psi}_s)ds$ is an $\{\mathcal{F}_t\}$-martingale.

Models with conditionally independent defaults II

Advantage: Easy to treat; in particular, the models lead to valuation formulae for credit derivatives with similar form as in default-free term-structure models.

Example: Consider vulnerable claim $H = X_T \cdot 1_{\{\tau_i \geq T\}}$ for X $\mathcal{F}_T^1$-measurable. Assume that spot rate equals $r_t = r(\mathbf{\Psi}_t)$ and that $\mathbf{Y}$ follows model with cond. independent defaults under equivalent martingale measure Q. Then price of H in t equals

$$H_t = E^Q\Big(\exp(-\int_t^T r(\mathbf{\Psi}_s)ds)\ H \mid \mathcal{F}_t\Big) \tag{1}$$

$$= 1_{\{\tau_i > t\}} E^Q\Big(\exp(-\int_t^T r\big(\mathbf{\Psi}_s) + \lambda_i(\mathbf{\Psi}_s)ds\big)\ X_T \mid \mathcal{F}_t^1\Big) \tag{2}$$

These expressions can be computed using techniques from default-free term structure models; computations are particularly easy in affine models

Affine Models: CIR Square Root Diffusions

Suppose that Ψ follows a CIR square root diffusion model, i.e. $d\Psi_t = \kappa(\bar{\theta} - \Psi_t)dt + \sigma\sqrt{\Psi_t}dW_t$ for constants $\kappa, \bar{\theta}, \sigma > 0$, so that drift and diffusion coefficients are affine functions of Ψ_t. Then we get for any constant γ

$$E\left(\exp(-\int_t^T \gamma\Psi_s ds) \mid \mathcal{F}_t\right) = \exp(\alpha(t,T) + \beta(t,T)\Psi_t) \qquad (3)$$

where α and β are functions only of time, which solve the ODEs

$$\dot{\alpha} = -\kappa\bar{\theta}\beta \quad \dot{\beta} = \gamma + \kappa\beta - \tfrac{1}{2}\sigma^2\beta^2\,,\ \alpha(T,T) = \beta(T,T) = 0.$$

(the ODE for β is a Ricatti equation and can be solved explicitly).

From this and relations (3), (1) prices of defaultable bonds are easy to compute. For extensions see the work of Duffie, Singleton and coworkers summarized eg. in [Duffie and Singleton, 2003]

Copula Model

Disadvantage of models with conditional independent defaults: Default correlations often not sufficiently high, and 'simultaneous defaults too rare compared to data if $\mathbf{\Psi}$ is identified with smooth macroeconomic factors, in particular in small portfolios. $\Rightarrow$ development of extensions such as the copula model.

Idea of copula model: In models with cond. independent defaults

$$\tau_i = \inf\{t \geq 0 : \int_0^t \lambda_i(\mathbf{\Psi}_{\boldsymbol{s}})ds \geq \Theta_i\}$$

$$= \inf\{t \geq 0 : 1 - \exp(\int_0^t \lambda_i(\mathbf{\Psi}_{\boldsymbol{s}})ds) \geq U_i := 1 - \exp(\Theta_i)\}\,,$$

where the U_i are independent and $\mathcal{U}(0,1)$ distributed. In copula models we generate other dependence structure by assuming $\mathbf{U} = (U_1, \ldots, U_m) \sim C$, where C is some copula function, i.e. a distribution function on $[0,1]^m$ with uniform marginals (but dependent components).

Copula models: some examples

Gaussian Copula: an implicit copula. Let $\mathbf{X}$ be standard multivariate normal with correlation matrix P.

$$
\begin{aligned}
C_P^{\mathsf{Ga}}(u_1, \ldots, u_d) &= P\left(\Phi(X_1) \le u_1, \ldots, \Phi(X_d) \le u_d\right) \\
&= P\left(X_1 \le \Phi^{-1}(u_1), \ldots, X_d \le \Phi^{-1}(u_d)\right)
\end{aligned}
$$

where Φ is df of standard normal. $P = I$ gives independence. Similar construction with other distributions such as multivariate t.

