Numeraire-invariant option pricing
&
american, bermudan, and trigger stream rollover[1]

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Abstract. Part I proposes a numeraire-invariant option pricing framework. It defines an option, its price process, and such notions as option indistinguishability and equivalence, domination, payoff process, trigger option, and semipositive option. It develops some of their basic properties, including price transitivity law, indistinguishability results, convergence results, and, in relation to nonnegative arbitrage, characterizations of semipositivity and consequences thereof. These are applied in Part II to study the Snell envelop and american options. The measurability and right-continuity of the former is established in general. The american option is then defined, and its pricing formula (for all times) is presented. Applying a concept of a domineering numeraire for superclaims derived from (the additive) Doob-Meyer decomposition, minimax duality formulae are given which resemble though differ from those in [R] and [H-K]. Multiplicative Doob-Meyer decomposition is discussed last. A part III is also envisaged.

1. Introduction

This paper proposes a new option pricing framework in Part I and applies it to the Snell envelop and american options in Part II. A third part is shortly envisaged, with application to bermudan options.

Part I develops a notion of an *option* as a pair consisting of an expiry and a payoff, and of its *price process* defined up to the expiry. The definition references an arbitrary numeraire, but is actually numeraire-independent due to the optional sampling theorem. Some basic properties such as price transitivity law, indistinguishability results, convergence results, and, emphasizing nonnegative arbitrage, properties of "semipositive options" are derived. Various other notions such as "domination", "payoff processes," and "trigger options" are pursued. The concepts and results of this part are integral to Parts II and III. The "dominated and trigger option convergence" theorems are used

[1]This paper is an expansion of an earlier draft entitled "Minimax optimality of bermudan and american claims and their Monte-Carlo upper bound approximation". I thank an anonymous referee for valuable comments on that draft.

repeatedly in Part II, and semipositivity aligns the analysis of general nonnegative options on the same footing as positive ones.

Part II, technically the most advanced of the parts, defines the Snell envelop of a right-continuous payoff process and proves it is adapted and right continuous. It then defines an american option and derives its pricing formula. Utilizing "domineering numeraires" resulting from the additive Doob-Meyer decomposition of "superclaims", it presents minimax duality formulae for positive, nonnegative and semipositive american options. The concept of a domineering numeraire, while derived from the additive Doob-Meyer decomposition, is also closely connected to the multiplicative Doob-Meyer decomposition, as we see. Accordingly, we refer to this result as multiplicative minimax duality, to distinguish it from a similar but different result in [R].

Part III picks up a more detailed study of the issues in Part II for the more concrete case of finite option streams. While still under continuous-time filtration, induction is available here. This simplification is partially made up for by a more elaborate definition of a bermudan option where the exercise dates themselves are allowed to be random rather than fixed times like semiannually. The main construct is the "rollover operator". Applied to a finite nonnegative option stream, it produces a "rollover option" connected to the multiplicative Doob-Meyer decomposition, and one with an intuitive financial interpretation as reinvestment of payoff at expiry in the next option.

Results in Part III include a multiplicative counterpart of minimax duality in [H-K] for bermudan options, further studied in [A-B], $[\text{K-S}]_1$, $[\text{K-S}]_2$, and applied in [J-T]. Particularly, a notion of a regenerative trigger option stream is pursued, one closely connected to the primal-dual exercise strategies in [A-B], though more explicitly fashioned after a notion of regeneration in $[\text{K-S}]_1$, extended in $[\text{K-S}]_2$ to an iterative construction of the bermudan Snell envelop by a convergent sequence of stopping times. Finite option streams are commonplace in fixed-income and credit markets. They might also serve as a base for further results on right-continuous payoff processes by convergence arguments.

The proposed framework is rooted in a "pricing model", consisting of a stochastic base and a state price density (or deflator), leading to notions of a (contingent) claim C and its price process (C_t), a numeraire β as a positive claim, and its associated numeraire measure $\mathbb{P}^\beta$. Options and their price processes are then defined in reference to a fixed but arbitrary numeraire, followed by demonstration of independence of their definition from choice of numeraire. Such manner of definition and derivation continues to the point where we can start to define constructs and manipulate formulae through intrinsic operations, bypassing reference to any numeraire and its expectation operator, while leaving that option open, as expedient.

The crux of our approach is to view an option as a pair $\mathcal{O} = (T, O)$, consisting of an expiry T and a payoff O paid at time T. The expiry T can of course be random, as is the case in american and bermudan options, trigger (barrier) options, and credit derivatives. The payoff O must of course be known at time T, more precisely be measurable with respect to the σ-algebra $\mathcal{F}_T$ of events at or before stopping time T.

But, an integrability condition is also needed, simply that O/β_T is integrable under numeraire measure $\mathbb{P}^\beta$ for some, or equivalently for all, numeraires β. This is our definition of an option. Its numeraire invariance, i.e., the independence of $\mathbb{P}^\beta$-integrability of O/β_T from choice of numeraire β, is attested by the optional sampling theorem. Thus internalized within an option (and in its price process), the theorem is transparently invoked by price transitivity law henceforth.

The definition of price of an option $\mathcal{O} = (T, O)$ at times and states before expiry T is modelled on that of the claim induced by investing (rolling over) at T the payoff O in a (any) numeraire. While such claim price depends on the choice of numeraire after expiry, it is independent of this choice before expiry, as is intuitive. More precisely, the price process $(\mathcal{O}_t)$ of option $\mathcal{O}$ lives as a measurable function on the stochastic interval $[[0, T]]$, dependent only on the underlying pricing model. After expiry, the option has ceased to exist and has no operational price.

But, as it is beneficial to view such a function on $[[0, T]]$ as a stochastic process in the normal sense of being defined at all times and states, we set the option price process $(\mathcal{O}_t)$ to zero after expiry T, an alternative easier to visualize, and more suitable for our purposes, than the stopped price process $(\mathcal{O}_{t\wedge T})$ - equivalent nonetheless. The option price process $(\mathcal{O}_t)$ is thus characterized as the unique process vanishing after expiry such that $\mathcal{O}_T = O$ a.s. and the stopped numeraire-deflated price process $(\mathcal{O}_{t\wedge T}/\beta_{t\wedge T})$ is a right-continuous martingale in the numeraire measure $\mathbb{P}^\beta$, for some, hence all, numeraires β.

Almost all our technical needs are met by Chapters 2, 4 and 8 of [E], with only few scattered references to other chapters.

1.1. **Detailed description of Part I.** Section 2 establishes notation, and sets up a more-or-less accepted notion of a (contingent) claim via a given state price density (or deflator) on a stochastic base. Positive claims, designated as numeraires, are granted special emphasis, deemed among the most readily observable, while being in one-to-one correspondence with equivalent probability measures (up to a positive multiplicative constant), the so-called numeraire measures.

By definition, the state price density (ξ_t) induces a linear isomorphism between claims C and martingales (namely, $C \rightsquigarrow (\xi_t C_t)$, where (C_t) is the claim price process). Martingales are interpreted as fair games, which trading and investment are also in some sense. Nonetheless, it is a claim that affords here financial interpretation as a specific traded instrument, not a martingale. Such setup, used also in [J], suffices here, though no doubt would benefit from amendments for more comprehensive studies that address hedging and replication.

Section 3 defines an option and its price process. A convenient and financially intuitive notation ensues that somewhat deflects the traditional focus on the expectation operator in price calculations. The simple expression $(T, O)_t$ now describes the price $\mathcal{O}_t$ at time t of an option $\mathcal{O} = (T, O)$ which pays O at stopping time expiry T.

Price transitivity law is the dividend - pricing a T-expiry option $\mathcal{O}$ at time S, then pricing the result to time τ, is the same as directly pricing $\mathcal{O}$ at τ, i.e., $(S, \mathcal{O}_S)_\tau = \mathcal{O}_\tau$ for $\tau \leq S \leq T$. This law encapsulates the law of iterated expectations: it is just a restatement of the formula $\mathbb{E}[M_T \mid \mathcal{F}_S \mid \mathcal{F}_\tau] = \mathbb{E}[M_T \mid \mathcal{F}_\tau] = M_\tau$ valid for a martingale (M_t) by the optional sampling theorem. But, to apply the latter to pricing, an additional step is needed, namely choice of a numeraire and division by that numeraire. Price transitivity takes care of that internally.

We call two options *equivalent* if the they have a.s. the same expiry and a.s. the same payoff, and *indistinguishable* if they have indistinguishable price processes. Equivalent options are indistinguishable, but the converse is not generally true. We show two options are indistinguishable if and only if they have a.s. equal payoffs and their expiries differ a.s. only where the payoff is zero. An interpretation of this is that postponing zero payoffs does not change an option in an essential way. Another observation is price additivity of certain event-triggered choice of two payoffs, generalizable inductively, and handy in Part III.

Section 4 presents our convergence results, simple, still effective in Part II. The main criteria is "domination". An option $\mathcal{O}$ dominates a process (Z_t) if a.s. $|Z_t| \leq \mathcal{O}_t$ all t. A "dominated option convergence theorem" makes a statement about convergence of prices of a sequence of uniformly dominated options with convergent expiries and payoffs.

The notion of a *payoff process*, a progressively measurable process (Z_t) dominated by some option, plays an important role here. Its main feature is that for any stopping time T, the pair (T, Z_T) is an option, what we call a *trigger option.*[1] A "trigger option convergence theorem" states that if (Z_t) is a right-continuous payoff process and $T_n \searrow T$, then a.s. $(T_n, Z_{T_n})_t \to (T, Z_T)_t$, all t. Some consequences are noted also.

Section 5 introduces semipositivity. A semipositive option is a nonnegative option that has positive price at any time before expiry. Remarkably, any nonnegative option is indistinguishable from a semipositive one. Indeed, as soon as the option price becomes zero, it is certain then that price will remain zero thereafter, yielding zero payoff. So, it makes no substantive difference to the counterparties to expire the option then, aside from the side benefit of easing maintenance costs.

[1]More strongly, domination implies (Z_t/β_t) is $\mathbb{P}^\beta$ class D for any numeraire β.

More precisely, let $\mathcal{O} = (T, O)$ be a nonnegative option, i.e., $O \geq 0$ a.s. Let T^0 be the first time that its price becomes zero. So, $\mathcal{O}_t$ is positive for $t < T^0$. This implies by price transitivity that option $(T^0, \mathcal{O}_{T^0})$ is semipositive. We show that this semipositive option is indistinguishable from option $\mathcal{O}$. Furthermore, it is equivalent to any other semipositive option that is indistinguishable from $\mathcal{O}$.

These assertions follow from *nonnegative arbitrage*, the statement that once the price of a nonnegative option becomes zero, it will stay zero for ever. This implies positivity properties, e.g., almost all sample paths of a positive option are positive at all times at or before expiry.

Semipositivity has an interesting characterization in terms of "tight events". Loosely speaking, the zero-payoff set $\{O = 0\}$ of a semipositive option is not previsible before expiry; nor is any portion of it. This indicates that semipositivity is a property of the zero-payoff set. It implies that a nonnegative option is semipositive if its zero-payoff set is contained in the zero-payoff set of some semipositive option.

Semipositivity proves an effective tool for reducing the study of nonnegative options to a more tractable class enjoying similar properties as positive options. It enables an extension of minimax duality to nonnegative and semipositive american and bermudan options, and leads to the associativity of the rollover operator and consequences thereof.

1.2. **Detailed description of Part II.** Section 6 defines the Snell envelop - a numeraire-invariant process (V_t) associated to any right-continuous payoff process (Z_t), and interpreted as the price at time t of the american option issued at time t, which whenever exercised at some (stopping) time $T_t \geq t$, yields Z_{T_t}.[2] Rather than simply assuming it, we venture on a result that falls more on a specialist's domain: the measurability and right-continuity of the Snell envelop (V_t). Reckoning that the convergence results of Part I supply the prerequisites for this assertion, we stand by the proof detailed in section 6.1.[3]

The definition next of the american option $\mathcal{A} := (T_0, Z_{T_0})$ hinges on an additional analytic technicality, that the supremum in the definition of Snell envelop is actually attained at a stopping time (for each t).[4] Counterexamples to three desirable properties are given to illustrate

[2]The numeraire-invariance of the Snell is recognized in [A-B], [JT], and $[\text{K-S}]_1$, but rather implicitly, and not exploited to the extent here. This seemingly obvious invariance follows from invariance of trigger option prices $(T, Z_T)_t$ for any stopping time T, in turn a consequence of general option price invariance.

[3]The proof partially relies on approximation by bermudan options, whose price measurability is well-known by backward induction. It also uses a well-known bermudan option pricing formula (one easily derived by induction as in part III), in terms of certain extension V_t^s of the Snell envelop with $V_t^t = V_t$. Right continuity utilizes this formula and additional trigger option convergence arguments.

[4]We also define the american stream as the curve of options $(\mathcal{A}^t), \mathcal{A}^t := (T_t, Z_{T_t})$, with T_t as described above. So, $\mathcal{A} = \mathcal{A}^0$.

the difficulties. Once riding the tide, we are back in calm waters, and may cruise along on sheer delight of algebra.

A main result is an american option pricing formula $\mathcal{A}_t$ for any time t, implying, as is well-known, that $\mathcal{A}_0$ equals V_0, the supremum of prices of all trigger options. The difference being that here we first define the american option, and then show this as a (nontrivial) consequence, rather than defining its time-0 price as V_0, without ever mathematically defining the american option itself, as hitherto been the case.

Section 7 is on minimax duality, a formula equating the Snell envelop to the infimum (or, suitably formulated, the minimum) over all numeraires (or semipositive claims) of the price of a "max lookback option" obtained by rolling the payoff process over the numeraire.

First we cast an immediate consequence of the Doob-Meyer decomposition theorem in a form suitable to our setting, namely, existence of "domineering numeraires" for "superclaims". (A superclaim, analogue of a class D supermartingale, is a payoff processes (V_t) satisfying $(s, V_s)_t \leq V_t$ for $t \leq s$. A domineering numeraire (at $t = 0$) is then a numeraire B such that $V_0 = B_0$ and $V_t \leq B_t$ for $t > 0$. The Snell envelop turns out to be a superclaim.) Then, we present the multiplicative minimax duality formula, first for the simpler case of positive options, and then more generally for nonnegative and semipositive options. There are two versions of this formula and other related results.

A comparison is made to the "additive minimax duality" in [R], and in the case of bermudan options, in [H-K]. Initially it seemed to us that the main distinction here is one between the additive versus multiplicative versions of the Doob-Meyer decomposition. But ours now rests on the additive decomposition too. The main difference is that the additive version of [R] and [H-K] requires a reference numeraire in its formulation, whereas ours doesn't. In this sense, ours is more numeraire-invariant. Its statement is accordingly simpler. It remains to be seen which version is more effective in upper bound numerical approximation of bermudan option prices by Monte-Carlo simulation.[5]

Section 8 returns to Doob-Meyer decomposition, this time the multiplicative one. While we bypassed this decomposition for minimax duality, it still seems very relevant to our study. In fact, its finite stream version subsumes much of part III. The multiplicative decomposition

[5] The recent book of [G] contains application of this and other Monte-Carlo techniques to bermudan options. Related algorithms are studied in [D]. [F-L-M-S-W] compare several of such Monte-Carlo approaches to evaluating american options. See also a discussion in [A-B] on the background of these techniques. [C-H] propose a forward integro-PDE framework for american options, incorporating jumps, including the variance gamma type and the variety in [C-G-M-Y]. Forward Dupire PDE and its integro-PDE generalization to jump-diffusion in [A-A], originally devised for european call and put option volatility smile, are extended there to price american options now. Option pricing in presence of jumps in general is dealt at length in the recent book of [C-T].

has a simpler formulation than the additive one, and by producing a numeraire as its output, qualifies as a major source for self-financing trading strategies, some of which may be useful for approximating bermudan option prices, as in [A-B], [H-K], $[\text{K-S}]_1$, $[\text{K-S}]_2$.

The section first defines a "local claim", an analogue of a local martingale. Then, it shows that any supernumeraire (positive superclaim) admits a unique decomposition as product of a local numeraire and a decreasing, predictable process valued 1 at time 0, a result doubtless well known but seldom discussed. When the local numeraire is actually the price process of a numeraire, there appear to be interesting consequences, a known one noted. We also give a Novikov-type criteria for this property in the continuous case and a borrowed counterexample.

Part I: A Numeraire-Invariant Option Framework

2. Claims and Numeraires

2.1. **Stochastic base notation.** Fix throughout a stochastic base $(\Omega, (\mathcal{F}_t)_{t\in[0,m]}, \mathbb{P})$, consisting of a state space Ω, a right continuous filtration $(\mathcal{F}_t)_{t=0}^m$ on Ω with finite final maturity $0 < m < \infty$, and a complete probability measure $\mathbb{P}$ on $\mathcal{F} := \mathcal{F}_m$, where it is assumed for simplicity that $\mathcal{F}_0$ consists of the events of probability 0 or 1. (The assumption of a compact time t domain $[0, m]$ is also for simplicity.)

Stopping times take (finite) value in $[0, m]$ in this paper. Specifically, the set $\mathcal{T}$ of all stopping times is defined by

$$\mathcal{T} := \{T : \Omega \longrightarrow [0, m] \ : \ \{T \leq t\} \in \mathcal{F}_t, \ \forall\, 0 \leq t \leq m\}.$$

Letters t and s are reserved for times in $[0, m]$ and letter T and sometimes S or τ stand for stopping times.

Stochastic processes are denoted as $Z_\cdot = (Z_t) = (Z_t)_{t\in[0,m]}$. Such dot and parenthetic notation is uniformly used as appropriate for other parameterized objects, sometimes with superscript index, other times for sequences, such as a discrete process.

Given a stopping time T, the σ-algebra of "events before T" is denoted $\mathcal{F}_T := \{\Lambda \in \mathcal{F} : \Lambda \cap \{T \leq t\} \in \mathcal{F}_t\}$, as customary. By Theorem 2.33 in [E], X_T is $\mathcal{F}_T$-measurable for any progressively measurable process (X_t) (e.g., an adapted right-continuous or a predictable process).

The *optional sampling theorem* states that $\mathbb{E}[M_T \,|\, \mathcal{F}_S] = M_{T\wedge S}$ a.s. for all right-continuous martingales (M_t) and stopping times S and T; so $\mathbb{E}[M_T \,|\, \mathcal{F}_S] = M_S$ a.s. when $S \leq T$ [6] (Ditto for supermartingales.[7])

[6] e.g., Theorem 4.12 in [E]. To see the first formula from the second, note $\mathbb{E}[1_{T>S} M_T \,|\, \mathcal{F}_S] = \mathbb{E}[1_{T>S} M_{T\vee S} \,|\, \mathcal{F}_S] = 1_{T>S}\mathbb{E}[M_{T\vee S} \,|\, \mathcal{F}_S] = 1_{T>S} M_S$, as $1_{T>S}$ is $\mathcal{F}_T$-measurable and by first formula. Since $1_{T\leq S} M_T$ is $\mathcal{F}_S$-measurable, it follows $\mathbb{E}[M_T \,|\, \mathcal{F}_S] = \mathbb{E}[1_{T\leq S} M_T + 1_{T>S} M_T \,|\, \mathcal{F}_S] = 1_{T\leq S} M_T + 1_{T>S} M_S = M_{T\wedge S}$.

