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# On Utility Maximization in Incomplete Markets

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based on two joint papers with
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## The problem

We are interested in the utility maximization problem:

$$\sup_{H \in \mathcal{H}} E[u(x + (H \cdot X)_T)]$$

$u : \mathbb{R} \to \mathbb{R}$ is strictly concave increasing differentiable

$$\lim_{x \to -\infty} u'(x) = +\infty, \qquad \lim_{x \to +\infty} u'(x) = 0,$$

$x \in \mathbb{R}$ is the initial endowment, $T \in (0, +\infty]$,

$X = (X_t)_{t \in [0,T]}$ is an $\mathbb{R}^d-$valued càd-làg semimartingale,

which models the discounted prices of $d$ assets,

$\mathcal{H} \subseteq \left\{ \mathbb{R}^d - \text{valued, predictable, } X - \text{integrable proc. } H \right\}$

is an appropriate class of “ADMISSIBLE” integrands

$H \cdot X = \int H_s \cdot dX_s$ is the stochastic integral

- $X$ IS NOT NECESSARILY LOCALLY BOUNDED
- new concept of admissibility

# Definition of "classical" admissible strategies

It is common knowledge that some restrictions must be put on the class of trading strategies.

In the literature (Harrison and Pliska (1981), Delbaen and Schachermayer (1994) and many others) it is used the following:

DEFINITION
A trading strategy $H$ is admissible (we will say $1-$ADMISSIBLE) if there exists a constant $c \in \mathbb{R}$ such that, $P - a.s.$,

$$(H \cdot X)_t \geq -c1 \quad \text{for all } t \in [0,T].$$

$\mathcal{H}^1$ is the class of these $1-$ADMISSIBLE strategies.

The financial interpretation of $c$ is a finite credit line which the investor must respect in her trading.

In the NON LOCALLY BOUNDED case it can happen that:

$$\mathcal{H}^1 = \{0\}$$

and this fact forces us to introduce the less restrictive notion of $W-$admissibility, in order to provide a non trivial enlargement of the class $\mathcal{H}^1$

# Motivation (1)

$X = (X_0, X_1)$ one period process with $X_0 = 0$ and $X_1$ normally distributed $N(\mu, \sigma^2)$.

$\mathcal{H} = \{H$ predictable and $X-$ integrable$\} = \mathbb{R}$

For $H \in \mathcal{H}$, the stochastic integral

$$(H \cdot X)_1 = \alpha X_1 \qquad \alpha \in \mathbb{R}$$

is not bounded from below, unless $\alpha = 0$. Hence:

$$\mathcal{H}^1 = \{0\}\,.$$

Take $u(x) = -e^{-x}$, then

$$\sup_{H \in \mathcal{H}^1} E[u(x + (H \cdot X)_1)] = u(x+0) = -e^{-x}.$$

If we optimize over the whole class $\mathcal{H}$ we get:

$$\begin{aligned}\sup_{H \in \mathcal{H}} E[u(x + (H \cdot X)_1)] &= \sup_{\alpha \in R} E[u(x + \alpha X_1)] \\ &= -e^{-(x + \frac{1}{2}\frac{\mu^2}{\sigma^2})} > -e^{-x}.\end{aligned}$$

The maximizer is given by

$$H^* = \alpha^* = \frac{\mu}{\sigma}$$

which is not in $\mathcal{H}^1$, if $\mu \neq 0$.

# Moral

Accepting a greater risk may increase the expected utility.

The investor may trade in such a risky market, where potential losses are unlimited, in order to increase his expected utility.

We translate this attitude into mathematical terms by employing a class $H^W$ of $W-$admissible strategies which depend on a loss random variable $W$.

We will impose two conditions on the random amount $W$ that controls the losses admitted in the trading.

(1) A first condition "$W$ is SUITABLE" will guarantee that the set $\mathcal{H}^W$ is not reduced to zero.

(2) The second condition imposes that the $W-$admissible trading strategies are COMPATIBLE with the preferences of the investor, i.e. it assures that the expected utility of terminal wealths $x + (H \cdot X)_T$ from all $W$-admissible trading strategies is never $-\infty$.

With $\mathcal{W}$ we will denote the set of LOSS variables, i.e. of SUITABLE and COMPATIBLE random variables $W$.

Our approach is based on two main points:

(1) The selection of a natural class of ADMISSIBLE INTEGRANDS (i.e. trading strategies) that are appropriate for not necessarily bounded semimartingales;

(2) The DUAL approach.