Clayton Copula: a parametric copula. Here

$$
C_\beta^{Cl}(u_1, \ldots, u_d) = \left(u_1^{-\beta} + \ldots + u_d^{-\beta} - 1\right)^{-1/\beta}
$$

for $\beta > 0$. $\beta \to 0$ gives independence ; $\beta \to \infty$ gives comonotonicity (perfect positive dependence) .

Copula models

Factor copulas. Often dependence between the U_i is generated by assuming that all components depend on common unobservable factor Z. For instance in Gaussian copula we may assume that $U_i = \Phi(X_i)$ and $X_i = \rho Z + \sqrt{1-\rho^2}\epsilon_i$ for Z, ϵ_i independent standard normal. Z models unobservable shock which hits all firms.

Advantages of copula model. In copula models we may have credit contagion,i.e. default of firm i affects conditional default probability of firm $j \neq i$.
Moreover, copula models are easy to calibrate to given term structure of credit spreads, as prices of corporate bonds can be computed as in model with conditionally independent defaults, at least in $t = 0$.

Drawbacks. Results of model are very sensitive to copula choice; Unintuitive parametrization of dependence.

3. Explicit Models for Counterparty Risk

General idea: Include counterparty risk by modelling default intensity as function $\lambda_i(\mathbf{\Psi}_t, \mathbf{Y}_t)$ of the state variables $\mathbf{\Psi}_t$ and of the state $\mathbf{Y}_t$ of other obligors at time t;

Advantages: Intuitive parametrization of dependence; accounting for counterparty risk; preserves ease of simulation
Disadvantages: Calibration potentially difficult; new techniques for evaluation needed;

Our presentation is based on [Frey and Backhaus, 2003]. Our contributions

- We employ Markov process techniques such as Kolmogorov equations
- We study models with mean-field interaction, leading to parsimonious, plausible and relatively tractable models

Related work

- [Davis and Lo, 2001]: Counterparty risk, uses Markov chains
- [Jarrow and Yu, 2001]: only very special types of interaction; model is studied using Cox process techniques. Extensions by [Yu, 2002], (proper model construction and study of default correlations.)
- [Kusuoka, 1999] and [Bielecki and Rutkowski, 2002]: mathematical aspects
- [Collin-Dufresne et al., 2002]: analytical evaluation of certain derivatives using a particular change of measure
- [Gieseke and Weber, 2002]: application of interacting particle systems literature to default contagion

4. The Model

Informal description: Evolution of macroeconomic variables modelled by d-dimensional state variable process $\mathbf{\Psi} = (\mathbf{\Psi}_t)_{t\in[0,\infty)}$;

For a given realization of the state variable process the default indicators $\mathbf{Y}_t$ are constructed (and simulated) as time inhomogeneous Markov chain using Markov process techniques. Transition rates of this chain correspond to default intensities. Formally the Markov chain is constructed using transition kernels.

Formal description:

Denote by $S := \{0,1\}^m$ state space of $\mathbf{Y}$; note $|S| = 2^m$.

Define $\Omega_1 := \mathbf{D}([0,\infty),\mathbb{R}^d)$, $\Omega_2 := \mathbf{D}\left([0,\infty), S\right)$, with standard filtration $\{\mathcal{F}_t^i\}$; put $(\Omega,\mathcal{F}) := (\Omega_1 \times \Omega_2, \mathcal{F}_\infty^1 \times \mathcal{F}_\infty^2)$, $\mathcal{F}_t = \mathcal{F}_t^1 \times \mathcal{F}_t^2$.

Let $\mathbf{\Psi}$ be coordinate process on Ω_1, $\mathbf{Y}$ coordinate process on Ω_2.

The Model ctd

Probability measure. Consider measures of the form $P = \mu \otimes K_{\mathbf{y}}(\omega_1, d\omega_2)$ for some measure μ on Ω_1 (the law of the factors) and a stochastic kernel $K_{\mathbf{y}} : \Omega_1 \times \mathcal{F}^2 \to [0, 1]$.

The next assumption determines dynamics of $\mathbf{Y}_t$ for given ω_1.