[7] Namely, $\mathbb{E}[M_T \,|\, \mathcal{F}_S] \leq M_{T\wedge S}$ a.s. A right-continuous supermartingale has a Doob-Meyer decomposition if it is of class D (c.f., Theorem 8.15 in [E]). Time domain being compact, all martingales are closed here. So a process (X_t) is of class

Often to certain objects with unsubscripted symbols, e.g., β standing for a numeraire or $\mathcal{O}$ standing for an option, we associate a stochastic process (their price processes), denoted as $\beta_. = (\beta_t) = (\beta_t)_{t\in[0,m]}$, etc.

We sometimes indicate indistinguishability by a short hand like "a.s. $X_t = Y_t$ all t". Should we write instead, "$X_t = Y_t$ a.s. all t", we'd mean the processes are merely modifications of each other. We employ such abbreviation in similar contexts, but more often than not on guard, we amend it with clarifying longer and precise statements.

For ease of notation, we often drop subscript bracelets around event indicator functions for short expressions, e.g., denote $1_{\{X>Y\}}$ by $1_{X>Y}$.

2.2. **State price density and Claims.** A *State Price Density* (or *Deflator*) is a process $\xi_. = (\xi_t)_{t=0}^m$ such that (1) it is adapted, (2) it is right-continuous and has left limits, (3) a.s. $\xi_t > 0$ all t, (4) a.s. $\xi_{t_-} > 0$ all t, (5) $\xi_0 = 1$, and (6) ξ_t is $\mathbb{P}$-integrable for all t.

These conditions imply that almost all sample paths of the deflator (ξ_t) are bounded above, and are bounded below strictly above zero.

The financial interpretation of our setting rests entirely on the following interpretation of the state price density (ξ_t). Let $\Lambda \in \mathcal{F}_t$ be an event. Then, $\int_\Lambda \xi_t(\omega)\mathbb{P}(d\omega) = \mathbb{E}[1_\Lambda \xi_t]$ equals the price at time 0 of the contingent claim that pays at time t one unit of base currency if event Λ occurs by time t and zero otherwise. In particular, time-0 price of the t-maturity zero-coupon bond equals $\mathbb{E}[\xi_t]$.

As is well-known, the state price density encapsulates both interest-rate and risk-premia information. (See, for instance, Corollary 8.4.)

We refer to the pair $\mathcal{M} = ((\Omega, \mathcal{F}_., \mathbb{P}), \xi_.)$ consisting of a stochastic base and a state price density on it as a *Pricing Model*.

Henceforth, we fix a pricing model $\mathcal{M}$ throughout the paper, and use the notation and terminology above without recall at will.

A *Claim* is an $\mathcal{F}$-measurable random variable C such that $\xi_m C$ is $\mathbb{P}$-integrable. The set of all claims is denoted $\mathcal{C}$.

The *Price Process* of a claim $C \in \mathcal{C}$ is the unique (up to indistinguishability) process $C_. = (C_t)_{t\in[0,m]}$ such that $C_m = C$ and $(\xi_t C_t)$ is a right-continuous $\mathbb{P}$-martingale. So, $C_t = \mathbb{E}[C_m \xi_m \,|\, \mathcal{F}_t]/\xi_t$, a.s., all t.

Claim price processes are right-continuous and have left limits because so do the deflator (ξ_t) and all right-continuous martingales.

A *Continuous Claim* is one with a continuous price process.

2.3. **Numeraires and numeraire measures.** An a.s. positive claim β is called a *Numeraire*; so $\beta > 0$ a.s. We write $\beta \in \mathcal{C}^+ \subset \mathcal{C}$.

Clearly, $\beta_0 > 0$ and $\beta_t > 0$ a.s., all t. More strongly, a.s. $\beta_t > 0$ all t.

D, (i.e., $\{X_T : T \in \mathcal{T}\}$ is uniformly integrable) if it is "dominated" by a martingale (M_t), i.e., a.s. $|X_t| \leq M_t$ all t. The converse holds for right-continuous supermartingales by Doob-Meyer decomposition. So here, a right-continuous supermartingale is of class D if and only if it is dominated by a martingale.

In fact, almost all sample paths of (β_t) are bounded below strictly above zero, i.e., for almost all ω, there exists an $\varepsilon > 0$ such that $\beta_t(\omega) > \varepsilon$ for all t. This is well-known for positive right-continuous martingales (Theorem 4.16 in [E]), and follows for numeraires by the positivity assumptions on (ξ_t). (See also Corollaries 5.4 and 7.4 below).

For any numeraire β, define the induced *Numeraire Measure* $\mathbb{P}^\beta$ by

$$\frac{d\mathbb{P}^\beta}{d\mathbb{P}} = \frac{\xi_m \beta}{\beta_0}.$$

Then $\mathbb{P}^\beta$ is a $\mathbb{P}$-equivalent probability measure because $\beta > 0$ a.s. By the Radon-Nikodym theorem, every $\mathbb{P}$-equivalent probability measure equals $\mathbb{P}^\beta$ for some numeraire $\beta \in \mathcal{C}^+$ (unique up to a multiple).

If B is another numeraire, then $d\mathbb{P}^B/d\mathbb{P}^\beta = (\beta_0/B_0)B/\beta$.

Clearly, $1/\xi_m$ is a numeraire, $(1/\xi_m)_t = 1/\xi_t$, and $\mathbb{P}^{1/\xi_m} = \mathbb{P}$.

Let β be a numeraire. As is well known, a random variable C is a claim if and only if C/β is $\mathbb{P}^\beta$-integrable, in which case, the β-deflated price process (C_t/β_t) is a right-continuous $\mathbb{P}^\beta$-martingale. Therefore,

$$C_t = \beta_t \mathbb{E}^\beta[\frac{C}{\beta} \mid \mathcal{F}_t]$$

a.s. all t, for any claim C and any numeraire β.

If β is a numeraire and T is a stopping time, then $\beta_T > 0$ a.s. This follows because a.s. $\beta_t > 0$ all t as mentioned above, or, alternatively, from the optional sampling theorem as $\xi_T \beta_T = \mathbb{E}[\xi_m \beta_m \mid \mathcal{F}_T] > 0$ a.s.

Proposition 2.1. *Let β and B be two numeraires and T be a stopping time. Then $d\mathbb{P}^B{}_{|\mathcal{F}_T}/d\mathbb{P}^\beta{}_{|\mathcal{F}_T} = (\beta_0/B_0)B_T/\beta_T$.*

Proof. Let X be a bounded $\mathcal{F}_T$-measurable random variable. Then

$$\frac{B_0}{\beta_0}\mathbb{E}^B[X] = \mathbb{E}^\beta[X\frac{B}{\beta}] = \mathbb{E}^\beta[\mathbb{E}^\beta[X\frac{B}{\beta} \mid \mathcal{F}_T]]$$
$$= \mathbb{E}^\beta[X\mathbb{E}^\beta[\frac{B}{\beta} \mid \mathcal{F}_T]] = \mathbb{E}^\beta[X\frac{B_T}{\beta_T}],$$

the last equality by the optional sampling theorem, as the process (B_t/β_t) is a right-continuous $\mathbb{P}^\beta$-martingale and $B/\beta = B_m/\beta_m$. ☐

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$$\frac{B_0}{\beta_0}\mathbb{E}^B[\frac{O}{B_T}] = \mathbb{E}^\beta[\frac{O}{B_T}\frac{B_m}{\beta_m}] \qquad \text{(change of numeraire)}$$

$$= \mathbb{E}^\beta[\mathbb{E}^\beta[\frac{O}{B_T}\frac{B_m}{\beta_m} \mid \mathcal{F}_T]] \qquad \text{(iterating expectation)}$$

$$= \mathbb{E}^\beta[\frac{O}{B_T}\mathbb{E}^\beta[\frac{B_m}{\beta_m} \mid \mathcal{F}_T]] \qquad \text{(by } \mathcal{F}_T \text{ measurability of } \frac{O}{B_T})$$

$$= \mathbb{E}^\beta[\frac{O}{B_T}\frac{B_T}{\beta_T}] \qquad \text{(optional sampling theorem)}$$

$$= \mathbb{E}^{\beta}[\frac{O}{\beta_T}] < \infty,$$

the last equality by the optional sampling theorem, as the process (B_t/β_t) is a right-continuous $\mathbb{P}^{\beta}$-martingale and $B/\beta = B_m/\beta_m$.

3. Options and their price processes

3.1. **Options.** Given a stopping time T, let $\mathcal{C}_T$ denote the set of all $\mathcal{F}_T$-measurable random variables O such that O/β_T is $\mathbb{P}^{\beta}$-integrable for some numeraire β. (So, $\mathcal{C} = \mathcal{C}_m$).

If this property holds for 1 numeraire then it holds for all numeraires:

Proposition 3.1. *If $O \in \mathcal{C}_T$, then O/B_T is P^B-integrable for all numeraires B.*

Proof. Say O/β_T is $\mathbb{P}^{\beta}$-integrable, and let B be another numeraire. By Proposition 2.1, $O/B_T = (O/\beta_T)(B_T/\beta_T)$ is P^B-integrable. □

It follows that $O \in \mathcal{C}_T$ if and only if $\xi_T O$ is $\mathbb{P}$-integrable.

An *Option* $\mathcal{O}$ is a pair $\mathcal{O} = (T, O)$, where T is a stopping time and $O \in \mathcal{C}_T$. Its *Expiry* is the stopping time T, also denoted $T_{\mathcal{O}}$. Its *Payoff* is the random variable O. The set of all options is denoted $\mathbf{O}$.

Note, if $\mathcal{O} = (T, O)$ is an option, then so is $|\mathcal{O}| := (T, |O|)$.

An option is *Positive* (resp. *Nonnegative*) if its payoff is positive (resp. nonnegative) a.s.

A *European* option is one whose expiry is deterministic.

The s-maturity zero-coupon bond $(s, 1)$ is an example. As $\mathcal{C}_m = \mathcal{C}$, we may identify a claim with an m-expiry european option.

3.2. **Discussion.** We will define and see that options are closed under such operations as addition (making a portfolio of options into an option, with linear pricing), 'rolling" one option over another (investing at expiry the payoff of first in the second), or more generally rolling over an "option stream" (a sequence or curve of options), (compound) call or put options on an option, or a swaption, i.e., an option to exchange two options. This makes the notion sufficiently comprehensive for modelling standard american and bermudan options, trigger options (e.g. trigger swaps or knockout options possibly with rebate), and credit derivatives (e.g., a default protection), among more exotic structures. Stocks and bonds can also be viewed as options.

Most options are nonnegative in practice. This includes an option to enter a trade, such as call and put options or swaptions. A Putable bond (from point view of the option holder, i.e., the investor) is a positive option. On the other hand, a callable bond (from the point of view of the option holder, i.e., the issuer) is a negative option.

The study of optimal exercise of a negative option (e.g. a bermudan callable bond) can be reduced to that of a positive option, namely its negative, (e.g., the bond from viewpoint of investor), but one with value minimization over stopping times rather than value maximization. An alternative is transformation to a nonnegative maximization problem. A bermudan callable bond is equivalent to a short position in the underlying bond and a long position on a bermudan bond call option. So, its valuation is reduced to that of the latter, a nonnegative maximization problem. The same can done with all other negative "payoff processes," for as defined later, they are "dominated" by a numeraire.

3.3. **Option price process.** For the rest of this subsection, let $\mathcal{O} = (T, O)$ be a T-expiry option with payoff O (in $\mathcal{C}_T$).

For any numeraire β, define the "rollover claim" $C^{\mathcal{O},\beta}$ by

$$C^{\mathcal{O},\beta} := \frac{O\beta}{\beta_T} \in \mathcal{C}.$$

Further define the process $\mathcal{O}^\beta_\cdot = (\mathcal{O}^\beta_t)_{t\in[0,m]}$ (on all of $[0,m]\times\Omega$) by

$$\mathcal{O}^\beta_t := 1_{t\le T} C^{\mathcal{O},\beta}_t \in \mathcal{C}_{t\wedge T}.$$

The following numeraire invariance property is key to our approach.

Theorem 3.2. *For any two numeraire β and B, the processes $(\mathcal{O}^\beta_t)$ and $(\mathcal{O}^B_t)$ are indistinguishable.*

The theorem will be established in stages, the main step being Lemma 3.4 which shows $\mathcal{O}^\beta_t = \mathcal{O}^B_t$ a.s. all t.

The rollover claim $C^{\mathcal{O},\beta}$ may be interpreted as investing at expiry T the option payoff O in O/β_T units of the numeraire β, and holding this until terminal maturity m to finally yield a payoff equal $C^{\mathcal{O},\beta}$.

Intuitively, the rollover claim $C^{\mathcal{O},\beta}$ is the same as the option $\mathcal{O}$ before expiry T, because it is not yet invested in the numeraire β . So, before expiry the option price should be the same as the rollover claim price, regardless of choice of numeraire β. This indicates that a suitably-defined option price process of $\mathcal{O}$ should equal the price process of the rollover claim $C^{\mathcal{O},\beta}$ on the stochastic interval $[[0,T]] := \{(t,\omega) : t \le T(\omega)\}$ (at least outside of an evanescent subset).

Note, the rollover claim price process $C^{\mathcal{O},\beta}_\cdot = (C^{\mathcal{O},\beta}_t)_{t\in[0,m]}$ is by definition (and a cancellation) the unique (up to indistinguishability) adapted right-continuous process such that for all t

$$C^{\mathcal{O},\beta}_t = \beta_t \mathbb{E}^\beta[\frac{O}{\beta_T} \mid \mathcal{F}_t] \text{ a.s.}$$

The price of the rollover claim at expiry equals the option payoff:

Lemma 3.3. *For any numeraire β, we have $C^{\mathcal{O},\beta}_T = O$ a.s.*

Proof. Since process $(C_t^{\mathcal{O},\beta}/\beta_t)$ is a $\mathbb{P}^\beta$-martingale, we have by the optional sampling theorem a.s.,

$$C_T^{\mathcal{O},\beta} = \beta_T \mathbb{E}^\beta[\frac{C_m^{\mathcal{O},\beta}}{\beta_m} \,|\, \mathcal{F}_T] = \beta_T \mathbb{E}^\beta[\frac{O}{\beta_T} \,|\, \mathcal{F}_T] = O.$$

(The last equality is because β_T hence O/β_T is $\mathcal{F}_T$ measurable.) □

Given a numeraire β, above we defined the process $\mathcal{O}_\cdot^\beta$ to be the rollover claim price process $C_\cdot^{\mathcal{O},\beta}$ cut off to zero after expiry T, i.e., $\mathcal{O}_t^\beta := 1_{t \leq T} C_t^{\mathcal{O},\beta}$. The reason for the cutoff is that the rollover claim price process $(C_t^{\mathcal{O},\beta})$ surely depends on the numeraire β after expiry - in fact $1_{t>T} C_t^{\mathcal{O},\beta} = 1_{t>T}(O/\beta_T)\beta_t$. But, not so with the cutoff:

Lemma 3.4. *For any two numeraires β and B, $\mathcal{O}_t^B = \mathcal{O}_t^\beta$ a.s. all t.*

Proof. By the definition of $\mathcal{O}_t^\beta$, we have

$$(*) \quad \mathcal{O}_t^\beta = \beta_t \mathbb{E}^\beta[1_{t \leq T} \frac{O}{\beta_T} \,|\, \mathcal{F}_t] = \beta_t \mathbb{E}^\beta[1_{t \leq T} \frac{O}{\beta_{t \vee T}} \,|\, \mathcal{F}_t] = 1_{t \leq T} \beta_t \mathbb{E}^\beta[\frac{O}{\beta_{t \vee T}} \,|\, \mathcal{F}_t].$$

Now, a.s.,

$$\mathcal{O}_t^B = 1_{t \leq T} B_t \mathbb{E}^B[\frac{O}{B_{t \vee T}} \,|\, \mathcal{F}_t] \qquad ((*) \text{ applied to } B)$$

$$= 1_{t \leq T} \beta_t \mathbb{E}^\beta[\frac{O}{B_{t \vee T}} \frac{B_m}{\beta_m} \,|\, \mathcal{F}_t]] \qquad (\text{change of numeraire})$$

$$= 1_{t \leq T} \beta_t \mathbb{E}^\beta[\frac{O}{B_{t \vee T}} \frac{B_m}{\beta_m} \,|\, \mathcal{F}_{t \vee T} \,|\, \mathcal{F}_t]] \qquad (\text{iter. expect. as } \mathcal{F}_{t \vee T} \supset \mathcal{F}_t)$$

$$= 1_{t \leq T} \beta_t \mathbb{E}^\beta[\frac{O}{B_{t \vee T}} \mathbb{E}^\beta[\frac{B_m}{\beta_m} \,|\, \mathcal{F}_{t \vee T}] \,|\, \mathcal{F}_t]$$

$$= 1_{t \leq T} \beta_t \mathbb{E}^\beta[\frac{O}{B_{t \vee T}} \frac{B_{t \vee T}}{\beta_{t \vee T}} \,|\, \mathcal{F}_t] \qquad (\text{optional sampling theorem})$$

$$= 1_{t \leq T} \beta_t \mathbb{E}^\beta[\frac{O}{\beta_{t \vee T}} \,|\, \mathcal{F}_t] = \mathcal{O}_t^\beta \qquad ((*) \text{ applied to } \beta)$$

□

The above shows that the process $(\mathcal{O}_t^\beta)$ and $(\mathcal{O}_t^B)$ are modifications of each other. To establish indistinguishability, we further need

Lemma 3.5. *Let β be a numeraire. Then, a stochastic process $\mathcal{O}_\cdot = (\mathcal{O}_t)$ is indistinguishable from the process $\mathcal{O}_\cdot^\beta = (\mathcal{O}_t^\beta)$ if and only if (a) process $(1_{t>T}\mathcal{O}_t)$ is indistinguishable from the zero process, (b) the stopped process $(\mathcal{O}_{t \wedge T})$ is right continuous, (c) $\mathcal{O}_T = O$ a.s., and (d) for each t, $\mathcal{O}_t = \mathcal{O}_t^\beta$ a.s.*

Proof. Clearly, process $\mathcal{O}^\beta_\cdot$ satisfies (a), (b), and (d). It also satisfies (c) by Lemma 3.3. Hence, if $\mathcal{O}_\cdot$ is indistinguishable from $\mathcal{O}^\beta_\cdot$, it too satisfies (a)-(d). Conversely, assume $\mathcal{O}_\cdot$ satisfies (a)-(d). Note, (d) implies $\mathcal{O}_\cdot$ is an adapted process, and (a) and (d) imply $\mathcal{O}_{t\wedge T} = \mathcal{O}^\beta_{t\wedge T}$ a.s. all t, because $\mathcal{O}_{t\wedge T} = 1_{t\le T}\mathcal{O}_t + 1_{t>T}\mathcal{O}_T$ and similarly for $\mathcal{O}^\beta_t$. So, process $(\mathcal{O}_{t\wedge T})$ is a modification of process $(\mathcal{C}^{\mathcal{O},\beta}_{t\wedge T})$. But, the former is right-continuous by (c). So is the latter, as $(\mathcal{C}^{\mathcal{O},\beta}_t)$ is so. But two right-continuous modifications are easily seen to be indistinguishable (e.g. Lemma 2.21 in [E]). Therefore, the processes $(\mathcal{O}_{t\wedge T})$ and $(\mathcal{C}^{\mathcal{O},\beta}_{t\wedge T})$ are indistinguishable, which by (a) implies $(\mathcal{O}_t)$ and $(\mathcal{C}^{\mathcal{O},\beta}_t)$ are indistinguishable. □

Proof. of Theorem 3.2: Clearly, the process $\mathcal{O}^B_\cdot$ satisfies the conditions (a), (b) in Lemma 3.5. And it satisfies conditions (c) and (d) respectively by Lemmas 3.3 and 3.4. Therefore, by Lemma 3.5 it is indistinguishable from process $\mathcal{O}^\beta_\cdot$. □

We define *the Price Process* of option $\mathcal{O}$ to be any stochastic process that is indistinguishable from process $\mathcal{O}^\beta_\cdot$ for some numeraire β. The theorem shows that any two option price processes of option $\mathcal{O}$ are indistinguishable. This justifies use of the word "the" in the definition.