# Definition of $W-$admissible strategies

DEFINITION

Let $W \in L^0(P)$ be a fixed random variable, $W \geq 1$ $P-$ a.s.

The $\mathbb{R}^d-$valued predictable $X-$integrable process $H$ is $W-$ADMISSIBLE, or it belongs to $\mathcal{H}^W$, if there exists a nonnegative constant $c$ such that, $P-$ a.s.,

$$(H \cdot X)_t \geq -cW \quad \forall t \leq T.$$

This natural extension of the notion of admissibility was already used in Schachermayer (1994) in the context of the fundamental theorem of asset pricing, as well as in Delbaen Schachermayer (1998), but was never used in the framework of utility maximization for general semimartingale models.

Note that

$$\mathcal{H}^1 \subseteq \mathcal{H}^W,$$

since $W \geq 1$.

## Will our results depend on the selection of $W$ ?

This question is of primarily importance.

Our aim is to find the set $\mathcal{W}$ of loss variables $W$ for which $\mathcal{H}^1 \subset \mathcal{H}^W$ and

- the optimal value and
- the optimal solution

do not depend on which particular $W$ is selected in $\mathcal{W}$.

Answer:

NO,

our results will depend only on the whole set

$$\mathcal{W}$$

of loss variables.

## $X-$SUITABLE

DEFINITION:
A random variable $W \in L^0(P)$ is $X-$SUITABLE (or simply suitable) if $W \geq 1\ P-$ a.s. and

for all $1 \leq i \leq d$ there exists a process $H^i$ such that:

- $(0, \cdots, H^i, \cdots, 0) \in \mathcal{H}^W \cap (-\mathcal{H}^W)$
- $P(\{\omega \mid \exists t \geq 0\ H_t^i(\omega) = 0\}) = 0$.

This means that:

$H^i \neq 0$ and both investments $H^i$ and $-H^i$ in the single asset $X^i$ are "$W$-admissible".

The next simple proposition shows that the notion of $W$-admissibility is indeed a generalization of the standard concept of admissibility.

PROPOSITION:
If $X$ is locally bounded, then

the constant 1 is suitable.

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# Compatibility with $u$

DEFINITION:
A random variable $W \in L^0(P)$ is $u-$COMPATIBLE if $W \geq 1$ $P-$ a.s. and

$$E[u(-\alpha W)] > -\infty \ \forall \alpha > 0.$$

DEFINITION OF $\mathcal{W}$
The set of LOSS variables $\mathcal{W}$ is the set of all $X-$suitable and $u-$compatible random variables.

In the sequel we will assume that

$$\mathcal{W} \neq \varnothing.$$

This assumption is needed because it may happen that the market model and the utility function are not compatible.

EXAMPLE
(1) Arbitrage free market (NFLVR holds true)
(2) $u(x) = -x^2$ if $x \leq -1$
(3) $\mathcal{W} = \varnothing$.

## On compatibility conditions

Let $W \in L^0$, $W \geq 1$ and consider

$$\begin{aligned} W &\in L^\infty, && (1) \\ \forall \alpha &> 0\, E[u(-\alpha W)] > -\infty, && (2) \\ \exists \alpha &> 0\, E[u(-\alpha W)] > -\infty. && (3) \end{aligned}$$

Obviously

$$(1) \Rightarrow (2) \Rightarrow (3)$$

The strongest condition (1) leads to the well established notion of 1-admissibility, i.e.: $\mathcal{H}^W = \mathcal{H}^1$.

The weaker compatibility condition (2) is studied in this paper and leads to uniformity w.r. to $W \in \mathcal{W}$.

The weakest condition (3) is left for future investigation.

## Definitions of $\mathcal{P}_\Phi$ and $M_\sigma$

For the dual approach we are going to follow, we will need the convex conjugate $\Phi : \mathbb{R}_+ \to \mathbb{R}$ of $u$:

$$\Phi(y) = \sup_{x \in \mathbb{R}} \{u(x) - xy\} .$$

DEFINITION

$$\mathcal{P}_\Phi = \{Q \ll P \mid E[\Phi(\frac{dQ}{dP})] < +\infty\}$$

is the set of $P-$a.c. probability measures with FINITE GENERALIZED ENTROPY.

