Dynamic Correlation Hedging in Copula Models for Portfolio Selection

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Abstract

In this paper we address the problem of solving for optimal portfolio allocation in a dynamic setting, where conditional correlation is modeled using observable factors, which allows us to isolate the demand for hedging correlation risk. We are able to analyse separately the impact of tail dependence through the unconditional distribution and that of conditional correlation on portfolio holdings. With those distinct ways of modeling dependence we aim at replicating the stylised fact of increased dependence during extreme market downturns, rising market-wide volatility, or worsening macroeconomic conditions. We find that both correlation hedging demands and intertemporal hedges due to increased tail dependence have distinct portfolio implications and cannot act as substitutes to each other. As well, there are substantial economic costs for disregarding both the dynamics of conditional correlation and the dependence in the extremes.

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1 Introduction

An increasing body of literature is interested in modeling time variations in the conditional dependence of asset returns in terms of conditional covariances and correlations (Bollerslev et al. (1988) or Engle (2002) to cite a few). From a modeling perspective, popular choices for the time-varying correlation phenomenon are the Dynamic Conditional Correlation model of Engle (2002) in a discrete-time setting, or the continuous-time Wischart process, introduced by Bru (1991) that gives rise to an affine model and tractable portfolio allocation rules.

The main theme behind those models is the fact that the correlation structure of world equity markets is not constant over time, but is highly time varying. A number of studies have addressed this issue, as well as the driving factors behind this time variation. Based on data from the last 150 years, Goetzmann et al. (2005) find that correlations between equity returns vary substantially over time and achieve their highest levels during periods characterized by highly integrated financial markets. As well peaks in correlations and not only volatility can be attributed to major market crashes, as for example the Crash of 1929. Longin and Solnik (1995) study shifts in global equity markets correlation structure and reject the hypothesis of constant correlations among international stock markets. Moreover, they find evidence that correlations increase during highly volatile periods. Using Extreme Value Theory, Longin and Solnik (2001) find that international stock markets tend to be highly correlated during extreme market downturns than during extreme market upturns, establishing a pattern of asymmetric dependence. Further, Ang and Chen (2002) confirm this finding for the US market for correlations between stock returns and an aggregate market index. Another strand of literature connects the variability of stock return correlations to the phase of the business cycle. Ledoit et al. (2003) and Erb et al. (1994) show that correlations are time-varying and depend on the state of the economy, tending to be higher during periods of recession. Similar evidence is brought forward by Moskowitz (2003) who links time variation of volatilities and covariances to NBER recessions.

The above empirical findings find theoretical justification in Ribeiro and Veronesi (2002) where in a Rational Expectations Equilibrium model time variations in correlations are obtained endogenously as a result of changes in agents' uncertainty about the state of the economy. Further, by relating recessionary periods to a higher level of uncertainty, excess co-movements across international stock markets are obtained during bad times when the global economy slows down.

The evidence of highly varying conditional correlations on equity markets has motivated us to propose a continuous time process for asset prices that incorporates the above mentioned stylized facts in two distinct ways. First, we allow for tail dependence between extreme realizations of asset returns by explicitly modeling the stationary distribution of the process using copula functions that incorporate dependence in the left or the right tail. This construction of a multivariate diffusion with a pre-specified stationary distribution relies on Chen et al. (2002) and follows the lines of Stefanova (2008). It allows us to obtain higher dependence when markets experience downturns than during upward moves. However, this approach does not exploit the conditional correlation structure of the process. To this end, we further propose a specification for modeling correlation dynamics using observed factors, including macroeconomic and market volatility factors. With those we aim at capturing the above mentioned features of asset returns, and namely the fact that correlations increase during extreme market downside moves, hectic periods and recessionary states of the economy.

This paper further concentrates on the portfolio implications of those distributional assumptions. Staying within a complete market framework, we are able to undertake the standard portfolio solution methodology of Cox and Huang (1989), further developed by Ocone and Karatzas (1991) and Detemple et al. (2003), which allows us to obtain in closed form up to numerical integration the optimal portfolio components in terms of mean-variance demand and intertemporal hedging demands. For the case where we model conditional correlation as a function of observed factors, we are able to isolate the hedging demands for correlation risk, due to stochastic changes in the factors. We use the solution for the optimal portfolio allocation in order to address the following issues:

a) We test whether the implications of allowing for tail dependence through the stationary distribution and for dynamic conditional correlation on the optimal portfolio hedging demands are similar in magnitude and direction. As those distributional assumptions aim at replicating the same stylized behavior, it is interesting to see whether the portfolio effects will share this similarity. For an in-sample market timing exercise along realized paths of the state variables over a 20-year investment horizon and two risky funds, we find that allowing for dynamic conditional correlation generally drives up the intertemporal hedging demands, while allowing for tail dependence in the stationary distribution diminishes them. There is also a distinction in the portfolio composition between the risky funds: in the presence of dynamic conditional correlation the spread between the hedging demands for the two funds increases, while in the presence of tail dependence it decreases, bringing about smaller hedging components in absolute value for the two funds. Those effects become more important when increasing the investment horizon.

b) We further investigate the evolution of the correlation hedging demands implied by the observable factors. Using a factor to capture market-wide volatility and another one to account for macroeconomic conditions, we find that the total correlation demands due to those factors are generally negative throughout the period we consider. The impact of the macroeconomic factor is more significant and directs the behavior of the hedging demands.

c) We test whether results are sensitive to the particular choice of investment period. We consider two sub-periods that differ in the level of stock market volatility and macroeconomic conditions, and we consider an investor with investment horizon set at the end of each of these sub-periods. We find that for a relatively calm period with almost no extreme events towards its end the impact of tail dependence disappears once we allow for a data generating process that incorporates dynamics in the conditional correlation behavior. To the contrary, for a hectic period with declining macroeconomic conditions and a number of extreme events, especially towards its end, the importance of modeling tail dependence for the optimal hedging demand cannot be overwritten by allowing for dynamically varying correlations.

d) We further test the economic importance of considering dynamic conditional correlation or tail dependence using the concept of the certainty equivalent cost and find substantial utility loss due to disregarding either form of dependence, which increases with the investment horizon and for low levels of the agent's relative risk aversion. As well, we find substantial utility loss for disregarding dependence between extreme realizations, even when dynamic conditional correlation has already been accounted for, and vice versa. We also compare different dynamic conditional correlation specifications that take into account or not observable factors and we find that there is utility loss related to disregarding observable factors, especially factors related to macroeconomic conditions.

e) As well we study the sensitivity of the optimal hedging behavior for different levels of average correlation and find higher hedging demands for high correlation levels, when the impact of stochastic changes in conditional correlation on investor's utility is expected to be the highest. This finding is confirmed by the certainty equivalent cost of disregarding dynamic conditional correlation: the utility loss increases for higher levels of average correlation. Alternatively, we study the impact of disregarding tail dependence for varying levels of tail dependence coefficients in the data generating process and find that there are far more significant costs of disregarding dependence between extreme realizations when its level increases, even when dynamic conditional correlation is already taken into account.

The present study is closely related to the work of Buraschi et al. (2007) who solve for the optimal portfolio hedging behavior in the presence of correlation risk in a setting where both volatilities and correlations are stochastic, giving rise to separate demands for volatility and correlation risk. They model covariance dynamics using the analytically tractable Wischart process and study the portfolio impact of stylized facts of asset returns such as volatility and correlation persistence and leverage effects. However they work in an incomplete market setting which allows them to obtain closed-form portfolio solutions for only the CRRA investor. While in Buraschi et al. (2007) the correlation between the risky assets is stochastic and is driven by its independent risk source, the model of Liu (2007) allows for stochastic correlations that however are deterministic functions of return volatilities, which does not allow disentangling the portfolio effect of correlation from that of volatility. Under this model's assumptions, including quadratic returns, for which the four elements, describing the investment opportunity set (the short rate, the maximal squared Sharpe ratio, the hedging coefficient vector, and the unspanned covariance matrix), are all Markovian diffusions with quadratic drift and diffusion coefficients, it is again possible to obtain explicit dynamic portfolio solutions for an investor with CRRA utility. The portfolio problem can be solved under either complete markets (when utility is defined over consumption and terminal wealth) or incomplete markets (when utility is defined only over terminal wealth).

The portfolio solution methodology that we consider allows us to identify the intertemporal hedging demands that arise from the need to hedge against changes in the stochastic investment opportunity set, and separate them from the mean-variance component. As well, we can solve under general utility preferences, that are not constrained to the CRRA case. We consider a case when conditional correlation is modeled as a deterministic function of the state variables driving volatility, and alternatively as a function of observed state variables, linked to market-wide volatility and macroeconomic conditions. In the second case we are able to isolate the correlation hedging demands that appear due to the need to hedge against fluctuations in the observed factors.