The definition of option price process and the theorem imply that $\mathcal{O}^\beta_\cdot$ is the price process of option $\mathcal{O}$ for any numeraire β. In particular, $\mathcal{O}^{1/\xi_m}_\cdot$ is the option price process. Therefore, the price process of option $\mathcal{O}$ exists. Further, it is unique up to indistinguishability by the theorem. We denote it (really its indistinguishability equivalence class) by $\mathcal{O}_\cdot = (\mathcal{O}_t)_{t\in[0,m]}$. So, for any numeraire β, the option price process $\mathcal{O}_\cdot$ is indistinguishable from process $\mathcal{O}^\beta_\cdot$, and for all t,[8]

$$\mathcal{O}_t = 1_{t\le T}\beta_t \mathbb{E}^\beta[\frac{O}{\beta_T} \,|\, \mathcal{F}_t] \text{ a.s.} \tag{3.1}$$

By Lemma 3.3, $\mathcal{O}_T = O$ a.s. So, for any option $\mathcal{O}$ we have

$$\mathcal{O} = (T_\mathcal{O}, \mathcal{O}_{T_\mathcal{O}}),$$

another way of saying an option's price at expiry equals its payoff.

Lemma 3.5 characterizes the option price process $\mathcal{O}_\cdot = (\mathcal{O}_t)$ as the unique (up to indistinguishability) process that (a) vanishes after expiry, (b) the stopped process $(\mathcal{O}_{t\wedge T_\mathcal{O}})$ is right continuous, (c) $\mathcal{O}_{T_\mathcal{O}}$ equals the option payoff, and (d) satisfies Eq. (3.1).

Note, if $\mathcal{O} = (T, O)$ is an option then

$$(T, O)_t = \mathcal{O}_t = 1_{t\le T}\mathcal{O}_t = (T, 1_{t\le T}O)_t.$$

Similarly, if (X_t) is any bounded adapted process, $(T, X_t O)_t = X_t(T, O)_t$.

Given an option $\mathcal{O}$ and a numeraire β, the stopped numeraire-relative price process $(\mathcal{O}_{t\wedge T_\mathcal{O}}/\beta_{t\wedge T_\mathcal{O}})$ is a right-continuous $\mathbb{P}^\beta$-martingale. In

[8] We also have $\mathcal{O}_t = 1_{t\le T}\beta_t\mathbb{E}^\beta[O/\beta_T \,|\, \mathcal{F}_{t\wedge T}]$ a.s.

particular, if $\mathcal{O}$ is european, i.e., $T_{\mathcal{O}}$ is deterministic, then the β-deflated option price process $(\mathcal{O}_t/\beta_t)$ is a $\mathbb{P}^\beta$-martingale on the interval $[0, T_{\mathcal{O}}]$.

Option price at a stopping time is given by a similar formula:

Proposition 3.6. *Let $\mathcal{O} = (T, O)$ be an option and τ a stopping time. Then, $\mathcal{O}_\tau \in \mathcal{C}_\tau$, and for any numeraire β, we have a.s.*

$$\mathcal{O}_\tau = 1_{\tau \leq T} \beta_\tau \mathbb{E}^\beta[\frac{O}{\beta_T} \,|\, \mathcal{F}_\tau].$$

Proof. By definition, $\mathcal{O}_\tau = 1_{\tau \leq T} C_\tau^{\mathcal{O},\beta}$. As $(C_t^{\mathcal{O},\beta}/\beta_t)$ is a $\mathbb{P}^\beta$-martingale, the optional sampling theorem implies $C_\tau^{\mathcal{O},\beta}/\beta_\tau$ is $\mathbb{P}^\beta$-integrable; hence so is $\mathcal{O}_\tau/\beta_\tau$ i.e., $\mathcal{O}_\tau \in \mathcal{C}_\tau$. It also implies a.s.,

$$\frac{C_{\tau\wedge T}^{\mathcal{O},\beta}}{\beta_{\tau\wedge T}} = \mathbb{E}^\beta[\frac{C_T^{\mathcal{O},\beta}}{\beta_T} \,|\, \mathcal{F}_\tau] = \mathbb{E}^\beta[\frac{O}{\beta_T} \,|\, \mathcal{F}_\tau].$$

Hence a.s.,

$$\mathcal{O}_\tau = 1_{\tau \leq T} C_\tau^{\mathcal{O},\beta} = 1_{\tau \leq T} C_{\tau\wedge T}^{\mathcal{O},\beta}$$
$$= 1_{\tau \leq T} \beta_{\tau\wedge T} \mathbb{E}^\beta[\frac{O}{\beta_T} \,|\, \mathcal{F}_\tau] = 1_{\tau \leq T} \beta_\tau \mathbb{E}^\beta[\frac{O}{\beta_T} \,|\, \mathcal{F}_\tau].$$

□

3.4. **Price transitivity law.** Let $\mathcal{O}$ be an option and $S \leq T_{\mathcal{O}}$ be a stopping time. As just shown, the time-S price $\mathcal{O}_S$ is in $\mathcal{C}_S$. So, the pair $(S, \mathcal{O}_S)$ is an option. The law of iterated expectation translates into a transitivity law for option prices. Namely, if we price $(S, \mathcal{O}_S)$ at an earlier time $t \leq S$, it should give the price $\mathcal{O}_t$ of option $\mathcal{O}$ at time t:

Theorem 3.7. *(Price Transitivity) Let $\mathcal{O}$ be an option, and $S \leq T_{\mathcal{O}}$ be a stopping time. Then, a.s., for all t,*[9]

$$(S, \mathcal{O}_S)_t = 1_{t \leq S} \mathcal{O}_t.$$

Further, if $\tau \leq S$ is another stopping time then, $(S, \mathcal{O}_S)_\tau = \mathcal{O}_\tau$ a.s.

Proof. The first statement follows from the second by setting $\tau = t \wedge S$ in the second, and invoking Lemma 3.5 to conclude indistinguishability. As for the second, let β be a numeraire. Applying Proposition 3.6 twice, then iterating expectation, and applying Proposition 3.6 once again,

$$(S, \mathcal{O}_S)_\tau = \beta_\tau \mathbb{E}^\beta[\frac{\mathcal{O}_S}{\beta_S} \,|\, \mathcal{F}_\tau]$$
$$= \beta_\tau \mathbb{E}^\beta[\mathbb{E}^\beta[\frac{\mathcal{O}_T}{\beta_T} \,|\, \mathcal{F}_S] \,|\, \mathcal{F}_\tau] = \beta_\tau \mathbb{E}^\beta[\frac{\mathcal{O}_T}{\beta_T} \,|\, \mathcal{F}_\tau] = \mathcal{O}_\tau.$$

□

[9]That is, processes $((S, \mathcal{O}_S)_t)$ and $(1_{t \leq S} \mathcal{O}_t)$ are indistinguishable.

Price transitivity facilitates option price manipulations, often obviating the need every time to choose a reference numeraire, translate prices as expectations of ratios in the numeraire measure, iterate expectations, and translate back to prices. Price transitivity does all this in one step, without even asking for a reference numeraire. It encapsulate the optional sampling theorem as built-in feature, automatically invoking it whenever used. This makes it handy for calculations, as exemplified by the theorem on indistinguishability in the next subsection.

Define the *sum of two options* $\mathcal{O}^1 = (T_1, O^1)$ and $\mathcal{O}^2 = (T_2, O^2)$ by

$$\mathcal{O}^1 + \mathcal{O}^2 := (T_1 \wedge T_2, \mathcal{O}^1_{T_1\wedge T_2} + \mathcal{O}^2_{T_1\wedge T_2}).$$

The price operator is linear: if $S \leq T_1 \wedge T_2$, then $(\mathcal{O}^1+\mathcal{O}^2)_S = \mathcal{O}^1_S + \mathcal{O}^2_S$. More generally, $(\mathcal{O}^1 + \mathcal{O}^2)_t = 1_{t\leq T_2}\mathcal{O}^1_t + 1_{t\leq T_1}\mathcal{O}^2_t$. The sum operator is associative: $\mathcal{O}^1 + (\mathcal{O}^2 + \mathcal{O}^3) = (\mathcal{O}^1 + \mathcal{O}^2) + \mathcal{O}^3$, as follows from price transitivity and price linearity.

Multiplication of a T-expiry option $\mathcal{O}$ by a scalar $a \in \mathbb{R}$ is defined by $a\mathcal{O} := (T, a\mathcal{O}_T)$. As such, as alluded to in Sect. 3.2, a portfolio of long and short positions in options is itself an option, and the portfolio price equals the algebraic sum of prices of the constituent options.[10]

Given a T_1-maturity option $\mathcal{O}^1$, a T_2-maturity option $\mathcal{O}^2$ and a stopping time $T \leq T_1 \wedge T_2$, the option $(T, (\mathcal{O}^1 - \mathcal{O}^2)^+_T)$ is called a *swaption*. It is a T-expiry option to swap (exchange) $\mathcal{O}^2$ with $\mathcal{O}^1$, i.e., the right to receive at T the option $\mathcal{O}^1$ and pay the option $\mathcal{O}^2$. If $\mathcal{O}^2$ (resp. $\mathcal{O}^1$) is a T-maturity zero-coupon bond (T, K) with face value $0 < K \in \mathbb{R}$, then, the swaption is a call option (resp. put) option on $\mathcal{O}^1$ (resp. on $\mathcal{O}^2$). Callable and putable assets can be defined similarly.

Note, for any two options $\mathcal{O}$ and $\mathcal{O}'$ and any two stopping times $\tau \leq T$, we have $\max(\mathcal{O}_\tau, \mathcal{O}'_\tau) \leq (T, \max(\mathcal{O}_T, \mathcal{O}'_T))_\tau$.

3.5. **Postponing zero payoffs: indistinguishable options.** It is certainly possible for two different options to have indistinguishable price processes. For example, if $S \neq T$ are two stopping times, then the S and T-expiry "zero options" $(S, 0)$ and $(T, 0)$ are different, yet their price processes are indistinguishable from the zero-process.

More generally, let $\mathcal{O}^1 = (T_1, O^1)$ be an option and $T_2 \geq T_1$ be stopping times. Set $T = 1_{O^1\neq 0}T_1 + 1_{O^1=0}T_2$. Then, the price processes of options (T, O^1) and $\mathcal{O}^1$ are indistinguishable. This is intuitive for, ignoring back office overheads, the counterparties care less if a zero payment due at T_1 is postponed to a later even unknown time T.

Two options $\mathcal{O}^1 = (T_1, O^1)$ and $\mathcal{O}^2 = (T_2, O^2)$ are *Equivalent* if $T_1 = T_2$ a.s. and $O^1 = O^2$ a.s.

[10]Obviously $a(\mathcal{O}^1 + \mathcal{O}^2) = a\mathcal{O}^1 + a\mathcal{O}^2$. However, these definitions of sum and scalar product do *not* make $\mathbf{O}$ into a vector space. Indeed, there is no "zero vector." For example, if $\mathcal{O}^1$ and $\mathcal{O}^2$ are two options, $\mathcal{O}^1 - \mathcal{O}^1 \neq \mathcal{O}^2 - \mathcal{O}^2$ unless $T_{\mathcal{O}^1} = T_{\mathcal{O}^2}$.

Two options $\mathcal{O}^1 = (T_1, O^1)$ and $\mathcal{O}^2 = (T_2, O^2)$ are *indistinguishable* if $O^1 = O^2$ a.s. and $\{T_1 \neq T_2\} \subset \{O^1 = 0\}$ a.s.

So, if $\mathcal{O}^1$ and $\mathcal{O}^2$ expire at different times, they both pay zero, and if they expire at the same time, they pay the same amount.

Obviously, equivalence implies indistinguishability. Not so the other way, e.g., the options $(0, 0)$ and $(m, 0)$ are indistinguishable, but not equivalent because $m > 0$. It is easy to see that option indistinguishability is an equivalence relation. This follows even more easily from

Theorem 3.8. *Two options are indistinguishable if and only if their price processes are indistinguishable.*

Proof. Let $\mathcal{O}^1 = (T_1, O^1)$ and $\mathcal{O}^2 = (T_2, O^2)$ be two options. First assume they are indistinguishable. Note this implies $O^1 = 1_{T_1=T_2} O^1$ a.s. Also by definition, a.s. $O^2 = O^1 =: O$. Hence a.s.,

$$\mathcal{O}^1_t = 1_{t \leq T_1} \mathcal{O}^1_t = 1_{t \leq T_1} (T_1, O)_t = (T_1, 1_{t \leq T_1} O)_t = (T_1, 1_{t \leq T_1 = T_2} O)_t.$$

By symmetry, we also conclude $\mathcal{O}^2_t = (T_2, 1_{t \leq T_1 = T_2} O)_t$ a.s. This shows $(\mathcal{O}^1_t)$ and $(\mathcal{O}^2_t)$ are indistinguishable processes. Indeed, if β is a numeraire, it shows a.s.,

$$\mathcal{O}^1_t = \beta_t \mathbb{E}^\beta[1_{t \leq T_1 = T_2} \frac{O}{\beta_{T_1}} \mid \mathcal{F}_t] = \beta_t \mathbb{E}^\beta[1_{t \leq T_1 = T_2} \frac{O}{\beta_{T_2}} \mid \mathcal{F}_t] = \mathcal{O}^2_t.$$

Conversely, assume price processes of $\mathcal{O}^1$ and $\mathcal{O}^2$ are indistinguishable from each other and from a process $\mathcal{O}_\cdot$. Since $\mathcal{O}_\cdot$ is the price processes of both options, we have $1_{t \leq T_1} \mathcal{O}_t = \mathcal{O}_t = 1_{t \leq T_2} \mathcal{O}_t$. Hence $\mathcal{O}_t$ vanishes on $\{t > T_1 \wedge T_2\}$, implying $\mathcal{O}_{T_1 \vee T_2} = 1_{T_1 = T_2} \mathcal{O}_{T_1}$. It follows from price transitivity that $\mathcal{O}_{T_1 \wedge T_2} = (T_1 \vee T_2, 1_{T_1 = T_2} \mathcal{O}_{T_1})_{T_1 \wedge T_2}$. Since $1_{T_1 \neq T_2}$ is $T_1 \wedge T_2$-measurable, multiplying both sides gives $1_{T_1 \neq T_2} \mathcal{O}_{T_1 \wedge T_2} = (T_1 \vee T_2, 1_{T_1 \neq T_2} 1_{T_1 = T_2} \mathcal{O}_{T_1})_{T_1 \wedge T_2} = 0$ a.s. In particular, $0 = 1_{T_1 < T_2} \mathcal{O}_{T_1 \wedge T_2} = 1_{T_1 < T_2} \mathcal{O}_{T_1}$. But, we also know $1_{T_1 > T_2} \mathcal{O}_{T_1} = 0$ because again, $\mathcal{O}_t$ being price process of $\mathcal{O}^2_t$ satisfies $1_{t > T_2} \mathcal{O}^2_t = 0$. Hence, $1_{T_1 \neq T_2} \mathcal{O}_{T_1} = 0$. This shows $\{T_1 \neq T_2\} \subset \{\mathcal{O}_{T_1} = 0\}$ a.s. By symmetry, we also conclude $\{T_1 \neq T_2\} \subset \{\mathcal{O}_{T_2} = 0\}$ a.s. So, $\mathcal{O}_{T_1}$ and $\mathcal{O}_{T_2}$ are a.s equal to zero on set $\{T_1 \neq T_2\}$. But, they are obviously equal on $\{T_1 = T_2\}$ too. So, $O^1 = \mathcal{O}_{T_1} = \mathcal{O}_{T_2} = O^2$, a.s. Indistinguishability follows. ☐

An option $\mathcal{O}$ is *Semipositive* if it is nonnegative and a.s. $\mathcal{O}_t$ positive on the set $\{t < T\}$. We will show in section 5, that for any nonnegative option $\mathcal{O}$ there exists a unique (up to equivalence) semipositive option indistinguishable from it. This is in some sense the opposite of postponing zero payoffs. Contrariwise, it settles a zero payoff at the earliest possible moment, namely, the first time price becomes zero.

3.6. **Event-triggered price additivity.** The following intuitive result usefully generalizes in an inductive fashion, lending itself to the underlining structure of regenerative trigger streams in Part III. Its simple proof is another illustration of calculation using intrinsic operations, rather than falling back on numeraires and expectation.