DEFINITION

$$M_\sigma = \{Q \ll P : X \text{ is a } \sigma - \text{martingale w.r.to } Q\}$$

is the set of SIGMA MARTINGALE MEASURE,

In our context, $M_\sigma$ replace the set of local martingale measures, that was adequate when $X$ was assumed locally bounded.

## Definition of a $\sigma-$MARTINGALE

An $\mathbb{R}$-valued semimartingale $X$ is called a $\sigma-$MARTINGALE (wrt $P$) if there exists an increasing sequence of predictable sets $\Sigma_n$ such that $\cup_n \Sigma_n = \Omega \times \mathbb{R}_+$ and if for any $n \geq 1$ the process:

$$I_{\Sigma_n} \cdot X \quad \text{is a } P - \text{uniformly integrable martingale.}$$

If $X$ is $\mathbb{R}^d$-valued, it is a $\sigma$-martingale if each of its components is a $\sigma$-martingale.

If $\Sigma_n = [0, T_n]$, where $T_n \uparrow +\infty$ is a sequence of stopping times, then the previous definition BOILS DOWN TO THE DEFINITION OF A LOCAL MARTINGALE.

THEOREM (Emery)
For a $d$-dimensional semimartingale $X$ the following are equivalent:
(1) $X$ is a $\sigma$-martingale;
(2) there exist a $d$-dimensional local martingale $M$ and (scalar) predictable, $M^i-$integrable processes $H^i$ such that $X^i = H^i \cdot M^i$.

This means that each component $X^i$ can be written as a stochastic integral of a local martingale $M^i$.

Nice mathematical properties when $W \in \mathcal{W}$

Let $W \in L^0(P)$ and define:

$$M_{\sigma,W} \triangleq \{Q \in M_\sigma \mid E_Q[W] < +\infty\};$$

$$M_{sup,W} \triangleq \{Q \ll P \,|\, E_Q[W] < +\infty \text{ and } H \cdot X \text{ is a } Q\text{-supermart. } \forall H \in \mathcal{H}^W\};$$

$$M_{T,W} \triangleq \{Q \ll P \,|\, E_Q[W] < +\infty \; (H \cdot X)_T \in L^1(Q), E_Q[(H \cdot X)_T] \leq 0 \,\forall H \in \mathcal{H}^W\}.$$

LEMMA:
Let $W \in L^0(P)$ and suppose that $M_{\sigma,W}$ is not empty.

(a) For all $H \in \mathcal{H}^W$, $H \cdot X$ is a local martingale and a supermartingale under each $Q \in M_{\sigma,W}$.

(b) If $W$ is $X-$suitable, then

$$M_{\sigma,W} = M_{sup,W} = M_{T,W}.$$

(c) If $W$ is $u-$compatible, then $W \in L^1(Q)$ for all $Q \in \mathcal{P}_\Phi$ and

$$M_{\sigma,W} \cap \mathcal{P}_\Phi = M_\sigma \cap \mathcal{P}_\Phi.$$

DEFINITION

Let $W \in \mathcal{W}$ and set:

$$\mathbf{K^W} = \left\{(H \cdot X)_T \mid H \in \mathcal{H}^W\right\};$$

$$\mathbf{K^{\mathcal{W}}} = \bigcup_{W \in \mathcal{W}} K^W;$$

$$\mathbf{K_\Phi} = \left\{ f \in \bigcap_{Q \in M_\sigma \cap \mathcal{P}_\Phi} L^1(Q) \mid E_Q[f] \leq 0 \, \forall Q \in M_\sigma \cap \mathcal{P}_\Phi \right\}$$

$$\mathbf{U^W(x)} = \sup_{k \in K^W} E[u(x+k)];$$

$$\mathbf{U^{\mathcal{W}}(x)} = \sup_{k \in K^{\mathcal{W}}} E[u(x+k)];$$

$$\mathbf{U_\Phi(x)} = \sup_{k \in K_\Phi} E[u(x+k)].$$

(all well defined)

WHY $\mathbf{K_\Phi}$ ?

EXAMPLE: We show what may go wrong:

(1) Arbitrage free market (NFLVR holds true)

(2) $\mathcal{W} \neq \varnothing$

(3) $U^W(x) < u(+\infty)$ for all $W \in \mathcal{W}$

(4) For each $W \in \mathcal{W}$ the problem

$$\sup_{H \in \mathcal{H}^W} E[u(x + (H \cdot X)_T)]$$

does NOT admit an optimal solution $H^* \in \mathcal{H}^W$.