Assumption. Under $K_{\mathbf{y}}(\omega_1, d\omega_2)$ $\mathbf{Y}$ is a time-inhomogeneous Markov chain with initial value $\mathbf{y}$ and infinitesimal generator

$$G_{\left[\boldsymbol{\Psi}_{t(\omega_1)}\right]} f(\mathbf{y}) = \sum_{i=1}^{m} (1 - \mathbf{y}(i))\, \lambda_i \left(\boldsymbol{\Psi}_t(\omega_1), \mathbf{y}\right)\right) \big(f(\mathbf{y}^i) - f(\mathbf{y})\big) \,,$$

where $f : S \to \mathbb{R}$, and $\mathbf{y}^i$ is obtained from $\mathbf{y}$ by flipping the ith coordinate.

Comments

Under previous assumption we have

- $\lambda_i(\mathbf{\Psi}_t, \mathbf{Y}_t)$ is the martingale default intensity of firm i, i.e. $Y_t(i) - \int_0^{t\wedge\tau_i} \lambda_i(\mathbf{\Psi}_s, \mathbf{Y}_s)ds$ is a martingale. (τ_i default time of obligor i).
- The form of $G_{[\mathbf{\Psi}_{t(\omega_1)}]}$ excludes joint defaults
- If $\mathbf{\Psi}$ is Markov, the pair $\mathbf{\Gamma}_t = (\mathbf{\Psi}_t, \mathbf{Y}_t)$ is jointly Markov
- The model is as easy to simulate from as the model with conditional independent defaults, using the standard approach for simulating Markov chains.

Examples for default intensity λ_i

Primary secondary framework. Type of interaction considered in [Jarrow and Yu, 2001]. Two types of firms

- primary firms: their default intensity depends only on $\mathbf{\Psi}$
- secondary firms: their default intensity depends on $\mathbf{\Psi}$ and on default-state of primary firms

Note that here a firm cannot simultaneously affect other firms and be affected by them.
Typical interpretation: primary firms big corporations, secondary firms commercial banks with credit exposure to primary firms.

Example: $m = 2$ and $d = 1$; Ψ_t is identified with short rate r_t; firm 1 is primary, firm 2 secondary. Put

$$\lambda_1(r, \mathbf{Y}) = \lambda_{1,0} + \lambda_{1,1} r \,, \quad \lambda_2(r, \mathbf{Y}) = \lambda_{2,0} + \lambda_{2,1} r + \lambda_{2,2} 1_{\{Y(1)=1\}}$$

Models with mean-field interaction

Primary secondary framework typical example of local interaction, where default intensity is affected by state of a few "neighbours". Alternatively consider global or mean-field interaction. Here default intensities depend on the overall proportion of defaulted firms at t.

Formally. Define empirical distribution $\rho(\mathbf{Y}_t, \cdot) = \frac{1}{m}\sum_{i=1}^m \delta_{Y_t(i)}(\cdot)$ of defaults at t and note that $\rho(\mathbf{Y}_t, \{1\})$ gives proportion of defaulted companies. Put $\lambda_i\left(\mathbf{\Psi}_t, \mathbf{Y}_t\right) = h\big(\mathbf{\Psi}_t, \rho(\mathbf{Y}_t, \{1\})\big)$ for some continuous function $h : \mathbb{R}^d \times [0,1] \to \mathbb{R}$;

Remarks. 1) Typically h will be increasing in its second argument.
2) Our results extend to models with different homogeneous groups.
3) Mean field interaction makes immediate economic sense; moreover, it is natural type of interaction for a portfolio consisting of different homogenous goups.

Kolmogorov equations

Kolmogorov equations are useful tools for analyzing the model. Define for $0 \le t \le s < \infty$ conditional transition function of $\mathbf{Y}$ by

$$p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1) := E^{K(\omega_1, \cdot)}\left(\mathbf{Y}_s = \mathbf{y} \mid \mathbf{Y}_t = \mathbf{x}\right), \, \mathbf{x}, \mathbf{y} \in S\,.$$

Backward equation. for $(t, \mathbf{x}) \to \tilde{p}(t, \mathbf{x}) = p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)$

$$\frac{\partial}{\partial t}\tilde{p}(t, \mathbf{x}) + \sum_{i=1}^{m}(1 - x(i))\lambda_i\left(\mathbf{\Psi}_t(\omega_1), \mathbf{x}\right)\left(\tilde{p}(t, \mathbf{x}^i) - \tilde{p}(t, \mathbf{x})\right) = 0$$

Forward equation. for $(s, \mathbf{y}) \to \bar{p}(s, \mathbf{y}) = p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)$

$$\begin{aligned}\frac{\partial}{\partial s}\bar{p}(s, \mathbf{y}) &= \sum_{k=1}^{m}(y(k))\lambda_k\left(\mathbf{\Psi}_s(\omega_1), \mathbf{y}^k\right)\bar{p}(s, \mathbf{y}^k) \\ &\quad - \sum_{k=1}^{m}(1 - y(k))\lambda_k\left(\mathbf{\Psi}_s(\omega_1), \mathbf{y}\right)\bar{p}(s, \mathbf{y})\end{aligned}$$

Note that both equations are ODE-systems of size $S = 2^m$.