Proposition 3.9. *Let* $\mathcal{O}^1 = (T_1, O^1)$ *and* $\mathcal{O}^2 = (T_2, O^2)$ *be two options with* $T_1 \leq T_2$*. Let* $\Lambda_1 \in \mathcal{F}_{T_1}$ *be an event. Set* $T = 1_{\Lambda_1} T_1 + 1_{\Lambda_1^c} T_2$ *and* $O = 1_{\Lambda_1} O^1 + 1_{\Lambda_1^c} O^2$*. Then* $O \in \mathcal{C}_T$ *and the price process of* (T, O) *is*

$$(1_{\Lambda_1} T_1 + 1_{\Lambda_1^c} T_2, 1_{\Lambda_1} O^1 + 1_{\Lambda_1^c} O^2)_t = (T_1, 1_{\Lambda_1} O^1)_t + (T_2, 1_{\Lambda_1^c} O^2)_t.$$

Proof. Using $1_{t \leq T} = 1_{\Lambda_1} 1_{t \leq T_1} + 1_{\Lambda_1^c} 1_{t \leq T_2}$, we calculate,

$$\begin{aligned}
&(1_{\Lambda_1} T_1 + 1_{\Lambda_1^c} T_2, 1_{\Lambda_1} O^1 + 1_{\Lambda_1^c} O^2)_t \\
&= (1_{\Lambda_1} T_1 + 1_{\Lambda_1^c} T_2, 1_{t \leq T}(1_{\Lambda_1} O^1 + 1_{\Lambda_1^c} O^2))_t \\
&= (1_{\Lambda_1} T_1 + 1_{\Lambda_1^c} T_2, (1_{\Lambda_1} 1_{t \leq T_1} + 1_{\Lambda_1^c} 1_{t \leq T_2})(1_{\Lambda_1} O^1 + 1_{\Lambda_1^c} O^2))_t \\
&= (1_{\Lambda_1} T_1 + 1_{\Lambda_1^c} T_2, 1_{\Lambda_1} 1_{t \leq T_1} O^1 + 1_{\Lambda_1^c} 1_{t \leq T_2} O^2))_t \\
&= (T_1, 1_{\Lambda_1} 1_{t \leq T_1} O^1)_t + (T_2, 1_{\Lambda_1^c} 1_{t \leq T_2} O^2)_t \\
&= (T_1, 1_{\Lambda_1} O^1)_t + (T_2, 1_{\Lambda_1^c} O^2))_t.
\end{aligned}$$

(This calculation can also be done using more familiar numeraire and expectation operator notation.[11]) □

The result has a simple interpretation. The stopping time $T = 1_{\Lambda_1} T_1 + 1_{\Lambda_1^c} T_2$ is constructed to stop at T_1 if event Λ_1 has occurred by then, and to stop at T_2 otherwise. Accordingly, the payoff $O = 1_{\Lambda_1} O^1 + 1_{\Lambda_1^c} O^2$ pays at T_1 the same as option O^1 if event Λ_1 has occurred, and otherwise, pays at T_2 the same as option O^2. It is not then surprising that the price process of option $O = (T, O)$ should be indistinguishable from the sum of price processes of a T_1-expiry option with payoff $1_{\Lambda_1} O^1$ and a T_2-expiry option with payoff $1_{\Lambda_1} O^2$. The crucial assumptions here are that $\Lambda_1 \in \mathcal{F}_{T_1}$ and $T_1 \leq T_2$.[12]

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[11]Indeed, we may repeat these calculations, with reference to a numeraire β. Using $1_{\Lambda_1}/\beta_T = 1_{\Lambda_1}/\beta_{T_1}$ and $1_{\Lambda_1^c}/\beta_T = 1_{\Lambda_1^c}/\beta_{T_2}$ and the definition of price,

$$\begin{aligned}
\mathcal{O}_t &= \beta_t \mathbb{E}^\beta[1_{t \leq T} \frac{O}{\beta_T} \,|\, \mathcal{F}_t] \\
&= \beta_t \mathbb{E}^\beta[(1_{\Lambda_1} 1_{t \leq T_1} + 1_{\Lambda_1^c} 1_{t \leq T_2})(1_{\Lambda_1} O^1 + 1_{\Lambda_1^c} O^2)/\beta_T \,|\, \mathcal{F}_t] \\
&= \beta_t \mathbb{E}^\beta[1_{\Lambda_1} 1_{t \leq T_1} \frac{O^1}{\beta_{T_1}} + 1_{\Lambda_1^c} 1_{t \leq T_2} \frac{O^2}{\beta_{T_2}} \,|\, \mathcal{F}_t] \\
&= 1_{t \leq T_1} \beta_t \mathbb{E}^\beta[1_{\Lambda_1} \frac{O^1}{\beta_{T_1}} \,|\, \mathcal{F}_t] + 1_{t \leq T_2} \beta_t \mathbb{E}^\beta[1_{\Lambda_1^c} \frac{O^2}{\beta_{T_2}} \,|\, \mathcal{F}_t]. \\
&= (T_1, 1_{\Lambda_1} O^1)_t + (T_2, 1_{\Lambda_1^c} O^2)_t.
\end{aligned}$$

[12]Using the definition of sum of two options previously, one can easily show that option (O, T) can be represented as the sum $(O, T) = (T, 1_{\Lambda_1} O^1) + (T_2, 1_{\Lambda_1^c} O^2)$. Linearity of the price operator then gives another demonstration of the proposition.

4. Dominated and trigger option convergence

4.1. **Dominated option convergence.** An option $\mathcal{O}$ *Dominates* a process (Z_t) if a.s. $|Z_t| \le \mathcal{O}_t$ for all $t \in [0, m]$.

We then interchangeably say (Z_t) is *Dominated* by option $\mathcal{O}$.

Note then $|Z_T| \le \mathcal{O}_T$ a.s. for any stopping time T.[13]

Note also if (Z_t) is a right continuous process and $|Z_t| \le \mathcal{O}_t$ a.s. for each t, then $\mathcal{O}$ dominates (Z_t).[14]

A process (Z_t) is *Dominated* if it is dominated by some option.

An option $\mathcal{O}$ *Dominates* another option $\mathcal{E}$ if $\mathcal{O}$ dominates $(\mathcal{E}_t)$.

Note, any option is dominated by a numeraire.[15] Therefore, a process (Z_t) is dominated if and only if it is dominated by some numeraire.

Note an option $\mathcal{O}$ dominates $\mathcal{E}$ if $|\mathcal{E}_{T_\mathcal{E}}| \le \mathcal{O}_{T_\mathcal{E}}$ a.s. For then $|\mathcal{E}_t| \le \mathcal{O}_t$ a.s. all t; so domination follows by right continuity of $(\mathcal{E}_t)$ on $\{t < T_\mathcal{E}\}$.

Theorem 4.1. *(Dominated Option Convergence) Let $(\mathcal{O}^n)_{n=1}^\infty, \mathcal{O}^n = (T_n, O^n)$ be a sequence of options such that (T^n) is decreasing, and $T^n \searrow T$ a.s. and $O^n \to O$ a.s. to some random variables T and O. Assume there exists a numeraire β that dominates options $\mathcal{O}^n$ all n. Then $\mathcal{O} = (T, O)$ is an option, and a.s. $\mathcal{O}^n_t \to \mathcal{O}_t$, all t.*

We prove the theorem simultaneously with a similar result that replaces the decreasing assumption on (T_n) with a continuity assumption. (Recall a claim is continuous if it has a continuous price process.)

Theorem 4.2. *Let $(\mathcal{O}^n)_{n=1}^\infty, \mathcal{O}^n = (T_n, O^n)$ be a sequence of options such that $T^n \longrightarrow T$ a.s. and $O^n \to O$ a.s. Assume there exists a continuous numeraire β that dominates options $\mathcal{O}^n$ for all n. Then $\mathcal{O} = (T, O)$ is an option, and a.s. $\mathcal{O}^n_t \to \mathcal{O}_t$, all t.*

Proof. Assumption of either theorem implies $O^n/\beta_{T_n} \to O/\beta_T$ a.s. (In the case (T_n) is decreasing by right continuity of (β_t).) But, $|O^n/\beta_{T_n}| \le 1$. So by (conditional) bounded convergence theorem, $\mathbb{E}^\beta[O^n/\beta_{T_n} \,|\, \mathcal{F}_\tau] \to \mathbb{E}^\beta[O/\beta_T \,|\, \mathcal{F}_\tau]$ a.s. for any stopping time τ. Thus,

$$\mathcal{O}^n_\tau = 1_{\tau \le T_n} \mathbb{E}^\beta[O^n/\beta_{T_n} \,|\, \mathcal{F}_\tau] \to 1_{\tau \le T} \mathbb{E}^\beta[O/\beta_T \,|\, \mathcal{F}_\tau] = \mathcal{O}_\tau \text{ a.s.}$$

To prove the stronger statement a.s. $\mathcal{O}^n_t \to \mathcal{O}_t$, all t, let $\mathbb{Q}$ be the subset of rational numbers $[0, m]$. Since $\mathbb{Q}$ is countable, by what we just showed there is an event Λ of probability 1 such that $\mathcal{O}^n_s(\omega) \to \mathcal{O}_s(\omega)$ for all $s \in \mathbb{Q}$ and $\omega \in \Lambda$. Since any t can be approximated on the right

[13] The definition of domination requires a.s. $|Z_t| \le \mathcal{O}_t$ all t, rather than the weaker property $|Z_t| \le \mathcal{O}_t$ a.s. all t. The latter property does *not* imply $|Z_T| \le \mathcal{O}_T$ a.s. for all stopping times T. Indeed, let T be a stopping time with a continuous probability distribution (e.g., exponentially distributed). Then, $Z_t = 0$ a.s. all t, where $Z_t = 1_{\{T=t\}}$. But, $Z_T = 1$. Note also, the definition of domination actually makes sense for any function (Z_t) on $[0, m] \times \Omega$.

[14] This follows easily by approximating any time t by a rational number to its right and then using right continuity of process $(\mathcal{O}_{t \wedge T_\mathcal{O}})$.

[15] Indeed, option (T, O) is dominated by numeraire $\beta(1 + |O|/\beta_T)$ for any $\beta \in \mathcal{C}^+$.

by some $s \in \mathbb{Q}$, it follows from this, and the right continuity of stopped option prices, and the inequality

$$|\mathcal{O}^n_t - \mathcal{O}_t| \le |\mathcal{O}^n_t - \mathcal{O}^n_s| + |\mathcal{O}^n_s - \mathcal{O}_s| + |\mathcal{O}_s - \mathcal{O}_t|,$$

that $\mathcal{O}^n_t(\omega) \to \mathcal{O}_t(\omega)$ for all $\omega \in \Lambda$ and any time t. □

The result resembles the dominated convergence theorem in its assumption. The conclusion is that a sequence of processes is everywhere pointwise convergent on $[0, m] \times \Omega$ less an evanescent set.

The following application is useful for semipositive minimax duality.

Lemma 4.3. *Let $\mathcal{O} = (T, O)$ be an option and B and β be numeraires. Then $|C^{\mathcal{O},B+\beta}| \le C^{|\mathcal{O}|,B} + C^{|\mathcal{O}|,\beta}$ a.s. (Recall $C^{\mathcal{O},\beta} := O\beta/\beta_T$.)*

Proof. . As $B_T > 0$ a.s. and $\beta_T > a.s.$, we have. a.s.

$$\frac{B+\beta}{B_T+\beta_T} < \frac{B}{B_T} + \frac{\beta}{\beta_T}.$$

Hence $|C^{\mathcal{O},B+\beta}| \le |O|(B/B_T + \beta/\beta_T) = C^{|\mathcal{O}|,B} + C^{|\mathcal{O}|,\beta}$ a.s. □

Corollary 4.4. *Let $\mathcal{O}$ be an option and B and β be two numeraires. Then, a.s. all t,*

$$\lim_{\varepsilon \searrow 0} C^{\mathcal{O},B+\varepsilon\beta}_t = C^{\mathcal{O},B}_t.$$

Proof. For any $\varepsilon > 0$, we have by the lemma,

$$|C^{\mathcal{O},B+\varepsilon\beta}| \le C^{|\mathcal{O}|,B} + C^{|\mathcal{O}|,\varepsilon\beta} = C^{|\mathcal{O}|,B} + C^{|\mathcal{O}|,\beta} =: A.$$

So numeraire A dominates all numeraires $B + \varepsilon\beta$. Since $B + \varepsilon\beta \to B$ a.s., as $\varepsilon \to 0$, the result follows from Theorem 4.1. □

4.2. **Payoff processes and trigger options.** A *Payoff Process* is a dominated progressively measurable process.

Proposition 4.5. *Let (Z_t) be a payoff process. Then for any numeraire β, the β-deflated process (Z_t/β_t) is $\mathbb{P}^\beta$-class D.*[16]

In particular, $Z_T \in \mathcal{C}_T$ for any stopping time T. So, (T, Z_T) is an option, one dominated by any option that dominates (Z_t).

Proof. Progressive measurability of process $Z.$ implies Z_T is $\mathcal{F}_T$-measurable for any stopping time T. By the domination assumption, a.s. $|Z_t| \le B_t$ all t for some numeraire B. Let β be any numeraire and T a stopping time. Then a.s. $|Z_t|/\beta_t \le B_t/\beta_t$ all t. So $|Z_T/\beta_T| \le B_T/\beta_T$ a.s. But B_T/β_T is $\mathbb{P}^\beta$-integrable as $B_T \in \mathcal{C}_T$. Thus Z_T/β_T is $\mathbb{P}^\beta$-integrable, i.e., $Z_T \in \mathcal{C}_T$. More strongly, the process (B_t/β_t) is $\mathbb{P}^\beta$ class D because it is a (closed) $\mathbb{P}^\beta$-martingale (closed by B/β). Hence so is process (Z_t/β_t), as $|Z_T/\beta_T| \le B_T/\beta_T$ a.s. for any stopping time T. □

If (Z_t) is a payoff process and T a stopping time, we refer to the option (T, Z_T) as a *$Z.$-Trigger Option.*

[16] That is, the set $\{Z_T/\beta_T : T \in \mathcal{T}\}$ is $\mathbb{P}^\beta$-uniformly integrable.

4.3. **Trigger option convergence.** This simple but important consequence of dominated option convergence will be used several times in Part II, both for approximating american options by bermudan options and for establishing the right-continuity of the Snell envelop. Its paramount assumption is the right continuity of the payoff process.

Theorem 4.6. *(Trigger Option Convergence) Let* (Z_t) *be a right continuous payoff process and* $(T_n)_{n=1}^{\infty}$ *be a sequence of decreasing stopping times converging to a stopping time* T. *Then, a.s.* $(T_n, Z_{T_n})_t \to (T, Z_T)_t$, *all* t.

Proof. By right continuity and the assumption that (T_n) is decreasing, $Z_{T_n} \to Z_T$ a.s. Since (Z_t) is dominated by some option $\mathcal{O}$, by previous proposition each (T_n, Z_{T_n}) is an option, and is dominated by same option $\mathcal{O}$. Hence the result follows from Theorem 4.1. □

Trigger convergence immediately implies certain right continuity:

Corollary 4.7. *Let* (Z_t) *be a right continuous payoff process. Then a.s.* $Z_t = \lim_{s \searrow t}(s, Z_s)_t$, *all* t.

The following version of trigger convergence is based on the dominated option convergence Theorem 4.2 rather than Theorem 4.1, replacing the assumption of (T_n) decreasing by a continuity assumption.

Proposition 4.8. *Let* (Z_t) *be a continuous adapted process that is dominated by a continuous numeraire, and* $(T_n)_{n=1}^{\infty}$ *be a sequence of stopping times converging to a stopping time* T. *Then, a.s.* $(T_n, Z_{T_n})_t \to (T, Z_T)_t$, *all* t.

5. Semipositivity

5.1. **Nonnegative arbitrage.** If $\mathcal{O}$ is a nonnegative option, then obviously $\mathcal{O}_\tau \geq 0$ a.s. for any stopping time τ. Slightly more strongly, we have a.s. $\mathcal{O}_t \geq 0$ all t. (This follows from the right continuity of option price process before expiry and approximation of any t on the right by a rational number). Also, clearly a nonnegative option whose time-0 price is 0 must have a.s. 0 payoff. These may be viewed as no-arbitrage statements, albeit quite weak ones.

To prepare for stronger no-arbitrage statements, for any option $\mathcal{O} = (T, O)$, define the stopping time $T^0_{\mathcal{O}}$, abbreviated T^0, by

$$T^0 =: T^0_{\mathcal{O}} := \inf\{t > 0 : \mathcal{O}_t = 0\}.$$

Clearly, $T^0 \leq T$. The following result is strengthened later in Theorem 5.12, but already the way it is will be adequate for us.

Theorem 5.1. *Let* $\mathcal{O} = (T, O)$ *be a nonnegative option and* $S \geq T^0$ *a stopping time. Then* $1_{T^0<T}\mathcal{O}_S = 0$ *a.s. In particular,* $1_{T^0<T}O = 0$ *a.s.*

Proof. By the right-continuity of the price process $(\mathcal{O}_t)$ on the set $\{t < T\}$ and definition of T^0, we have $1_{T^0<T}\mathcal{O}_{T^0} = 0$ a.s. Note also, $1_{T^0=T}$ is $\mathcal{F}_S$-measurable for it is $\mathcal{F}_{T^0}$-measurable and $S \geq T^0$. Also, as $1_{S>T}\mathcal{O}_S = 0$, we may assume $S \leq T$, by replacing S with $S \wedge T$ if necessary. Using these three observations and price transitivity, a.s.,

$$(S, \mathcal{O}_S)_{T^0} = \mathcal{O}_{T^0} = 1_{T^0<T}\mathcal{O}_{T^0} + 1_{T^0=T}\mathcal{O}_{T^0}$$

$$= 1_{T^0=T}\mathcal{O}_{T^0} = 1_{T^0=T}(S, \mathcal{O}_S)_{T^0} = (S, 1_{T^0=T}\mathcal{O}_S)_{T^0}.$$

Therefore, a.s. (again employing price transitivity a few of times)

$$(S, \mathcal{O}_S)_0 = \mathcal{O}_0 = (T^0, (S, \mathcal{O}_S)_{T^0})_0$$

$$= (T^0, (S, 1_{T^0=T}\mathcal{O}_S)_{T^0})_0 = (S, 1_{T^0=T}\mathcal{O}_S)_0.$$

But $\mathcal{O}_S \geq 1_{T^0=T}\mathcal{O}_S$. It follows, $\mathcal{O}_S = 1_{T^0=T}\mathcal{O}_S$ a.s., i.e., $1_{T^0<T}\mathcal{O}_S = 0$ a.s. Applied to $S = T$ gives $1_{T^0<T}O = 0$ a.s. □

The three consequences below are more-or-less immediate.