PROPOSITION:
If $W \in \mathcal{W}$ then

$$\mathbf{K}^{\mathbf{W}} \subseteq \mathbf{K}^{\mathcal{W}} \subseteq \mathbf{K}_{\mathbf{\Phi}} \quad \text{and} \quad \mathbf{U}^{\mathbf{W}}(\mathbf{x}) \leq \mathbf{U}^{\mathcal{W}}(\mathbf{x}) \leq \mathbf{U}_{\mathbf{\Phi}}(\mathbf{x}).$$

In our main result we will assume:

ASSUMPTION (1):
The utility $u$ has Reasonable Asymptotic Elasticity - RAE($u$) - as introduced by Schachermayer.

Consider the following other conditions:

$$E_P[\Phi(\lambda \frac{dQ}{dP})] \ < \ +\infty, \ \forall \lambda > 0, \text{-} \forall Q \in P_\Phi. \qquad \text{(1a)}$$

$$E_P[\Phi(\lambda \frac{dQ}{dP})] \ < \ +\infty, \forall \lambda > 0, \forall Q \in M_\sigma \cap P_\Phi \qquad \text{(1b)}$$

Then:

$$RAE(u) \Rightarrow (1a) \Rightarrow (1b).$$

Examples show that the reverse implications do not hold.

We only need the weaker condition (1b).

ASSUMPTION (2):

There exist $W_0 \in \mathcal{W}$ and $x_0 \in \mathbb{R}$ such that

$$\sup_{k \in K^{W_0}} E[u(x_0 + k)] \triangleq U^{W_0}(x_0) < u(+\infty).$$

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From our main Theorem we will show that:

Assumption (2) is equivalent to

$$\mathcal{W} \neq \emptyset \quad \text{and} \quad M_\sigma \cap \mathcal{P}_\Phi \neq \varnothing.$$

If Assumption (2) does not hold true then, even with very small $x_0$ ($x_0 \downarrow -\infty$),

$$\sup_{k \in K^{W_0}} E[u(x_0 + k)] = u(+\infty)$$

with:

$$E_Q[k] \leq 0$$

for all $k \in \mathbf{K}^{W_0}$, under each $Q \in M_\sigma \cap P_\Phi$.

Assumption (2) can be regarded as a hypothesis of ABSENCE OF UTILITY BASED ARBITRAGE OPPORTUNITIES.

# THEOREM (A)

Suppose that $u$ satisfies RAE($u$) and that there exists $W_0 \in \mathcal{W}$ and $x_0 \in \mathbb{R}$ such that $U^{W_0}(x_0) < u(+\infty)$.

(a) $M_\sigma \cap \mathcal{P}_\Phi \neq \varnothing$;

(b) For all $W \in \mathcal{W}$ and all $x \in \mathbb{R}$, $U^W(x) < u(+\infty)$;

(c) $U^W(x)$ does NOT depend on $W \in \mathcal{W}$, and

$$U^W(x) = U^{\mathcal{W}}(x) = \min_{\lambda>0, Q \in M_\sigma \cap \mathcal{P}_\Phi} \lambda x + E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right];$$

(d) $\forall x \in \mathbb{R}$ there exists the optimal solution $f_x \in K_\Phi$ :

$$\max \{E[u(x+f)] \mid f \in K_\Phi\} = E[u(x+f_x)] = U_\Phi(x) < u(\infty)$$

and

$$U_\Phi(x) = U^{\mathcal{W}}(x);$$

(e) If $\lambda_x, Q_x$ is the optimal solution in (c), then:

$$u'(x+f_x) = \lambda_x \frac{dQ_x}{dP};$$

(f) There exists a $\mathbb{R}^d-$valued predictable $X-$integrable process $H^x$ such that

$$f_x = (H^x \cdot X)_T \quad Q_x - a.s.$$

and $H^x \cdot X$ is a $Q_x$-uniformly integrable martingale.

(g) SUPERMARTINGALE PROPERTY OF $H^x \cdot X$:

If $Q_x \sim P$ then the optimal process

$H^x \cdot X$ is a supermatingale wrt each $M_\sigma \cap P_\Phi$.