Pricing credit risky bonds and vulnerable claims

Assumptions: Model structure satisfied under equivalent martingale measure Q; $\boldsymbol{\Psi}$ is Markov under Q; default free short-rate equals $r_t = r(\boldsymbol{\Psi_t})$.
Consider vulnerable claim $H = f(\boldsymbol{\Psi}_T)g(\mathbf{Y}_T)$. An example is a defaultable zero coupon bond with zero recovery where $f \equiv 1$, $g(\mathbf{Y}_T) = 1_{\{Y_T(i)=0\}}$. Price of H in $t < T$:

$$\begin{aligned}
H_t &= E^Q\Big(\exp(-\int_t^T r(\boldsymbol{\Psi}_s)ds\Big)H \mid \mathcal{F}_t\Big) \\
&= E^Q_{(\boldsymbol{\Psi}_t,\mathbf{Y}_t)}\Big(\exp(-\int_0^{T-t} r(\boldsymbol{\Psi}_s)ds\Big)f(\boldsymbol{\Psi}_{T-t})g(\mathbf{Y}_{T-t})\Big) \\
&= E_{\boldsymbol{\Psi}_t}\Big(\exp(-\int_0^{T-t} r(\boldsymbol{\Psi}_s)ds\Big)f(\boldsymbol{\Psi}_{T-t})E^{K_{\mathbf{Y}_t}(\omega_1,d\omega_2)}(g(\mathbf{Y}_{T-t}))\Big)
\end{aligned}$$

and the inner expectation can be computed via Kolmogorov.

5. Models with mean field interaction

For large portfolios general types of interaction too complex due to exponential growth of state space. Model with homogeneous groups and mean field interaction economically reasonable way to reduce size of state space.

Model: Put $\overline{M}_t := \rho(\mathbf{Y}_t, \{1\})$ (proportion of defaulted firms), and assume $\lambda_i\left(\mathbf{\Psi}_t, \mathbf{Y}_t\right) = h\big(\mathbf{\Psi}_t, \overline{M}_t\big)$. (only one group for simplicity)

Dynamics of $\overline{M}_t$: Under our assumptions $\overline{M}_t$ follows a conditional Markov chain with generator $G^{\overline{\mathbf{M}}}_{[\mathbf{\Psi}_{\boldsymbol{t}}(\omega_1)]}$ given by

$$G^{\overline{\mathbf{M}}}_{[\mathbf{\Psi}_t(\omega_1)]} f\left(\bar{l}\right) = m(1-\bar{l})h\left(\mathbf{\Psi}_t(\omega_1), \bar{l}\right)\left(f\left(\bar{l} + \frac{1}{m}\right) - f\left(\bar{l}\right)\right)$$

and state space $S^{\overline{M}} = \{0, \frac{1}{m}, \ldots, 1\}$. Note that $|S^{\overline{M}}| = m+1$.

Models with mean field interaction ctd

Default probabilities and correlations. These quantities can be computed from distribution of $\overline{M}_t$ using **exchangeability**. We have

$$P\left(Y_T(i) = 1\right) = E(\overline{M}_T) \text{ and } P\left(Y_T(i) = 1, Y_T(j) = 1\right) \approx E\left(\left(\overline{M}_T\right)^2\right)$$

Limits for large portfolios. Suppose that $m \to \infty$, and consider sequence of models as above with distribution of factors $\mathbf{\Psi}$ identical for all m and with $h^m \to h^\infty$ uniformly on compacts. Define $\overline{\mathbf{M}}_t^{(\infty)}(\omega_1)$ as solution of the following system of ODE's with random coefficients

$$\frac{d}{dt}\overline{M}_t^{(\infty)}(\omega_1) = \left(1 - \overline{M}_t^{(\infty)}\right)(\omega_1)\, h^{(\infty)}\left(\mathbf{\Psi}_t(\omega_1), \overline{M}_t^{(\infty)}(\omega_1)\right). \quad (4)$$

Limits for large portfolios

Proposition. As $m \to \infty$ the sequence $(\boldsymbol{\Psi}, \overline{\mathbf{M}}^{(m)})$ converges in distribution to $(\boldsymbol{\Psi}, \overline{\mathbf{M}}^{(\infty)})$.