Corollary 5.2. *Let $\mathcal{O} = (T, O)$ be a nonnegative option and S be a stopping time with $T^0 \leq S \leq T$. Then, $\mathcal{O}_S = O$ a.s. Moreover, options $\mathcal{O}$ and $(S, \mathcal{O}_S)$ are indistinguishable.*

Proof. By the theorem, $1_{T^0<T}O = 0$, a.s., so $O = 1_{T^0=T}O$ a.s. Again by the theorem, $1_{T^0<T}\mathcal{O}_S = 0$. As $1_{T^0=T} = 1_{T^0=S=T}$, this implies

$$\mathcal{O}_S = 1_{T^0=T}\mathcal{O}_S = 1_{T^0=S=T}\mathcal{O}_S = 1_{T^0=T}O = O.$$

Since, $1_{T^0<T}\mathcal{O}_S = 0$ a.s., we also have $1_{S<T}\mathcal{O}_S = 0$ a.s. But, $\mathcal{O}_S = O$ a.s. Hence, $1_{S<T}O = 0$ It follows $\{S \neq T\} = \{S < T\} \subset \{O = 0\}$ a.s. Thus $\mathcal{O}$ and $(S, \mathcal{O}_S)$ are indistinguishable. □

In particular, the zero-payoffs of a nonnegative option are previsible at time T^0, that is, $\{O = 0\} \in \mathcal{F}_{T^0}$.

The following two consequences of $1_{\{T^0<T\}}O = 0$ make a definitive positivity statement without reference to the stopping time $T^0_{\mathcal{O}}$.

Corollary 5.3. *Let $\mathcal{O} = (T, O)$ be a nonnegative option. Then there exist an event Λ of probability 1 such that such that $\mathcal{O}_t(\omega) > 0$ for all states $\omega \in \Lambda$ with $O(\omega) > 0$ and all times $t \leq T(\omega)$.*

Equivalently, then the option price process $\mathcal{O}.$ is positive on $[[0, T]] \cap [0, m] \times \{O > 0\}$ outside of an evanescent subset.

Proof. By the proposition, $1_{\{T^0<T\}}O = 0$ a.s. So $\{O > 0\} \subset \{T^0 = T\}$ a.s., i.e., there is an event Λ of probability 1 be such that for any $\omega \in \Lambda$ with $O(\omega) > 0$, we have $T^0(\omega) = T(\omega)$. Now let $\omega \in \Lambda$ with $O(\omega) > 0$, and let $t \leq T(\omega)$. If $t = T(\omega)$ then obviously $\mathcal{O}_t(\omega) = O(\omega) > 0$. Otherwise, if $t < T(\omega)$, then $t < T^0(\omega)$ (because $T^0(\omega) = T(\omega)$); hence by the definition of T^0, $\mathcal{O}_t(\omega) > 0$. □

The following conclusion for positive options is immediate.

Corollary 5.4. *Let $\mathcal{O} = (T, O)$ be a positive option. Then there exist an event Λ of probability 1 such that such that $\mathcal{O}_t(\omega) > 0$ for all states $\omega \in \Lambda$ and times $t \leq T(\omega)$. Put another way, the price process of a positive T-expiry option is positive on $[[0, T]]$ less an evanescent set.*

5.2. **Semipositive options.** A nonnegative option $\mathcal{O} = (T, O)$ is *Semipositive* if its price process $(\mathcal{O}_t)$ is positive on the stochastic interval $[[0, T[[:= \{(t, \omega) : 0 \leq t < T(\omega)\}$ outside an evanescent set, or equivalently, if there exists an event Λ of probability 1 such that $\mathcal{O}_t(\omega) > 0$ for all states $\omega \in \Lambda$ and all times $t < T(\omega)$.

As short way of saying this goes like: "a.s. $\mathcal{O}_t > 0$ on $\{t < T\}$ all t."

Note, this definition requires that the event Λ above can be chosen independent of time t. An apparently weaker property would allow Λ to depend on t, which is like saying for each t, $\mathcal{O}_t$ is positive a.s on the set $\{t < T\}$. The result below shows that this apparently weaker positivity statement actually implies the strong positivity statement of the definition. (Recall $T^0 =: T^0_{\mathcal{O}} := \inf\{t > 0 : \mathcal{O}_t = 0\}$.)

Theorem 5.5. *Let $\mathcal{O}$ be a nonnegative option. Then the following conditions are equivalent.* (a) *$\mathcal{O}$ is semipositive,* (b) *$T^0_{\mathcal{O}} = T_{\mathcal{O}}$ a.s.,*
(c) *$\{S < T\} \subset \{\mathcal{O}_S > 0\}$ a.s for any stopping time S (or $S \leq T$),*
(d) *for each t, $\{t < T\} \subset \{\mathcal{O}_t > 0\}$ a.s.*

Proof. That (a) implies (c) and (c) implies (d) is clear. That (b) implies (a) is easy too. Indeed assume (b) and set $\Lambda = \{T^0 = T\}$. By assumption, $P[\Lambda] = 1$. Let $\omega \in \Lambda$ and $t < T(\omega)$. Then $t < T^0(\omega)$ as $T^0(\omega) = T(\omega)$. But, then definition of T^0 gives $\mathcal{O}_t(\omega) > 0$. It remains to show (d) implies (b).

Assume (d). Let $\mathbb{Q}$ be the set of rationals in $[0, m]$. As $\mathbb{Q}$ is dense,

$$\{T^0 < T\} = \bigcup_{t \in \mathbb{Q}} \{T^0 < t < T\}$$

As $\mathbb{Q}$ is countable, it suffices to show $P[T^0 < t < T] = 0$ for each $t \in \mathbb{Q}$. By (d), $\{T^0 < t < T\} \subset \{\mathcal{O}_t > 0\}$ a.s. But, Theorem 5.1 applied to $S = t \wedge T^0$ implies $\{T^0 < t < T\} \subset \{\mathcal{O}_t = 0\}$ a.s.[17] Hence, $P[T^0 < t < T] = 0$ a.s.. □

Proposition 5.6. *Let $\mathcal{O}$ be a nonnegative option. Then the option $(T^0, \mathcal{O}_{T^0})$ is semipositive.*

Proof. By part (d) above, it suffices to show $(T^0, \mathcal{O}_{T^0})_t$ is a.s. positive on the set $\{t < T^0\}$ for any t. But, by price transitivity, $(T^0, \mathcal{O}_{T^0})_t = \mathcal{O}_t$ on this set, and $\mathcal{O}_t$ is positive on this set by the definition of T^0. □

Proposition 5.7. *Any two indistinguishable semipositive options $\mathcal{O}$ and $\mathcal{O}'$ are equivalent.*

[17] In fact, more strongly it implies $\{T^0 < T\} \cap \{T^0 \leq t\} \subset \{\mathcal{O}_t = 0\}$ a.s.

Proof. Say $\mathcal{O} = (T, O)$ and $\mathcal{O}' = (T', O)$. Since $\mathcal{O}'$ is semipositive, $\mathcal{O}'_T > 0$ on $\{T < T'\}$. But $\mathcal{O}'_T = \mathcal{O}_T$ by Theorem 3.8 because $\mathcal{O}$ and $\mathcal{O}'$ are indistinguishable. Hence $O = \mathcal{O}_T = \mathcal{O}'_T > 0$ on $\{T < T'\}$. By symmetry, it follows $O > 0$ on the set $\{T \neq T'\}$. But as $\mathcal{O}$ and $\mathcal{O}'$ are indistinguishable, $\{T \neq T'\} \subset \{O = 0\}$ a.s. Thus, $T = T'$ a.s. □

Corollary 5.2 says $\mathcal{O}$ and $(T^0, \mathcal{O}_{T^0})$ are indistinguishable options. Proposition 5.6 says the latter is semipositive. Proposition 5.7 states uniqueness. We conclude

Theorem 5.8. *Let $\mathcal{O} = (T, O)$ be nonnegative option. Then, there exists up to equivalence a unique semipositive option that is indistinguishable from $\mathcal{O}$, namely the option $(T^0, O) = (T^0, \mathcal{O}_{T^0})$.*

This result will prove effective for deriving some properties for nonnegative option which are more easily established for positive options, such as minimax duality for american options in Part II and the associativity of the rollover operator and its consequences in Part III.

5.3. **Tight events and semipositivity.** Our purpose here to show semipositivity is a property only of the zero set of the payoff - not of the entire payoff. A useful result in this connection is

Theorem 5.9. *Let $\mathcal{O} = (T, O)$ be a nonnegative option and $\mathcal{E} = (T, E)$ a semipositive option. If $\{O = 0\} \subset \{E = 0\}$, then $\mathcal{O}$ is semipositive.*

The theorem, proved below, follows readily from the proposition below, which is immediate, and the lemma following it, which gives an interesting new characterization of semipositivity. Basically it says, an option is semipositive if its *zero-payoff set is tight at expiry.*

Let T be a stopping time. "Tightness" is a property of events in $\mathcal{F}_T$:

An event $\Gamma \in \mathcal{F}_T$ is *Tight at T* if $\Lambda \subset \{S = T\}$ a.s. for every subset $\Lambda \subset \Gamma$ and every stopping time $S \leq T$ such that $\Lambda \in \mathcal{F}_S$.

Note then, $\Gamma \notin \mathcal{F}_S$ for any stopping time $S < T$ unless Γ is null.

In effect, the definition states that no portion of Γ is previsible at any time before T. It is designed so that a subevent of a tight event is tight - and this is obvious and requires no proof.

Proposition 5.10. *Let T be a stopping time, and Γ, Λ be two $\mathcal{F}_T$-measurable events, $\Lambda \subset \Gamma$. If Γ is tight at T then so is Λ.*

We now tie this notion of tightness to semipositivity:

Lemma 5.11. *A nonnegative option $\mathcal{O} = (T, O)$ is semipositive if and only if its zero-payoff set $\{O = 0\}$ is tight at expiry T.*

Proof. Assume $\{O = 0\}$ is tight at T. This then implies $\{O = 0\} \subset \{T^0 = T\}$ a.s. because $\{O = 0\} \in \mathcal{F}_{T^0}$ by Corollary 5.2. Set $T^0 = T^0_{\mathcal{O}}$. But $\{T^0 < T\} \subset \{O = 0\}$ a.s. by Theorem 5.1. It follows $T^0 = T$ a.s., which by Theorem 5.5 part (b) implies $\mathcal{O}$ is semipositive.

Conversely assume $\mathcal{O}$ be semipositive. Let $\Lambda \subset \{O = 0\}$ and $S \le T$ be a stopping time with $\Lambda \in \mathcal{F}_S$. Then, $1_\Lambda O = 0$ a.s. Hence, $1_\Lambda \mathcal{O}_S = 1_\Lambda (T, O)_S = (T, 1_\Lambda O)_S = 0$ a.s., i.e., $\mathcal{O}_S = 0$ on Λ a.s. But by definition of semipositivity, $\mathcal{O}_S > 0$ on $\{S < T\}$ a.s. So $\Lambda \subset \{S = T\}$ a.s. □

Proof. of Theorem 6.9. By the lemma $\{E = 0\}$ is tight at T. By assumption $\{O = 0\} \subset \{E = 0\}$. The proposition then implies $\{O = 0\}$ is tight at T. By the lemma again, $\mathcal{O}$ is semipositive now. □

5.4. **Discussion: settling zero payoffs as soon as possible.** Non-semipositive nonnegative options are *not* oddities and often arise in practice. It is just that they are indistinguishable from semipositive ones. To clarify this point, let $s < m$, and B be a numeraire such that $0 < \mathbb{P}[B \le 1] < 1$. Consider the european s-expiry call option $\mathcal{E} = (s, (B_s - 1)^+)$. Let β be a numeraire. Then the nonnegative rollover claim $C = C^{\mathcal{E},\beta} := (B_s - 1)^+ \beta/\beta_t$ is certainly not semipositive. In fact if $B_s(\omega) \le 1$, then $C_t(\omega) = 0$ for all $t \in [s, m]$. Now, assume $\mathcal{E}$ is semipositive. The indistinguishable semipositive version of C is then given by (T^0, C), where $T^0 = 1_{B_s \le 1} s + 1_{B_s > 1} m$.

Of course, a.s. $C_t = (T^0, C)_t$ all t, because claim C and option (T^0, C) are indistinguishable. For the study of nonnegative option price processes, we can choose either version, but as we shall see, a semipositive version displays superior analytic and algebraic properties.

Semipositivity is achieved by settling zero payoffs and expiring the option as soon as its price becomes zero, for it is then certain that payoff will end zero at expiry as well. This is the opposite of postponing zero payoffs as discussed in section 3. It is a welcome relief on fragile back office settlement systems to mark the trade as deleted and archive it then, rather than keeping it live as a trade with no cashflow.

For example consider a 9-year knockout call option which knockouts with 0 payoff when a level is first crossed. Suppose this actually comes to pass in year 2. Efficient settlement calls for legally expiring the contract then at year 2, rather than wait until year 9 and then do it.

5.5. **Further on nonnegative arbitrage.** The theorem below states in a rather strong way that once the price of a nonnegative option $\mathcal{O}$ becomes zero it stays zero. Actually, it is already subsumed by Corollary 5.2 saying $\mathcal{O}$ is indistinguishable from (T^0, O), which combined with Theorem 3.8, implies a.s., $\mathcal{O}_t = (T^0, O)_t$ all t, for such property clearly holds for the semipositive option (T^0, O). However, for pedagogical reasons we provide a more direct proof using Theorem 5.1. The result strengthens Theorem 5.1, though no doubt, it can further be improved along the lines of Theorem 4.16 in [E]. (See also Corollary 7.4.)

Theorem 5.12. (*Nonnegative arbitrage*) *Let $\mathcal{O} = (T, O)$ be a nonnegative option. Then a.s.*

$$1_{T^0 < T} 1_{T^0 \le t} \mathcal{O}_t = 0, \ \forall t.$$

In particular, a.s. $1_{\{T^0 \le t < T\}} \mathcal{O}_t = 0$ *all* t.

Equivalently, there exists an event Λ *of probability 1 such that* $\mathcal{O}_t(\omega) = 0$ *for all states* $\omega \in \Lambda$ *with* $T^0(\omega) < T(\omega)$ *and all times* $t \ge T^0(\omega)$*, i.e., such that* $\Lambda \cap \{T^0 < T\} \cap \{T^0 \le t\} \subset \{\mathcal{O}_t = 0\}$ *for all* $t \in [0, m]$.

Proof. By Theorem 5.1, for any s, there exists an event Λ_s of probability 1 such that $\Lambda_s \cap \{T^0 < T\} \subset \{\mathcal{O}_{s \vee T^0} = 0\}$. This implies

$$\Lambda_s \cap \{T^0 < T\} \cap \{T^0 \le s\} \subset \{\mathcal{O}_s = 0\}. \tag{5.1}$$

Now set $\Lambda = \bigcap_{s \in \mathbb{Q}} \Lambda_s$, where $\mathbb{Q}$ is the set of rational numbers in $[0, m]$. Note $P[\Lambda] = 1$ as $\mathbb{Q}$ is countable. Let $t \in [0, m]$. Choose a decreasing sequence $(t_n)_{n=1}^{\infty}$, $t_n \in \mathbb{Q}$ such that $t_n \searrow t$. Then,

$$\begin{aligned}
&\Lambda \cap \{T^0 < T\} \cap \{T^0 \le t\} \\
&\subset \bigcap_{n=1}^{\infty} \Lambda \cap \{T^0 < T\} \cap \{T^0 \le t_n\} \qquad \text{(since } t \le t_n) \\
&\subset \bigcap_{n=1}^{\infty} \Lambda_{t_n} \cap \{T^0 < T\} \cap \{T^0 \le t_n\} \qquad \text{(since } \Lambda_{t_n} \subset \Lambda) \\
&\subset \{T^0 < T\} \cap \bigcap_{n=1}^{\infty} \{\mathcal{O}_{t_n} = 0\} \qquad \text{(by Eq. 5.1)} \\
&\subset \{T^0 < T \le t\} \bigcup \left(\{t < T\} \cap \bigcap_{n=1}^{\infty} \{\mathcal{O}_{t_n} = 0\}\right) \\
&\subset \{\mathcal{O}_t = 0\},
\end{aligned}$$

where the last step follows because $\{T^0 < T \le t\} \subset \{\mathcal{O}_t = 0\}$ by again Theorem 5.1, and because $\{t < T\} \cap \bigcap_{n=1}^{\infty} \{\mathcal{O}_{t_n} = 0\} \subset \{\mathcal{O}_t = 0\}$ since price process $(\mathcal{O}_t)$ is right-continuous at t on the set $\{t < T\}$. □

For pedagogical reasons, we also give a direct proof of another result subsumed by Corollary 5.2

Proposition 5.13. *Let* $\mathcal{O} = (T, O)$ *be a nonnegative option. Then* $\{O = 0\} \in \mathcal{F}_{T^0_{\mathcal{O}}}$.

Proof. We must show that $\{O = 0\} \cap \{T^0_{\mathcal{O}} \le t\} \in \mathcal{F}_t$, $\forall t$. Since $T^0_{\mathcal{O}} \le T$, $\{T \le t\} \subset \{T^0_{\mathcal{O}} \le t\}$. So, $\{T^0_{\mathcal{O}} \le t\} = \{T \le t\} \cup (\{T^0_{\mathcal{O}} \le t\} \cap \{T > t\})$. But $\{O = 0\} \cap \{T \le t\} \in \mathcal{F}_t$ since $O \in \mathcal{F}_T$. Hence, to show $\{O = 0\} \in \mathcal{F}_{T^0_{\mathcal{O}}}$, it is enough to show $\{O = 0\} \cap \{T^0_{\mathcal{O}} \le t \ < T\} \in \mathcal{F}_t\ \forall t$.

Since $\mathcal{O}$ is nonnegative, $\{\mathcal{O}_t = 0\} \cap \{t \ < T\} \subset \{O = 0\}$ by Theorem, 5.1. Hence by definition of $T^0_{\mathcal{O}}$, $\{T^0_{\mathcal{O}} \le t \ < T\} \subset \{O = 0\}$. Therefore $\{O = 0\} \cap \{T^0_{\mathcal{O}} \le t \ < T\} = \{T^0_{\mathcal{O}} \le t \ < T\}$, and this set is $\mathcal{F}_t$ measurable because $T^0_{\mathcal{O}}$ and T are stopping times. □

Part II: Snell Envelop, American Options, Minimax Duality

6. Snell envelop and the american stream

6.1. **Snell Envelop.** Throughout this subsection, let $Z. = (Z_t)_{t \in [0,m]}$ be a right-continuous, payoff process. Equivalently, (Z_t) is adapted, right-continuous, and $|Z_t| \le \beta_t$ a.s. all t for some numeraire β.

The *Snell Envelop* of (Z_t) is the function $V. :=: (V_t^{Z.}) = (V_t)_{t \in [0,m]}$ on $[0,m] \times \Omega$ defined by

$$V_t :=: V_t^{Z.} := \sup_{t \le T \in \mathcal{T}} (T, Z_T)_t.$$

Clearly $Z_t \le V_t$. Also, any numeraire that dominates (Z_t) also dominates (V_t), as it dominates all $Z.$-trigger options (T, Z_T).