(h) Let $V_\Phi(\lambda) = \min_{Q \in M_\sigma \cap \mathcal{P}_\Phi} E[\Phi(\lambda \frac{dQ}{dP})]$ and let $Q_\lambda$ attain the minimum.

$$U_\Phi(x) = \inf_\lambda \{\lambda x + V_\Phi(\lambda)\}$$

$V_\Phi$ and $U_\Phi$ are cont. differentiable and:

$$V_\Phi'(\lambda) = E[\Phi'(\lambda \frac{dQ_\lambda}{dP}) \frac{dQ_\lambda}{dP}]$$

$$xU_\Phi'(x) = E[u'(x + f_x)(x + f_x)]$$

# Uniformity over $W \in \mathcal{W}$

The theorem provides a desirable uniformity over all the $W \in \mathcal{W}$.

It is then clear that in the locally bounded case, the "old" admissible strategies are the right ones to work with.

In fact, the associated $W$ is the minimal one: it is simply $1$.

Hence, these strategies are the ones that guarantee the minor losses among all the other $\mathcal{H}^W$ and at the same time the related maximization leads to the optimal value $U^{\mathcal{W}}(x)$.

# ON NFLVR

COUNTEREXAMPLE (1):

$$\text{NFLVR} \not\Rightarrow \text{Assumption (2).}$$

There exists a continuous complete market satisfying:

(i) $NFLVR$

(ii) $U^W(x) = u(+\infty)$ for all $x \in \mathbb{R}$ and all $W \in \mathcal{W} \neq \varnothing$.

COUNTEREXAMPLE (2) (from Schachermayer):

$$\text{Assumption (2)} \not\Rightarrow \text{NFLVR.}$$

There exists a continuous market where:

(i) there is precisely one martingale measure $Q$,
(ii) $\frac{dQ}{dP} = 2I_A$, $P(A) = \frac{1}{2}$.

Since $Q \not\sim P$, there are FLVR.

$u(x) = -e^{-x}$, $W = 1, x = 0$. Then

$$U^1(0) = -\frac{1}{2} < u(+\infty)$$

and the optimal solution is:

$$f_0 = \begin{cases} 0 & \text{on } A \\ +\infty & \text{on } A^c \end{cases} .$$

# Existence of the optimal solution even with FLVR

The existence of a FLVR does not preclude that Assumption (2) holds true, nor the existence of the optimal solution $f_x \in K_\Phi$.

This optimal solution

$$f_x = +\infty \text{ on the set } \{dQ_x/dP = 0\},$$

which can have positive $P$ measure when $Q_x \not\sim P$.

Under Assumption (2), a FLVR $g$ will not be considered by the investor as an interesting opportunity, since $g$ will not increment the optimal utility:

$$x + f_x = x + f_x + g \quad P - a.s.$$

since $P(\{f_x < +\infty\} \cap \{g > 0\}) = 0$.

## Example 1 (Merton)

We consider a Black Scholes market with an EXPONENTIAL utility maximizer agent.

$$dX_t = \mu X_t dt + \sigma X_t dB_t \quad 0 \le t \le T < +\infty,$$

where $B$ is the standard Brownian motion.

Here the process is continuous (hence locally bounded) and the hypotheses of the Theorem are satisfied with $W_0 = 1$, $x$ arbitrary, so that:

$$U^W(x) = U^1(x) \quad \text{for any} \quad W \in \mathcal{W}.$$

Let $Z_t = B_t + \frac{\mu}{\sigma} t$ be the Brownian motion under the unique martingale measure $Q$.

It is widely known that

$$U^1(x) = \sup_{k \in K^1} E[u(x+k)] = E[-e^{-(x+\frac{\mu}{\sigma}Z_T)}],$$

However, the function

$$f_x = \frac{\mu}{\sigma} Z_T$$

does not belong to $K^1$, because it is unbounded, and no optimal solution exists in $K^1$.

But if we take $W' = 1 - \inf_{t \le T} Z_t$, then:

$$W' \in \mathcal{W} \quad \text{and} \quad f_x \in K^{W'}.$$

Indeed:

$$f_x = \frac{\mu}{\sigma} \int_0^T \frac{1}{\sigma X_t} dX_t \quad \text{with} \quad H' = \frac{\mu}{\sigma^2 X} \in \mathcal{H}^{W'}$$

This classic setup provides an example in which:

(1) $\mathcal{H}^1$ is strictly contained in $\mathcal{H}^{W'}$,

(2) $U^1(x) = U^{W'}(x)$.

(3) There exists an optimal solution in $\mathcal{H}^{W'}$, but not in $\mathcal{H}^1$.

THIS ENLARGEMENT OF THE STRATEGIES DOES NOT INCREASE THE MAXIMUM,

BUT IT IS NECESSARY TO CATCH THE OPTIMAL SOLUTION.