Comments

- Note that $\left[t \mapsto \overline{\mathbf{M}}_t^{(\infty)}(\omega_1)\right]$ is deterministic given ω_1. Hence for $m \to \infty$ the proportion of defaulted companies is fully determined by the evolution of the economic factors. Similar result in static Bernoulli mixture models; see [Frey and McNeil, 2003] or [Gordy, 2001].

- Impact of fluctuations of $\boldsymbol{\Psi}_t$ on proportion of defaulted firms is increased by mean-field interaction.

An affine reference model

Given constants $\lambda_j \geq 0$, $j = 0, \ldots, d+1$ and "average default intensity" $\bar{\lambda}$ we put

$$h(t, \boldsymbol{\psi}, l) = \left[\lambda_0 + \sum_{j=1}^{d} \lambda_j \psi_j + \lambda_{d+1}(l - (1 - e^{-\bar{\lambda} t}))\right]^+.$$

We assume that factor processes follow square-root diffusions.

Interpretation. $\lambda_{d+1} > 0 \Rightarrow$ default intensity is increased (decreased) if proportion of defaulted companies higher (lower) than expected. $\lambda_{d+1} = 0 \Rightarrow$ standard framework.

Form of limit model:

$$\frac{d}{dt}\overline{M}_t^{(\infty)} = (1 - \overline{M}_t^{(\infty)})\left[\lambda_0 + \sum_{j=1}^{d} \lambda_j \Psi_{t,j} + \lambda_{d+1}\left(\overline{M}_t^{(\infty)} - (1 - e^{-\bar{\lambda} t})\right)\right]^+.$$

Default correlations in affine model

We now study default correlation and quantiles of $\overline{M}_T$ in our reference model (joint work with J. Backhaus, Leipzig).

Fix horizon $T > 0$. Default correlation of two firms i, j then given by $\mathrm{corr}(Y_T(i), Y_T(j))$; we have (as $Y_T(i), Y_T(j)$ are exchangeable)

$$\mathrm{corr}(Y_T(i), Y_T(j)) = \rho_Y(T) := \frac{E(Y_T(i)Y_T(j)) - \pi^2}{\pi - \pi^2}, \text{ where}$$

$\pi = E(\overline{M}_T)$ is default probability of individual firm.

We carried out a simulation study in our affine model with one factor, varying the strength of mean-field interaction as given by λ_2. Parameters for factor model were taken from Yu (2002) and adjusted so that default probabilities remain unaltered as λ_2 increases.

Some results from a simulation study

100 firms						
λ_2	$P(Y_1(i)=1)$	ρ_Y	Quantile			
			90%	95%	97.5%	99%
0	0.031987	0.000416	0.06	0.06	0.07	0.08
1	0.031989	0.020918	0.07	0.09	0.11	0.13
3	0.031997	0.19118	0.12	0.21	0.29	0.38

The case $m=\infty$						
λ_2	$P(Y_1(i)=1)$	ρ_Y	Quantile			
			90%	95%	97.5%	99%
0	0.03199	0.00042	0.0367	0.0380	0.0393	0.0408
1	0.03199	0.00041	0.0390	0.0409	0.0429	0.0452
3	0.031982	0.00043	0.0503	0.0554	0.0611	0.0669

Conclusion and Outlook

- High quantiles and (to a lesser extent) default correlation are increased by mean-field interaction. Effect of varying λ_2 more pronounced for smaller portfolios
(Default dependency where it is most needed).

- Mean-field interaction might be possible way to generate clustering of defaults observed in real data. More generally: some concepts from statistical physics could be useful for modelling interactive phenomena in finance and economics.

- Next steps: consider pricing of basket credit derivatives.

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