The measurability of (V_t) is not evident because the supremum is taken over an uncountable set. A related apparent problematic is with versioning: if we replace each trigger option price process $(T, Z_T)_t$ by an indistinguishable process, then, again because of uncountability of $\mathcal{T}$, it is not apparent that we a get a modification of (V_t), let alone an indistinguishable version of it. But, in fact, regardless

Theorem 6.1. *The Snell envelop* $(V_t^{Z.})$ *of a right-continuous payoff process* $Z.$ *is a right-continuous payoff process. Further, for* $t \le s$*, a.s.*

$$(s, V_s)_t = \sup_{s \le T \in T} (T, Z_T)_t =: V_t^s, \qquad t \le s.$$

In particular, since V_t^s *is decreasing in* s*,* $V_t \ge (s, V_s)_t$ *a.s. all* $t \le s$*.*

The proof is in stages and given below. It approximates by bermudan options to establish the measurability of (V_t), and uses additional trigger convergence arguments for its right-continuity. To emphasize the delicacy of the latter, we first give an example of a *continuous* payoff process whose Snell envelop is *not* (left) continuous.

Continuity Counterexample: This is an example of an american put option on an stock with a stochastic volatility that jumps when an event occurs. Choose $\xi_t = \exp(-rt)$, $r > 0$. Let (w_t) be $\mathbb{P}$-Brownian motion and τ be the first jump of a Poisson process with $0 < \epsilon := P[\tau < m] < 1$. Let $0 \le \sigma_2 < \sigma_1$ (e.g., $\sigma_1 = 0.2$, $\sigma_2 = 0.05$, $\varepsilon = 0.01$, $r = 0.05$,). Set $\sigma_t = 1_{t<\tau}\sigma_1 + 1_{t \ge \tau}\sigma_2$. Consider the stock with volatility σ_t: $dS_t/S_t = r\ dt + \sigma_t dw_t$, $S_0 = 1$, and the american put $Z_t = (1 - S_t)^+$ (struck at 1). Then, (Z_t), is a continuous payoff process, but we claim $V_{\tau_-} < V_\tau$. (The reverse inequality holds when $\sigma_2 > \sigma_1$.). Indeed, let V_t^1 (resp. V_t^2) denote the price at time t of the american put option issued at time t on a stock with constant volatility σ_1 (resp. σ_2). Choose σ_1 and σ_2 such that $V_t^1 > V_t^2$ (This is especially clear with $\sigma_2 = 0$). Clearly, $V_\tau = V_\tau^2$. Further, by choosing ε small enough V_t become close to V_t^1 and away from V_τ^2 for $t < \tau$ sufficiently close to τ. This gives $V_{\tau_-} < V_\tau$, as claimed.

The rest of this subsection is devoted to the proof of the theorem.

A *Finite Exercise Date Approximation* is an increasing sequence $U_\cdot = (U_n)_{n=1}^\infty$ of *finite* subsets of $[0,m]$ whose union is dense in $[0,m]$. So, $U_n \subset U_{n+1}$ all n, and $\bigcup_n U_n \subset [0,m]$ is dense.

(Think of successively approximating an american option with a sequence of ever closer bermudan options.)

For example the *dyadic* approximation is defined by $U_n = \{jm/2^n, j = 1, \cdots, 2^n\}$. (So, $U_1 = \{m/2, m\}$, $U_2 = \{m/4, m/2, 3m/4, m\}$, etc.)

Fix a finite exercise approximation $(U_n)_{n=1}^\infty$, e.g., the dyadic one.

Lemma 6.2. *For any stopping time T, the sequence of stopping times $(T_n^{U_\cdot})_{n=1}^\infty$ is decreasing and converges to T, where $T_n^{U_\cdot} := \min\{t \in U_n : t \geq T\}$. Hence by trigger convergence, $(T_n^{U_\cdot}, Z_{T_n^{U_\cdot}})_t \to (T, Z_T)_t$, all t.*

Proof. That $(T_n^{U_\cdot})$ is decreasing follows because (U_n) is increasing, and the convergence to T is because $\bigcup_n U_n$ is dense. The last statement follows by Theorem 4.6 because (Z_t) is assumed right continuous. □

For each n, set $\mathcal{T}^n := \{T \in \mathcal{T} : T(\omega) \in U_n \ \forall \omega\}$. And for $t \leq s$, set

$$V_t^{s,n} := \sup_{s \leq T \in \mathcal{T}^n} (T, Z_T)_t, \quad t \leq s.$$

Obviously $V_t^{s,n} \leq V_t^s$ and $V_t^{s,n}$ is decreasing in s. Also, $V_t^{s,n}$ is increasing in n because (U_n) is an an increasing sequence of sets.

Corollary 6.3. *We have $V_t^s = \lim_{n\to\infty} V_t^{s,n}$ a.s. for $t \leq s$. In particular, $V_t = \lim_{n\to\infty} V_t^{t,n}$ a.s.*

Proof. As $V_t^{s,n}$ is increasing in n and bounded by V_t^s, the sequence $(V_t^{s,n})_{n=1}^\infty$ is convergent and $V_t^s \geq \lim_{n\to\infty} V_t^{s,n}$. On the other hand, Lemma 6.2 implies that $(T, V_T)_t \leq \sup_n (V_t^{s,n})$ for any stopping time $T \geq s$, because $T_n^{U_\cdot} \in \mathcal{T}_n$. Hence $V_t^s \leq \sup_n (V_t^{s,n}) = \lim_{n\to\infty} V_t^{s,n}$. □

It follows from the corollary that (V_t) is progressively measurable if we knew that $(V_t^{t,n})$ is so for each n. But, the latter is well known as $(V_t^{t,n})$ represents the time t-price of bermudan option issued at time t with exercise dates in U_n, and this will be shown in Part III under a more general setting. Specifically, for each $s \in [0,m]$ and each n, set

$$T_s^n := \inf\{s \leq t \in U_n : Z_t = V_t^{t,n}\}.$$

Then T_s^n is a stopping time as $(V_t^{t,n})$ is progressively measurable. Hence, $\mathcal{B}^{s,n} := (T_s^n, Z_{T_s^n})$ is an option as (Z_t) is a payoff process. We will also show that price of this bermudan option $\mathcal{B}^{s,n}$ is given by $\mathcal{B}_t^{s,n} = V_t^{s,n}$.[18]

[18]This assertion is easy for bermudan options because of (backward) induction.

As $(\mathcal{B}_t^{s,n})$ and $(\mathcal{B}_t^{t,n})$ are progressively measurable, it follows from previous corollary that $(V_t^s)_{t\in[0,s]}$ and (V_t) are progressively measurable too.[19] As (V_t) is dominated, it is therefore a payoff function. Next,

Proposition 6.4. *For all n and $t \le s$, we have $(s, V_s^{s,n})_t = V_t^{s,n}$ a.s.*

Proof. Applying the pricing formula $\mathcal{B}_t^{s,n} = V_t^{s,n}$ and the definition of $\mathcal{B}^{s,n}$, each twice, and price transitivity once,

$$(s, V_s^{s,n})_t = (s, \mathcal{B}_s^{s,n})_t := (s, (T_s^n, Z_{T_s^n})_s)_t$$
$$= (T_s^n, Z_{T_s^n})_t =: \mathcal{B}_t^{s,n} = V_t^{s,n}.$$

□

Corollary 6.5. *For all $t \le s$, we have $(s, V_s)_t = V_t^s$ a.s.*

Proof. Using the above corollary and above proposition,

$$V_t^s = \lim_{n\to\infty} V_t^{s,n} = \lim_{n\to\infty} (s, V_s^{s,n})_t.$$

By above corollary and dominated option convergence Theorem 4.1,

$$(s, V_s)_t = (s, \lim_{n\to\infty} V_s^{s,n})_t = \lim_{n\to\infty} (s, V_s^{s,n})_t.$$

Hence $(s, V_s)_t = V_t^s$, as desired. □

It remains to show that the snell envelop (V_t) is right continuous.

Lemma 6.6. *Snell envelop (V_t) has a right limit (V_{t_+}) and $V_t \ge V_{t_+}$ a.s. all t.*

Proof. Corollary 6.5 and the fact that V_t^s is decreasing in s immediately imply that $(s, V_s)_t \le V_t$ a.s. for all $t \le s$. This in turn immediately implies $X_t := (V_t/\beta_t)$ is a $\mathbb{P}^\beta$-supermartingale for any numeraire β. (See also Proposition 7.2). Now, Theorem 4.6 of [E] says any supermartingale (X_t) has a right limit and its satisfies $X_{t_+} \le X_t$ a.s. As numeraire price (β_t) is right-continuous, the desired result follows. □

It remains to show $V_t \le V_{t_+}$ a.s. all t. For this we first show a related right-continuity result of independent interest.

Proposition 6.7. *Almost all sample paths $V_t^s(\omega)$ are right continuous in s for fixed $t \le s$.*[20]

Proof. Let $s_n \searrow s$. As $V_t^{s_n}$ increases with n and $V_t^{s_n} \le V_t^s$, (the limit exists and) $V_t^s \ge \lim_{n\to\infty} V_t^{s_n}$. Next, let $s \le T \in \mathcal{T}$. Set $T_n = T \vee s_n$. Then $T_n \searrow T$; hence by trigger option convergence,

$$(T, Z_T)_t = \lim (T_n, Z_{T_n})_t \le \sup_{s_n \le T \in \mathcal{T}} (T, Z_T)_t =: V_t^{s_n},$$

[19]Note however, $\mathcal{B}_t^{t,n} = V_t^{t,n}(\omega)$ is not right-continuous at $t = T_t^n(\omega)$.

[20]A related and even easier statement is: $V_t = \sup_{t<T\in\mathcal{T}}(T, Z_T)_t$. Indeed, $V_t = \max(Z_t, \sup_{t<T\in\mathcal{T}}(T, Z_T)_t)$. But, Corollary 5.5, $Z_t \le \sup_{t<T\in\mathcal{T}}$.

where the inequality followed because $T_n \geq s_n$. Hence,

$$V_t^s := \sup_{s \leq T \in \mathcal{T}} (T, Z_T)_t \leq \sup_n V_t^{s_n} = \lim_{n \to \infty} V_t^{s_n}.$$

(The last equality is again because $V_t^{s_n}$ is increasing in n.) □

Using the proposition, and the fact that (V_t^s) is decreasing in s, we now show $V_t \leq V_{t_+}$, completing the proof of the theorem.

Lemma 6.8. *We have $V_t \leq V_{t_+}$ a.s. all t.*

Proof. It is sufficient to show that $\forall \varepsilon > 0, \exists \delta > 0$ such that $t < s < t+\delta$ implies (*) $V_t \leq V_s + \varepsilon$. Let $\varepsilon > 0$. By the above proposition, there is a $u > t$ so that $0 \leq V_t - V_t^u < \varepsilon/2$. Next, by Corollary 6.5, (V_t^u) is right continuous at t, so there is a $0 < \delta < u - t$ such that $|V_t^u - V_s^u| < \varepsilon/2$ whenever $t < s < t + \delta$. Combining the two inequalities, it follows that $|V_t - V_s^u| < \varepsilon$ for $t < s < t + \delta$. There are two possibilities. If $V_s^u \leq V_t$, then we get $V_t \leq V_s^u + \varepsilon \leq V_s + \varepsilon$, and (*) holds. While if $V_s^u > V_t$, then (*) holds trivially, because then $V_s \geq V_s^u > V_t$. □

This concludes the proof of Theorem 6.1.[21] □

6.2. **The american option stream.** Let $Z. = (Z_t)$ be a right-continuous payoff process, and $(V_t) =: (V_t^{Z.})$ be its Snell envelope.

For each t, define the *Post-t-Optimal Exercise Time* associated to payoff process (Z_t) by

$$T_t =: T_t^{Z.} := \inf\{t \leq s \leq m : Z_s = V_s\}, \qquad t \in [0, m].$$

Note, $t \leq T_t \leq T_s$ for $t \leq s$. The adaptability and right continuity of (V_t) shown in Theorem 6.1 implies each T_t is a stopping time.

Proposition 6.9. (a) *Each T_t is a stopping time.*
(b) *a.s. $Z_{T_t} = V_{T_t}$ all t.*
(c) *The (non-adapted) process (T_t) is "regenerative": for $t \leq s$,*

$$1_{s \leq T_t} T_t = 1_{s \leq T_t} T_s, \qquad (\text{or, } 1_{s \leq T_t} = 1_{s \leq T_t = T_s}) \qquad t \leq s.$$

Proof. (a) Since T_t is a hitting time of the right-continuous adapted process $(V_t - Z_t)$, it is a stopping time.[22] (b) For any ω, there is a sequence $t_n \to T_t(\omega)$ such that $t \leq t_n$ and $Z_{t_n}(\omega) = V_{t_n}(\omega)$. Hence, $Z_{T_t(\omega)}(\omega) = V_{T_t(\omega)}(\omega)$ at all ω where $(Z_t - V_t)$ is right continuous for all t,

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[21] Lemma 6.6 above used Theorem 4.6 of [E]. We now outline another proof for right continuity of (V_t) that does not depend on that result. Lemma 6.8 actually shows $V_t \leq \lim_{s \searrow t} \inf(V_s)$. So, it suffices to show $V_t \geq \lim_{s \searrow t} \sup V_s$. But,

$$\lim_{s \searrow t} \sup(V_s) = \lim_{s \searrow t} \sup(\sup_n V_s^{s,n}) = \sup_n(\lim_{s \searrow t} \sup V_s^{s,n}) \leq \sup_n V_t^{t,n} = V_t.$$

The second equality above follows from interchanging the order of supremum. Indeed, chasing definition, it is easy to show that for any doubly indexed sequences $(a_ij)_{i,j=1}^{\infty}$ increasing in j, we have $\limsup_i(\sup_j \ a_{ij}) = \sup_j(\limsup_i \ a_{ij})$.

[22] Actually, according to Corollary 6.12 of [E], all we need to know is that $(V_t - Z_t)$ is progressively measurable to conclude T_t is a stopping time.

a set of measure 1. (c) Set $u = T_t(\omega)$. Note $u \le T_s(\omega)$ as $t \le s$. By (b) $Z_u(\omega) = V_u(\omega)$. So $T_s(\omega) \le u$ if $u \ge s$, implying thus $T_s(\omega) = u$. □

Counterexample to right continuity of process (T_t): Consider a sequence $t_n \searrow t$. Then sequence (T_{t_n}) is decreasing and above T_t, so it has a limit and $\lim_{n\to\infty} T_{t_n} \ge T_t$. If $(V_t - Z_t)$ is a semimartingale with infinite variation, we expect equality to hold, but otherwise not in general. For instance, take $\xi_t = 1$ all t, and let (Z_t) be deterministic and smooth. Then $V_t = \max_{s\in[t,m]} Z_s$. Now assume $Z_1 = 1 = Z_2$ and $Z_t < 1$ for $t \notin \{1, 2\}$. Then $T_t = 1$ for $t \le 1$, while $T_t = 2$ for $1 < t \le 2$.

The right-continuous payoff process (Z_t) is *American* if $V_t = (T_t, Z_{T_t})_t$ a.s. all t. This means that the supremum in the definition of the Snell envelop (V_t) is reached at T_t, implying $V_t = \max_{t\le T\in\mathcal{T}}(T, Z_T)_t$.

The "superclaim property" of Snell envelop is particularly easy for american payoffs: for $t \le s$, using price transitivity, then that $t \le T_s$,

$$(s, V_s)_t = (T_s, Z_{T_s})_t \le \sup_{t\le T\in\mathcal{T}} (T, Z_T)_t =: V_t.$$

Given an american payoff process (Z_t), for any s, the $Z.$-trigger option $\mathcal{A}^s := (T_s, Z_{T_s})$ is the *Post-s American Option* associated to (Z_t):

$$\mathcal{A}^s =: \mathcal{A}^{s,Z.} := (T_s, Z_{T_s}).$$

As (Z_t) is assumed american, this gives immediately $\mathcal{A}^t_t = V_t$ a.s. all t.

The following gives a pricing formula for $\mathcal{A}^s$ at *any* time t.

Theorem 6.10. *Let* (Z_t) *be an american payoff process. Then a.s.*

$$\mathcal{A}^s_t = \sup_{s\le T\in\mathcal{T}} (T, Z_T)_t =: V^s_t, \quad t \le s;$$

$$\mathcal{A}^s_t = 1_{t\le T_s} V_t, \quad s \le t.$$

So, for all t *and* s *we have, a.s.,*

$$\mathcal{A}^s_t = 1_{t\le s} V^s_t + 1_{s<t\le T_s} V_t.$$

Moreover, for $s \le u \le t$ *we have the following regenerative property:*

$$\mathcal{A}^s_t = 1_{t\le T_s}\mathcal{A}^t_t = 1_{t\le T_s}\mathcal{A}^u_t, \qquad s \le u \le t.$$

Proof. Since (Z_t) is american, $V_s = \mathcal{A}^s_s$ by definition. Hence, by price transitivity, $\mathcal{A}^s_t = (s, V_s)_t$ for $t \le s$. But, $(s, V_s)_t = V^s_t$ for $t \le s$ by Theorem 6.1. The first formula follows. As for second, assume $s \le t$. Then $1_{t\le T_s} = 1_{t\le T_s = T_t}$ by above proposition part (c). Hence,

$$\mathcal{A}^s_t = 1_{t\le T_s}\mathcal{A}^s_t = 1_{t\le T_s=T_t}\mathcal{A}^s_t = 1_{t\le T_s=T_t}(T_s, Z_{T_s})_t$$

$$= (T_s, 1_{t\le T_s=T_t} Z_{T_s})_t = (T_t, 1_{t\le T_s=T_t} Z_{T_t})_t$$

$$= 1_{t\le T_s=T_t}(T_t, Z_{T_t})_t = 1_{t\le T_s} V_t.$$

This gives the second formula. The third is now immediate. The fourth follows from twice application of the second:

$$1_{t\le T_s}\mathcal{A}^u_t = 1_{t\le T_s}1_{t\le T_u}V_t = 1_{t\le T_s}V_t = \mathcal{A}^s_t.$$

□

We next show a continuous payoff process (Z_t) is american.

But, first we give an example of a right-continuous (but not left continuous) payoff process that is not american.

Counterexample to (Z_t) *being american*: Let S be a numeraire, such that $P[S_t > 1] > 0$, all t, e.g., a lognormally distributed stock. Set $Z_t = 1_{t<1}(S_t - 1)^+$. Clearly, (Z_t) is a right-continuous payoff process (dominated by S). But, it is not an american payoff process if we assume positive interest rates, meaning $(s,1)_t \le 1$ for $t \le s$. Indeed then, by put-call parity, $Z_t < (s, Z_s)_t$ for any $t < s < 1$. Hence $Z_t < V_t$ for $t < 1$. On the other hand, $Z_1 = V_1 = 0$. Hence $T_t = 1$ for any $t < 1$. It follows that $Z_{T_t} = 0$. But, $V_t > 0$ for $t < 1$ because $P[S_t > 1] > 0$.