Example 2 (not locally bounded price process)

Consider the price process:

$$X_t = V \, I_{\{\tau \leq t\}}$$

which consists of one jump of size $V$ at the stopping time $\tau$.

Suppose $V \sim N(\mu, \sigma^2)$, $\mu \neq 0$, and $V$ and $\tau$ are $P-$independent.

Let $u(x) = -e^{-x}$.

Then:

$$\mathcal{H}^1 = \{0\}, \qquad K^1 = \{0\}$$

$$U^1(x) = \sup_{k \in K^1} E[-e^{-(x+k)}] = -e^{-x}.$$

Note that the constant $1$ is NOT $X-$suitable, hence:

$$1 \notin \mathcal{W}.$$

PROPOSITION:
(1) $W' \triangleq (1 + |V|) \in \mathcal{W} \neq \emptyset$

(2) $M_\sigma \cap \mathcal{P}_\Phi \neq \emptyset$

(3) For all $x \in \mathbb{R}$, $U^{W'}(x) < 0 = u(+\infty)$ and

$$\sup_{k\in K^{W'}} E[-e^{-(x+k)}] = \min_{y>0, Q\in M_\sigma\cap\mathcal{P}_\Phi} \left\{ xy + E[\Phi(y\frac{dQ}{dP})] \right\}$$

where $\Phi(z) = z\ln z - z$.

(4) the supremum in the primal problem is a maximum, the optimal solution is

$$f^* = \frac{\mu}{\sigma^2}V \in K^{W'}$$

and the optimal value

$$U^{W'}(x) = -e^{-(x+\frac{\mu^2}{2\sigma^2})} > -e^{-x}$$

is strictly bigger than $-e^{-x}$, which is the optimal value of the maximization on the trivial domain $K^1 = \{0\}$.

Similar results can be obtained in a model with infinitely many jumps: take a Compound Poisson process on $[0, T]$:

$$X_t = \sum_{j\le N_t} V_j,$$

where the jumps $V_j$ are unbounded (i.e.:$V_j \sim N(m, \sigma^2)$, with $m \neq 0$) and $N_t$ is a Poisson process independent from $(V_j)_j$.

# Interpretation of $K_\Phi$

THEOREM (B. & F. AAP-2004):
Suppose that $W \in \mathcal{W}$, $M_\sigma \cap \mathcal{P}_\Phi \neq \varnothing$ and (1b) holds true. Then

$$\bigcap_{Q \in M_\sigma \cap \mathcal{P}_\Phi} \overline{K^W - L^1_+(Q)}^Q = \bigcap_{Q \in M_\sigma \cap \mathcal{P}_\Phi} \overline{K^{\mathcal{W}} - L^1_+(Q)}^Q = K_\Phi$$

(where $\overline{C}^Q$ denotes the $L^1(Q)-$closure of a set $C$).

- $K_\Phi$ admits a representation directly based on $K^{\mathcal{W}}$

THEOREM (B. & F. AAP-2004):
Suppose that (1b) holds true.
The weak super replication price $\widehat{f}$ of each fixed $f \in \bigcap_{Q \in M_\sigma \cap \mathcal{P}_\Phi} L^1(Q)$ admits the dual representation:

$$\widehat{f} \triangleq \inf \{x \in \mathbb{R} \mid f - x \in K_\Phi\} = \sup \{E_Q[f] \mid Q \in M_\sigma \cap \mathcal{P}_\Phi$$

- $f \in K_\Phi$ if and only if $\widehat{f} \leq 0$.

- $K_\Phi$ is the set of claims in $\bigcap_{Q \in M_\sigma \cap \mathcal{P}_\Phi} L^1(Q)$ having “weak super replication price” less than or equal to zero.

The supermartingale property, for general semimartingales, of the optimal process $H^x \cdot X$

$H^x \cdot X$ is a supermatingale wrt each $M_\sigma \cap P_\Phi$.

DEFINITION:

$$\mathcal{H}^{\text{sup}} = \{H \in L(X)(P) \mid H \cdot X \text{ is a supermart wrt each } Q \in M_\sigma \cap P_\Phi\}$$

THEOREM (B):

Suppose RAE($u$), $M_\sigma \cap P_\Phi \neq \emptyset$ and $\mathcal{W} \neq \varnothing$.