As in section 6.1, let $(U_n)_{n=1}^\infty$ be a finite exercise date approximation, and recall the definition of bermudan options $\mathcal{B}^{s,n} := (T^n_s, Z_{T^n_s})$, where $T^n_s := \inf\{s \le t \in U_n : Z_t = V^{t,n}_t\}$ with $V^{t,n}_t := \sup_{t\le T\in\mathcal{T}^n}(T, Z_T)_t$.

Lemma 6.11. *If* (Z_t) *is right continuous, then* $T^n_s \to T_s$ *a.s. all* s. [23]

Proof. We show (a) $T_s \ge \limsup_n T^n_s$ and (b) $T_s \le \liminf_n T^n_s$. It then follows sequence (T^n_s) has a limit, and that limit equals T_s. (a): Since $U = \bigcup_n U_n$ is dense, it suffices to show any $s \le t \in U$ which satisfies $t < \limsup T^n_s$ also satisfies $t < T_s$. If these conditions hold, then by definition of T^n_s, there exists an integer n_0 such that $Z_t < V^{t,n_0}_t$. As $V^{t,n}_t$ is increasing in n, it follows $Z_t < V^{t,n}_t$ for all $n \ge n_0$. Hence, by Corollary 6.3, $Z_t < \lim V^{t,n}_t = V_t$. Thus $t < T_s$ by definition of T_s. (b): Set $Y_t = V_t - Z_t$. Note $Y_{T^n_s} = 0$ for all s and n. Now let $\omega \in \Omega$. Set $u = u_\omega = \inf_n T^n_s(\omega)$. There is a subsequence $(T^{n_i}_s(\omega))_{i=1}^\infty$ such that $T^{n_i}_s(\omega)) \searrow u$. Since (Y_t) is right-continuous, it follows $Y_u(\omega) = 0$. Since $u \ge s$, this implies $u \ge T_s(\omega)$ by definition of T_s. □

Theorem 6.12. *Let* (Z_t) *be a continuous payoff process which is dominated by a continuous numeraire. Then* (Z_t) *is american.*

Proof. Using the lemma and the continuous version of trigger option convergence (Prop. 4.8), we get a.s. $\mathcal{B}^{s,n}_t \to (T_s, Z_{T_s})_t$ all t. In particular, $\mathcal{B}^{t,n}_t \to (T_t, Z_{T_t})_t$ a.s. But, as in section 6.1, $\mathcal{B}^{t,n}_t = V^{t,n}_t$, and Corollary 6.3 gives $\mathcal{B}^{t,n}_t \to V_t$. Hence, $(T_t, Z_{T_t})_t = V_t$ a.s., as desired. □

Remark: Assume (Z_t) is american. It would be desirable that the (non-adapted) process (T_s) be right continuous. For then trigger option

[23] It is not the case that T^n_s is monotone in n.

convergence would imply $\mathcal{A}^s_t$ is right continuous in s. Using this and that $\mathcal{A}^s_t$ is decreasing in s, an argument similar to Corollary 6.8 implies $(\mathcal{A}^t_t)$ is right-continuous. Now, by the definition of the american option, $(\mathcal{A}^t_t)$ is a modification of (V_t). As both are now right-continuous, it would follow $(\mathcal{A}^t_t)$ and (V_t) are indistinguishable.

Of special interest is $\mathcal{A} :=: \mathcal{A}^0 :=: \mathcal{A}^{Z.}$, the *American Option* associated to (Z_t). As the above analysis suggests, its properties are intertwined with those of the other american options $\mathcal{A}^s$, $s > 0$.

This tentatively prompts for a new notion, a generalization of a stochastic process, where an *Option Stream* is defined as a curve $\mathcal{O}^\cdot : [0, m] \longrightarrow \mathbf{O}$ of options, $s \mapsto \mathcal{O}^s = (T_s, O^s)$, such that the stopping time process (T_s) is increasing. A *Right-Continuous Stream* may then be defined as a stream $\mathcal{O}^\cdot$ such that the processes (T_s) and (O^s) are right continuous (but in general not necessarily adapted, although $O^s \in \mathcal{C}_{T_s}$).

Right-continuous option streams are beyond the scope of this paper. But, Part III is devoted to finitely supported option streams.

7. Multiplicative minimax duality

The minimax-duality formulae of this section are valid for any right-continuous payoff processes, not just american ones. But, first we need a consequence of the (additive) Doob-Meyer decomposition theorem.

7.1. **Superclaims and Domineering numeraires.** A *Superclaim* $(V_t) = (V_t)_{t=0}^m$ is a right-continuous payoff process such that $V_t \geq (s, V_s)_t$ a.s. for all $t \leq s$. Clearly, (price process of) any claim is a superclaim.

Proposition 7.1. *The Snell envelop of a right continuous payoff process is a superclaim, as evident in the statement of Theorem 6.1.*

An obvious reformulation in terms of numeraires justifies the jargon.

Proposition 7.2. *A right-continuous payoff process (V_t) is a superclaim if and only if process (V_t/β_t) is a right-continuous $\mathbb{P}^\beta$ supermartingale for some (hence all) numeraire β.*

Proof. Let $t \leq s$. For any numeraire β, we have $(s, V_s)_t = \beta_t \mathbb{E}^\beta[V_s/\beta_s \,|\, \mathcal{F}_t]$. Therefore $(s, V_s)_t \leq V_t$ if and only if $\mathbb{E}^\beta[V_s/\beta_s \,|\, \mathcal{F}_t] \leq V_t/\beta_t$. □

In particular, almost all sample paths of a superclaim have left limits.

Below is a formulation of the additive Doob-Meyer decomposition for superclaims. Like a later multiplicative version, it involves a claim and a decreasing process, but references an additional numeraire.

Corollary 7.3. (*Additive Doob-Meyer Decomposition*) *Let (V_t) be a superclaim and β be a numeraire. Then there exist a unique claim C and decreasing predictable process (A_t) with $A_0 = 0$ such that a.s. $V_t = \beta_t A_t + C_t$ all t. Moreover, $C_0 = V_0$, a.s. $V_t \leq C_t$ all t, and if $V_m > 0$ a.s., then C is a numeraire dominating (V_t).*

Proof. By above proposition (V_t/β_t) is a $\mathbb{P}^\beta$ supermartingale, and it is $\mathbb{P}^\beta$-class D by Proposition 4.5. Hence, it has a Doob-Meyer decomposition, i.e., there exists a unique decomposition $V_t/\beta_t = A_t + M_t$, where (M_t) is a $\mathbb{P}^\beta$-martingale and (A_t) is a decreasing predictable process with $A_0 = 0$. The claim $C := \beta M_m$ has the requisite properties. □

A *Supernumeraire* (V_t) is a superclaim such that $V_m > 0$ a.s.

Then $V_t \geq (m, V_m)_t > 0$ a.s. all t. In fact a well-known property of right-continuous nonnegative supermartingales (c.f., [E] Theorems 4.3 and 4.16) coupled with Proposition 7.2 immediately implies

Corollary 7.4. *Almost all sample paths of a supernumeraire* (V_t) *are bounded below strictly above zero. So, a.s.* $V_t > 0$ *and* $V_{t\_} > 0$*, all* t*.*[24]

Minimax duality hinges on the following notion. A numeraire B *Domineers* a superclaim (V_t) if B dominates (V_t) and $B_0 = V_0$.

The additive Doob-Meyer decomposition supplies plenty of domineering numeraires, in fact one for each numeraire. Namely, let β be a numeraire and B be the unique numeraire with $B_0 = V_0$ and $((V_t - B_t)/\beta_t)$ decreasing and predictable. Then clearly B domineers (V_t), as also stated in the last part of above Corollary 7.3.

But the above notion suffices only for the minimax duality formula at time 0. More generally, given any time t, say a numeraire B *Domineers* a superclaim $V.$ *at* t, if $V_t = B_t$ a.s. and $V_s \leq B_s$ a.s. for all $s > t$.

Again there are plenty of them:

Proposition 7.5. *Let* (V_t) *be supernumeraire and* β *be a numeraire. Let* B *be the numeraire in the Doob-Meyer decomposition* $V_t = A_t\beta_t + B_t$ *of* (V_t) *given by Corollary 7.3. For any time* t*, set* $B^t = B + A_t\beta$*. Then,* B^t *is a numeraire that domineers the supernumeraire* (V_t) *at* t*.*

Proof. Clearly for $s \geq t$ (though not for $s < t$), we have $B^t_s = B_s + A_t\beta_s$. Thus, $B^t_t = B_t + A_t\beta_t = V_t$, and $B^t_s \geq B_s + A_s\beta_s = V_s > 0$ for $s > t$. □

7.2. **Positive minimax duality.** Throughout this subsection, let (Z_t) be a *positive* right-continuous payoff process, that is $Z_t > 0$ a.s. all t.

As before, $(V_t) =: (V_t^{Z.})$ denotes the Snell envelope associated to payoff process (Z_t). It is *supernumeraire* because $Z_t > 0$.[25]

The main step leading to the multiplicative dual formula is

Proposition 7.6. *Let* B *be a numeraire that domineers* $V.$ *at* t*. Then*

$$\sup_{s \geq t}\left(\frac{Z_s}{B_s}\right) = 1 \text{ a.s.}$$

[24]In particular, this hold for (price process of) a numeraire, furnishing in this case a stronger positivity statement than that of Corollary 5.4.

[25]It is worth recalling that numeraire B dominates (V_t) if (and only if) it dominates (Z_t) (because if it dominates (Z_t) it dominates all trigger options (T, Z_T), hence their supremum (V_t)).

Proof. Since B domineers $V_\cdot$ at t and $Z_s \leq V_s$.

$$\sup_{s\geq t}(\frac{Z_s}{B_s}) \leq \sup_{s\geq t}(\frac{Z_s}{V_s}) \leq 1.$$

To prove equality, it suffices to show $\mathbb{E}^B[\sup_{s\geq t}(Z_s/B_s)\,|\,\mathcal{F}_t] \geq 1$. Clearly $\mathbb{E}^\beta[Z_T/B_T\,|\,\mathcal{F}_t] \leq \mathbb{E}^\beta[\sup_{s\geq t}(Z_s/\beta_t)\,|\,\mathcal{F}_t]$ for any stopping time $T \geq t$. Thus by definition of option price with B chosen as numeraire,

$$B_t = V_t = B_t \sup_{t\leq T\in\mathcal{T}} \mathbb{E}^B[\frac{Z_T}{B_T}\,|\,\mathcal{F}_t] \leq B_t\mathbb{E}^B[\sup_{s\geq t}(\frac{Z_s}{B_s})\,|\,\mathcal{F}_t].$$

Hence, $1 \leq \mathbb{E}^B[\sup_{s\geq t}(Z_s/B_s)\,|\,\mathcal{F}_t]$, as desired. □

For reference in the next subsections, we will refer to the two formulae below as the *first and second minimax duality formula*.

Corollary 7.7. *(Multiplicative Minimax Duality) We have a.s. all t,*

$$V_t = \inf_{\beta\in\mathcal{C}^+} \beta_t\mathbb{E}^\beta[\sup_{s\geq t}(\frac{Z_s}{\beta_s})\,|\,\mathcal{F}_t].^{26}$$

Moreover, the infimum is attained at any numeraire B that domineers $V_\cdot$ at t. Since such numeraires exist for any t, we may also write a.s.

$$V_t = \min_{\beta\in\mathcal{C}^+} \beta_t\mathbb{E}^\beta[\sup_{s\geq t}(\frac{Z_s}{\beta_s})\,|\,\mathcal{F}_t].$$

Proof. Clearly $\mathbb{E}^\beta[Z_T/\beta_T\,|\,\mathcal{F}_t] \leq \mathbb{E}^\beta[\sup_{s\geq t}(Z_s/\beta_t)\,|\,\mathcal{F}_t]$ for any stopping time $T \geq t$. Hence, $V_t \leq \beta_t\mathbb{E}^\beta[\sup_{s\geq t}(Z_s/\beta_t)\,|\,\mathcal{F}_t]$. Next, let B be any numeraire that domineers $V_\cdot$ at t. The above proposition gives $\sup_{s\geq t}(Z_s/B_s) = 1$ for numeraire B. So, $B_t = B_t\mathbb{E}^\beta[\sup_{s\geq t}(Z_s/B_s)\,|\,\mathcal{F}_t]$. But, $B_t = V_t$ because B is domineering at t. Hence the infimum is actually attained by this numeraire B, implying equality. □

Comparison with Additive Minimax Duality. [R], and in the bermudan case [H-K], formulate the original additive version under a fixed numeraire in terms of martingales rather than claims. In our context, a formulation, noted similarly in [A-B], is as follows. Let (Z_t) be a right-continuous payoff process (not necessarily positive) and β be a numeraire. Then, $V_0 = \inf_{C\in\mathcal{C}}(C_0 + \beta_0\mathbb{E}^\beta[\sup_t((Z_t - C_t)/\beta_t)])$, with the infimum attained at the claim C in Corollary 7.3.[27]

[26] For any t and numeraire β, $X^* := \sup_{s\geq t}(Z_s/\beta_s)$ is measurable. Indeed, by right continuity, $X^* = \sup_{s\geq t\in\mathbb{Q}}(Z_s/\beta_s)$, where $\mathbb{Q}$ is the set of rationals. And, whenever, as here or in the sequel, (Z_t) is nonnegative, $\mathbb{E}^\beta[X^*\,|\,\mathcal{F}_t]$ is well-defined and satisfies $0 \leq \mathbb{E}^\beta[X^*\,|\,\mathcal{F}_t] \leq \infty$. Of course, $X^*\beta$ is a claim if and only if $\mathbb{E}^\beta[X^*] < \infty$, in which case then also $\mathbb{E}^\beta[X^*\,|\,\mathcal{F}_t] = (X^*\beta)_t/\beta_t < \infty$. Whether or not $\mathbb{E}^\beta[X^*\,|\,\mathcal{F}_t] < \infty$, the formula remains valid. (The *infumum* over β of these expectation is taken, and numeraires inducing infinite exepectation do not contribute to infumum.) Note further, by Doob's L^p inequality, $X^*\beta$ is a claim if there exist a $p > 1$ and a numeraire B dominating (Z_t) such that $\sup_s \mathbb{E}^\beta[(B_s/\beta_s)^p] < \infty$.

[27] This follows by an argument similar to that of Corollary 7.7.

By contrast, our result (applicable only to nonnegative payoffs) states rather more simply, $V_0 = \inf_{B\in\mathcal{C}^+} B_0\mathbb{E}^B[\sup_t(Z_t/B_t)]$. While closely related, these two statements are evidently different. The main difference is that the additive version relies on a reference numeraire β, while the multiplicative version does not. In this sense, the additive version is not as numeraire-invariant as the multiplicative one. In the additive version, selection of a good approximating claim C depends not just on the payoff process (Z_t) but also on the choice of reference numeraire β.

7.3. **Preliminary nonnegative minimax duality.** Here we extend the first minimax duality formula to nonnegative payoff processes as preparation for next the subsection.[28]

Proposition 7.8. *Let C be a claim. If (Z_t) is a right-continuous payoff process, then so is $(Z_t + C_t)$, and for all t*

$$T_t^{Z.+C.} = T_t^{Z.}; \quad V_t^{Z.+C.} = V_t^{Z.} + C_t.$$

Moreover, if (Z_t) is american, then so is $(Z_t + C_t)$, and for all s, t,

$$\mathcal{A}_t^{s,Z.+C.} = \mathcal{A}_t^{s,Z.} + C_t.$$

Proof. Using price linearity and price transitivity

$$V_t^{Z.+C.} := \sup_{t\le T\in\mathcal{T}}(T, Z_T + C_T)_t = \sup_{t\le T\in\mathcal{T}}((T, Z_T)_t + (T, C_T)_t)$$

$$= \sup_{t\le T\in\mathcal{T}}((T, Z_T)_t + C_t) = \sup_{t\le T\in\mathcal{T}}((T, Z_T)_t) + C_t = V_t^{Z.} + C_t.$$

The remaining statements are immediate from this.[29] □

Using this, the *first* minimax duality formula in Corollary 7.7 for positive options easily generalizes to nonnegative options.

Corollary 7.9. *Let (Z_t) be a nonnegative right-continuous payoff process, i.e., $Z_t \ge 0$ a.s. all t. We then have a.s. all t,*

$$V_t^{Z.} = \inf_{\beta\in\mathcal{C}^+} \beta_t\mathbb{E}^\beta[\sup_{s\ge t}(\frac{Z_s}{\beta_s}) \mid \mathcal{F}_t].$$

Proof. Let $\varepsilon > 0$ and B be any numeraire. Then,

$$\inf_{\beta\in\mathcal{C}^+} \beta_t\mathbb{E}^\beta[\sup_{s\ge t}(\frac{Z_t}{\beta_s}) \mid \mathcal{F}_t] \le \inf_{\beta\in\mathcal{C}^+} \beta_t\mathbb{E}^\beta[\sup_{s\ge t}(\frac{Z_s + \varepsilon B_t}{\beta_s}) \mid \mathcal{F}_t]$$

$$= V_t^{Z.+\varepsilon B.} \qquad \text{(by Corollary 7.7)}$$

$$= V_t^{Z_t} + \varepsilon B_t \qquad \text{(by above proposition).}$$

[28]Curiously, when $Z_t \ge 0$ a.s. all t, we have $V_t = \sup_{T\in\mathcal{T}}(T, Z_T)_t$. Indeed, then $1_{t\le T}Z_T \le (T\vee t, Z_{T\vee t})_T$. Hence by price transitivity, $(T, Z_T)_t = (T, 1_{t\le T}Z_T)_t \le (T\vee t, Z_{T\vee t})_t$. So $\sup_{T\in\mathcal{T}}(T, Z_T)_t \le V_t$, as $t \le (T\vee t) \in \mathcal{T}$.

[29]Similarly if (Z'_t) is another right-continuous payoff process, we clearly have $V_t^{Z.+Z'_.} \le V_t^{Z.} + V_t^{Z'_.}$.

Since $\varepsilon > 0$ was arbitrary, $\inf_{\beta\in\mathcal{C}^+} \beta_t \mathbb{E}^\beta[\sup_{s\geq t}(\frac{Z_s}{\beta_s}) \,|\, \mathcal{F}_t] \leq V_t$. The reverse inequality is clear as before by the definition of $(V_t^{Z\cdot})$. □

7.4. **Semipositive minimax duality.** Let $\mathcal{C}_+$ denote the set of all semipositive claims. So, $\mathcal{C}^+ \subset \mathcal{C}_+$.

We first show that in the first minimax duality formula for nonnegative options above, we can replace the infimum over $\mathcal{C}^+$ by infimum over this larger set $\mathcal{C}_+$. Then we present our main minimax duality result, that the infimum is achieved if the m-expiry european option (m, Z_m) is semipositive. As any nonnegative option is indistinguishable from a semipositive one, this is not a stringent restriction at all.

Let $\mathcal{O} = (T, O)$ be an option and $B \in \mathcal{C}_+$ be a semipositive claim. Define the rollover claim $C^{\mathcal{O},B}$ by [30]

$$C^{\mathcal{O},B} := O(1_{T=m} + 1_{T<m}\frac{B}{B_T}), \qquad B \in \mathcal{C}_+.$$

Note, the ratio is well-defined, as, B being a semipositive claim, $B_T > 0$ a.s. on $\{T < m\}$. Note also, the definition agrees with the earlier one when B is a numeraire. It is easy to see that $C^{\mathcal{O},B}$ is a claim.

Proposition 7.10. *Let $\mathcal{O}$ be an option and B and B' be two semipositive claims. Then, a.s. all t,*

$$\lim_{\varepsilon\searrow 0} C_t^{\mathcal{O},B+\varepsilon B'} = C_t^{\mathcal{O},B}.$$

Proof. A similar argument as in Proposition 4.3 gives

$$|C^{\mathcal{O},B+\varepsilon B'}| \leq C^{|\mathcal{O}|,B} + C^{|\mathcal{O}|,\varepsilon B'} = C^{|\mathcal{O}|,B} + C^{|\mathcal{O}|,B'}.$$

The desired result now follows by dominated option convergence. □

Clearly, a nonnegative random variable F is *not* a claim if and only if $\mathbb{E}^\beta[F/\beta] = \infty$ for some hence all numeraire β. As such, for any nonnegative random variable $F \geq 0$, define $F_0 = \beta_0\mathbb{E}^\beta[F/\beta]$. If F is a claim, then this is just its time-0 price as before. If not, then $F_0 = \infty$.

Now, let (Z_t) be a nonnegative payoff process and B be a semipositive claim. For each s, $C^{(s,Z_s),B}$ is a nonnegative claim. Set

$$C^{*B} := \sup_{0\leq s\leq m} C^{(s,Z_s),B} = \max(Z_m, \sup_{0\leq s<m} C^{(s,Z_s),B}).$$

If B happens to dominate (Z_t), then clearly C^{*B} is a claim dominated by B. If C^{*B} is not a claim, then $C_0^{*B} = \infty$ because $C^{*B} \geq 0$.[31]

In the present notation, Corollary 7.9 for $t = 0$ states $V_0 = \inf_{\beta\in\mathcal{C}^+} C_0^{*\beta}$. It can be improved by replacing $\mathcal{C}^+$ with $\mathcal{C}_+$:

[30]More generally in Part III, will define the "rollover option" $\mathcal{O} \Rsh \mathcal{O}'$ for any option $\mathcal{O}$ and any nonnegative option $\mathcal{O}'$, and study its associativity.

[31]See also footnote 26.

Proposition 7.11. *Let (Z_t) be a nonnegative right-continuous payoff process. Then*

$$V_0 = \inf_{B \in \mathcal{C}_+} C_0^{*B}.$$

Proof. By Corollary 7.9 $V_0 = \inf_{\beta \in \mathcal{C}^+} C_0^{*\beta}$. Since $\mathcal{C}^+ \subset \mathcal{C}_+$, it follows $V_0 \geq \inf_{B \in \mathcal{C}_+} C_0^{*B}$. As for the reverse equality, let $B \in \mathcal{C}_+$. By Proposition 7.10, $C_0^{*B} = \lim_{\varepsilon \searrow 0} C_0^{*B+\varepsilon} \geq \inf_{\beta \in \mathcal{C}^+} C_0^{*\beta}$, the inequality following since $B + \varepsilon \in \mathcal{C}^+$ for $\varepsilon > 0$. The reverse inequality follows. □

We now show the infimum is attained if the claim Z_m is semipositive.

Theorem 7.12. *Let (Z_t) be a nonnegative right-continuous payoff process. If the european option (m, Z_m) is semipositive, then*

$$V_0 = \min_{B \in \mathcal{C}_+} C_0^{*B}.$$

Proof. Since claim Z_m is semipositive, a.s. $V_t \geq (m, Z_m)_t > 0$ for all $t < m$. Let β be a numeraire and $V_t = D_t\beta_t + B_t$ be the additive Doob Meyer decomposition of superclaim (V_t) given in Corollary 7.3. We have $B_t \geq V_t > 0$ a.s. for $t < m$. Hence, (B_t) is semipositive. The same argument as in Theorem 7.6 yields $\sup_{s<m}(Z_s/B_s) = 1$. Now, let β a numeraire. Then, by this and the definition of time-0 price,

$$\begin{aligned} C_0^{*B} &= \beta_0 \mathbb{E}^\beta[\frac{1}{\beta} \max(Z_m, \sup_{0 \leq s < m} C^{(s,Z_s),B})] \\ &= \beta_0 \mathbb{E}^\beta[\frac{1}{\beta} \max(Z_m, \sup_{0 \leq s < m} (\frac{Z_s}{B_s} B)] \\ &= \beta_0 \mathbb{E}^\beta[\frac{1}{\beta} \max(Z_m, B)] = \beta_0 \mathbb{E}^\beta[\frac{B}{\beta}] = B_0 = V_0. \end{aligned}$$

(For the third equality we used $Z_m \leq V_m \leq B$). □

A similar statement is readily formulated for V_t at any time $t > 0$, as done previously, or even by a time translation argument.

8. Multiplicative Doob-Meyer decomposition of supernumeraires

8.1. **Discussion.** In addition to the additive Doob-Meyer decomposition, supernumeraries satisfy a unique local multiplicative decomposition as product of a decreasing process and a "local numeraire" defined below. When this local numeraire is actually a numeraire, it can for instance be used as a domineering numeraire for minimax duality.

For the finite nonnegative option streams in Part III, this numeraire has a simple and interesting financial interpretation as a self-financing trading strategy which we call "stream rollover." There, its construction is explicit, resulting in properties which could possibly carry over to right-continuous option streams by trigger option convergence arguments similar to those we employed for the american stream.

The multiplicative Doob-Meyer decomposition is discussed much less than the additive version, and sources are scare. A 1965 paper of Ito is hard to find. Its application to financial mathematics seems growing though. One known application, repeated below in the present context, is extraction of the instantaneous interest rates and risk-neutral measure from the state price density. Another example is its use by [D-Y] in analysis of the passport option, a complex american-style option.

8.2. **Local claims and the multiplicative decomposition.** A process $C. = (C_t)_{t=0}^m$ is a *Local Claim* if it is adapted, right continuous with left limits, and there exists a increasing sequence $(T_n)_{n=1}^{\infty}$ of stopping times converging to m such that $1_{t \le T_n} C_t = (T_n, C_{T_n})_t$ a.s all t.

The terminology is justified by choosing a reference numeraire.

Proposition 8.1. *An adapted, right continuous process with left limits* (C_t) *is a local claim if and only if process* (C_t/β_t) *is a* $\mathbb{P}^\beta$*-local martingale for some (hence all) numeraire* β.

Proof. Suppose (C_t) is a local claim and set $\mathcal{O}^n = (T_n, C_{T_n})$, as in the definition. The condition $1_{t \le T_n} C_t = \mathcal{O}^n_t$ is equivalent to $C_{t \wedge T_n} = \mathcal{O}_{t \wedge T_n}$. But, since $\mathcal{O}$ is an option, process $(\mathcal{O}_{t \wedge T_n}/\beta_{t \wedge T_n})$ is a $\mathbb{P}^\beta$-martingale, implying (C_t/β_t) is a $\mathbb{P}^\beta$-local martingale. The converse is similar. □

It follows easily from price transitivity that (the price process of) a claim is a local claim. Conversely,

Proposition 8.2. *A local claim is a claim if it is dominated.*[32]

Proof. This is the counterpart of the statement that a uniformly integrable local martingale is a martingale. □

A *Local Numeraire* is an a.s. positive local claim process.[33]

Theorem 8.3. *(Local Multiplicative Doob-Meyer Decomposition) Let* (V_t) *be a supernumeraire. Then there exists (up to indistinguishability) a unique decomposition* $V_t = D_t B_t$*, where* (D_t) *is a decreasing predictable process with* $D_0 = 1$ *and* (B_t) *is a local numeraire.*[34]

[32]Put another way, a local claim is a claim if (and only if) it is a payoff process.

[33]As positive local martingales are supermartingales, a local numeraire (B_t) satisfies (a) $B_t \in \mathcal{C}_t$ all t, and (b) $B_t \ge (s, B_s)_t$ a.s. all $t \le s$. (Superclaim property requires (b) plus domination). If a local numeraire (B_t) is the (price process of) a numeraire then clearly a.s. $B_t = (m, B_m)_t$ all t. Conversely, if a local numeraire (B_t) satisfies $B_0 \le (m, B_m)_0$ then it is a numeraire. Indeed, we must show that $\mathcal{O}^t := (t, B_t - (m, B_m)_t)$ is a zero option for each t . As each option $\mathcal{O}^t$ is nonnegative by (b), it suffices to show $\mathcal{O}^t_0 = 0$. But, using (b) twice and price transitivity once, $(t, B_t)_0 \le B_0 \le (m, B_m)_0 = (t, (m, B_m)_t)_0 \le (t, B_t)_0$. Hence $\mathcal{O}^t_0 = 0$.

[34]Actually, the conclusion holds for any positive right-continuous, adapted, process (V_t) satisfying $(s, V_s) \le V_t$ a.s. all $t \le s$. This is weaker than being supernumeraire as it does not require (V_t) be dominated. The proof shows this, for it only relies on $X_t := V_t/\beta_t$ to have a (unique) *local* Doob-Meyer decomposition

Proof. Let β be a numeraire, and set $X_t := V_t/\beta_t$. Note, a.s., $X_t > 0$, $X_{t_-} > 0$, all t. As (X_t) is a $\mathbb{P}^\beta$-class D supermartingale, there exists a $\mathbb{P}^\beta$-martingale M_t and a decreasing predictable process A_t, $A_0 = 0$, such that $X_t = A_t + M_t$. Letting $\mathcal{E}_t(Y_\cdot)$ denote the stochastic exponential of any semimartingale (Y_t), we have

$$X_t = \mathcal{E}_t(\int_0^\cdot \frac{dX_s}{X_{s_-}}) = \mathcal{E}_t(\int_0^\cdot \frac{dA_s}{X_{s_-}})\mathcal{E}_t(\int_0^\cdot \frac{dM_s}{X_{s_-}}),$$

where, for the second equality we used $[\int_0^\cdot dA_s/X_{s_-}, \int_0^\cdot dM_s/X_{s_-}] = 0$, because $\int_0^\cdot dA_s/X_{s_-}$ is decreasing and predictable and $\int_0^\cdot dM_s/X_{s_-}$ is a $\mathbb{P}^\beta$ local martingale.[35] Stochastic exponentiation preserve both these properties, so, the desired decomposition is given by $D_t = \mathcal{E}_t(\int_0^\cdot dA_s/X_{s_-})$ and $B_t = \beta_t\mathcal{E}_t(\int_0^\cdot dM_s/X_{s_-})$. Uniqueness is similar.[36] □

We refer to the local numeraire (B_t) given by the theorem as the *Rollover Local Numeraire* associated to supernumeraire (V_t).

The following relates our approach to a more traditional formulation.

Corollary 8.4. *Assume that the discount factors satisfy* $(s, 1)_t \leq 1$*, all* $t \leq s$ *(which implies they are decreasing in maturity* s*), and that there exists a numeraire* B *such that a.s.* $B_t \geq 1$ *all* t*. Then there exists a unique increasing, predictable, local numeraire valued* 1 *at time* 0.[37]

Proof. The assumption is equivalent to saying that identically one process is a superclaim (which in turn is equivalent to state price density (ξ_t) being a $\mathbb{P}$-supermartingale of class D). The desired result now follows by applying the theorem to the identically one process. □

The increasing, predictable, local numeraire alluded to above is often called the *continuous money market numeraire*. It is the rollover local numeraire associated to the identically one process. When this local numeraire is an actual numeraire, its associated numeraire measure is often called the *risk-neutral measure*. So, the risk-neutral measure exists if there exists a numeraire whose price is increasing and predictable. Further assumption of absolute continuity of this increasing numeraire yields the *instantaneous interest rate* by logarithmic differentiation.

$A_t + M_t$ (with M_t a $\mathbb{P}^\beta$-*local* martingale), which decomposition is guaranteed for any right-continuous supermartingale (class D or not) by Theorem 8.22 in [E].

[35]Generally, $\mathcal{E}(A + B + [A, B]) = \mathcal{E}(A)\mathcal{E}(B)$ (c.f. Corollary 13.8 in [E].) . So, $\mathcal{E}(A + B) = \mathcal{E}(A)\mathcal{E}(B)$ when $[A, B] = 0$.

[36]Indeed say $X_t = D_tN_t$ with (D_t) a positive, decreasing, predictable process valued 1 at 0, and (N_t) a positive $\mathbb{P}^\beta$-local martingale. Both processes are semimartingales with positive left limits. Taking stochastic logarithm, we get a decomposition of stochastic logarithm of (X_t) as sum of a local martingale and a decreasing, predictable process, valued 0 at 0. But such a decomposition is unique.

[37]Actually, because of footnote 34, the conclusion holds without assuming the existence of such a numeraire $B_t \geq 1$, or equivalently without requiring ($\mathbb{P}$-supermartingale) (ξ_t) be class D.

8.3. **Multiplicative supernumeraires.** If D_t is a positive decreasing process and B is a numeraire, then the product process $(D_t B_t)$ is obviously a supernumeraire. We call such a supernumeraire multiplicative. Note, if $D_0 = 1$, then numeraire B domineers supernumeraire $(D_t B_t)$.

Formally, a supernumeraire (V_t) is *Multiplicative* if its rollover local numeraire (B_t) is actually (the price process of) a numeraire, i.e., if a.s. $B_t = (B_m)_t$, all t. ($(B_m)_t$:=: $(m, B_m)_t$.) We then call the numeraire B_m the *Rollover Numeraire* associated to supernumeraire (V_t).

By definition, the rollover local numeraire (B_t) of a multiplicative supernumeraire is the price process of its rollover numeraire B_m.

Multiplicativity is easily characterized by anyone of several relationships between the supernumeraire (V_t) and the numeraire B_m:

Proposition 8.5. *Let (V_t) be a supernumeraire and (B_t) be its rollover local numeraire. Then the following conditions are equivalent.*
(a) (V_t) is multiplicative; (b) Numeraire B_m dominates process (V_t); (c) $V_0 \leq (B_m)_0$; (d) $V_0 = (B_m)_0$.[38]

Proof. It is obvious that (a) implies (b) and (d), (b) implies (c), and (d) implies (c). Also, (c) implies implies (a) by footnote 33. Indeed, any positive local numeraire (B_t) satisfies a.s. $(m, B_m)_t \leq B_t$ all t, and moreover, because of this, it is (the price process of a) numeraire if $B_0 \leq (m, B_m)_0$. Applied to the rollover local numeraire $B_t = \beta_t \mathcal{E}_t(\int_0^{\cdot} dM_s/X_{s_-})$, the result follows, as $B_0 = V_0$. □

I am indebted to Freddy Delbaen for the following example of a continuous supernumeraire which is not multiplicative.

Counterexample to Multiplicativity . Let β be a numeraire. Let (M_t) be a positive $\mathbb{P}^\beta$ local martingale (hence also a $\mathbb{P}^\beta$-supermartingale) that is *not* a martingale (e.g., $M_t = 1/|x_t|$, where (x_t) is a three-dimensional $\mathbb{P}^\beta$ Brownian motion, $x_0 \neq 0$). Set $D_t = \exp(\sup_{s\in[0,t]}(M_0 - M_s))$. Then, (D_t) is decreasing, so $(D_t M_t)$ is a $\mathbb{P}^\beta$-supermartingale as (M_t) is so. Moreover, $D_t M_t \leq 1$. Set $V_t := D_t M_t \beta_t$. Then (V_t) is a supernumeraire (dominated by β) that is not multiplicative.[39]

The following provides a sufficient Novikov-type integrability condition in the continuous case for a supernumeraire to be multiplicative.

Proposition 8.6. *Let (V_t) be a continuous supernumeraire. Assume there exists a continuous numeraire β such that $\mathbb{E}^\beta[\exp(\int_0^m \frac{d[X_t,X_t]}{2X_t^2})] < \infty$, where $X_t = V_t/\beta_t$. Then (V_t) is multiplicative.*

[38] Note, by Theorem 8.3, (c) is equivalent to $V_0 \leq \beta_0 \mathbb{E}^\beta[\mathcal{E}_m(\int_0^{\cdot} dM_s/X_{s_-})]$ for some, hence all, numeraire β, where $X_t = V_t/\beta_t$, and (M_t) is the $\mathbb{P}^\beta$- local martingale in the additive Doob-Meyer decomposition of (X_t) with $M_0 = X_0$.

[39] For otherwise $V_t = D'_t B_t$ for some predictable decreasing process D'_t and numeraire B, which would imply that $D_t M_t = D'_t M'_t$, where $M'_t = B_t/\beta_t$. But, (M'_t) is $\mathbb{P}^\beta$-martingale, and uniqueness of the (local) decomposition of this kind implies implies $M_t = M'_t$, contradicting the assumption that (M_t) is not a $\mathbb{P}^\beta$-martingale.

Proof. Set $N_t = \int_0^t dM_s/X_s$, where (M_t) is the $\mathbb{P}^\beta$-martingale in the additive Doob-Meyer decomposition of the $\mathbb{P}^\beta$ supermartingale (X_t). By Theorem 8.3, we must show $(\mathcal{E}_t(N.))$ is a $\mathbb{P}^\beta$-martingale. As (X_t) is continuous by assumption, (M_t) and hence also (N_t) are continuous. Thus, by Novikov Theorem (c.f. [E], Theorem 13.27) $(\mathcal{E}_t(N.))$ will be a $\mathbb{P}^\beta$-martingale if $\mathbb{E}^\beta[\exp([N., N.]_m/2)] < \infty$. But, this holds by assumption, because $[N., N.]_m = \int_0^m d[M., M.]_t/X_t^2] = \int_0^m d[X., X.]_t/X_t^2]$. □

Unfortunately, the above integrability condition is *not* numeraire invariant - if it holds for one numeraire, it need not hold for another.

Part III: Bermudan and regenerative trigger stream rollover

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