(a) For all $x \in \mathbb{R}$:

$$\begin{aligned} U^{sup}(x) &\triangleq \sup_{H \in \mathcal{H}^{\text{sup}}} E[u(x + (H \cdot X)_T)] \\ &= \min_{\lambda > 0, Q \in M_\sigma \cap \mathcal{P}_\Phi} \lambda x + E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right] < u(+\infty), \end{aligned}$$

(b) Let $Q_x$, $\lambda_x$ be the dual optimal solutions.

If $Q_x \sim P$ then for all $x \in \mathbb{R}$:

$$\exists \text{ optimal } H_x \in \mathcal{H}^{\text{sup}}$$

and $H_x \cdot X$ is a u.i. martingale under $Q_x$.

# A bit of history

The supermartingale property of the optimal portfolio process for general semimartingales can be seen as the fourth point in the following list, concerning the case $X$ LOCALLY BOUNDED:

1. Six Authors' paper.
   When $u(x) = -e^{-x}$ and the REVERSE HOLDER INEQUALITY holds,
   it was proved that the optimal wealth process is a TRUE MARTINGALE wrt every loc. mart. meas. $Q$ with finite entropy.
2. Kabanov and Stricker removed the superfluous RHI;
3. Schachermayer proved that if $Q_x \sim P$, then
   the $H_x \cdot X$ is a SUPERMARTINGALE under every loc. mart. meas. with finite entropy
   (the true martingale property of the solution is lost for general $u$).

- We proved in Theorem (B) that this supermartingale property holds even for unbounded semimartingales.

Steps to the proof of Theorem (B)

a] The dual formula is a straightforward derivation of the results in Theorem A, as well as the integral representation $f_x = (H_x \cdot X)_T$.

In fact: let $K^{\sup} = \{(H \cdot X)_T \mid H \in H^{sup}\}$.

Since for all $W \in W$, the integrands in $H^W$ give rise to supermartingales wrt every $Q \in M_\sigma \cap P_\Phi$,

$$K^W \subseteq K^{\mathcal{W}} \subseteq K^{\sup} \subseteq K_\Phi$$

so that

$$U^W(x) \leq U^{\mathcal{W}}(x) \leq U^{\sup}(x) \leq U_\Phi(x)$$

and finally

$$U^{sup}(x) = \min_{\lambda > 0, Q \in M_\sigma \cap \mathcal{P}_\Phi} \lambda x + E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right].$$

When $Q_x \sim P$, we have $f_x = (H_x \cdot X)_T$

(which in general is not well controlled).

b] So, we are left to prove the supermartingale property, i.e.

$$H_x \in H^{sup}.$$

b.1] If $u(x) = -e^{-x}$,
the true martingale property of the optimal portfolio is preserved (Kabanov-Stricker argument can be applied);

b.2] General $u$:
we follow the original Schachermayer 03 approach.

However,

$$M_\sigma \text{ is not necessarily } L^1(P) \text{ closed},$$

while $M_{loc}$ was closed in the loc. bound case.

$\Rrightarrow$ Some extra work is needed!

On $M_\sigma(\tau;1)$

Consider the set of $\sigma$- martingale measures for process "$X$ after $\tau$]:

$$M_\sigma(\tau) \triangleq \{Q \mid (X_t - X_{t\wedge\tau})_t \text{ is a } Q - \sigma \text{ mart}\}$$

and indicate with

$$M_\sigma(\tau;1) = \{Q \in M_\sigma(\tau) \mid E^\tau[\frac{dQ}{dP}] = 1\}$$

those $\sigma$-mart measures that are constantly $1$ at time $\tau$ .

Note that these sets are not empty:

1- $M_\sigma \subseteq M_\sigma(\tau)$ for all $\tau$;

2- since $Q^x \sim P$, if we call $\widehat{Z}$ the density process,

$$\widehat{Y}_T = \frac{\widehat{Z}_T}{\widehat{Z}_\tau} \in M_\sigma(\tau;1).$$

THE DYNAMIC DUAL PROBLEM:

Given a s.t. $\tau \in [0,T]$ and a $\xi > 0$ $F_\tau$-measurable, define the

$$v_\tau(\zeta) = \text{ess inf}_{Y_T \in M_\sigma(\tau;1)} E^\tau[\Phi(\zeta Y_T)]$$

When $\tau = 0$,

$$v_0(\lambda) = \inf_{Y_T \in M_\sigma} E[\Phi(\lambda Y_T)].$$