The present study is also related to another strand of literature that studies the implications of asset co-movements on dynamic portfolio choice. Ang and Bekaert (2002) consider a regime-switching model of asset returns that accounts for asymmetries in their dependence structure by including a 'bear' regime with low expected returns, coupled with high volatilities and correlations, and a 'normal' regime with high expected returns, low volatilities and correlations. They find that the asymmetric correlation structure between the two regimes becomes important for an international investor only when she is allowed to trade in the risk-free asset. Only in this case there are any significant economic costs of disregarding regime switching. Liu et al. (2003) model event related jumps in prices and volatility in the double-jump framework, introduced by Duffie et al. (2000). The presence of event jumps renders the optimal portfolio holdings similar to those that could be obtained for an investor faced with short-selling and borrowing constraints. As well, event risk has a larger impact on the portfolio composition of investors with low levels of risk aversion. However, these results are obtained for a single risky asset portfolio. Das and Uppal (2004) consider the impact of systemic risk on dynamic portfolio choice by introducing a jump component in asset prices that is common for all assets. They work in a constant investment opportunity set and find that investors who ignore systemic risk would have larger holdings of the risky assets. As well, there is higher cost associated to ignoring systemic risk for investors with low levels of risk aversion and levered portfolios. In this setting there are portfolio effects due to higher moments that arise from the inclusion of jumps. Alternatively, Cvitanic et al. (2008) develop optimal allocation rules under higher moments when risky assets are driven by a time-changed diffusion of the Variance Gamma type, and find that ignoring skewness and kurtosis leads to overinvestment in the risky assets and a substantial wealth loss, especially for high volatility levels.

In this study we consider an alternative way to model asset co-movement asymmetries through the stationary distribution of the process for the state variables, driving the prices of the risky assets. We introduce an asymmetric dependence structure of the distribution explicitly by using copula functions that allow us to isolate the effect of the marginal distributions from that of the dependence structure itself. This allows us to model the above mentioned stylized facts without reverting to an incomplete market through the inclusion of jumps, which allows us to have a tractable portfolio solution for a general utility function specification. We chose between copula functions that incorporate dependence between extreme realizations of the state variables and copulas that imply no tail dependence and study the differences in the intertemporal hedging demands entailed by the alternative data generating processes.

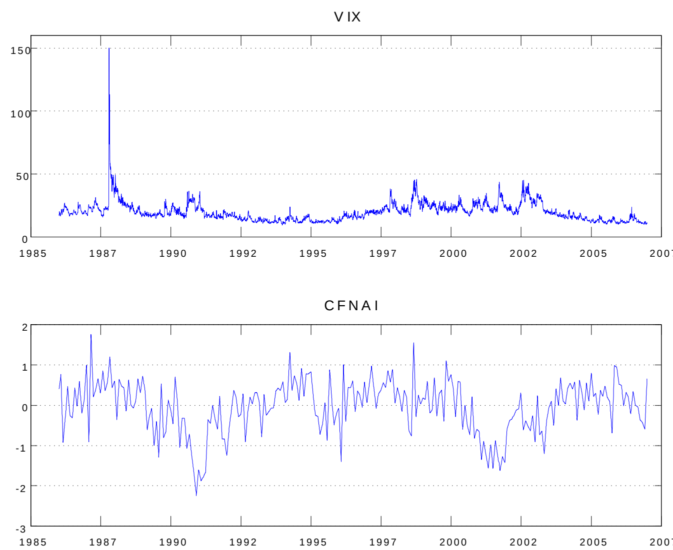
The remainder of the paper is organized as follows. Section 2 discusses several stylized facts of dynamic correlation and motivates the possibility to model it using observable factors. Section 3 describes the model, the solution to the portfolio choice problem, and the correlation hedging demands that appear due to observable factors driving correlation. Section 4 discusses the particular portfolio holdings for a bivariate application. In Section 5 we present numerical results used to gauge the importance of hedging demands that arise due to dynamic correlation or tail dependence. Section 6 concludes. Technical details are provided in the appendix.

2 Dynamic correlation and exogenous factors

Established empirical findings point towards several stylized facts that characterize conditional correlation dynamics of asset returns. It tends to increase in periods of high market volatility, or in cases of extreme downside market moves. As well, it appears to be linked to the business cycle and is higher in recessionary states of the economy.

We approach the above mentioned facts in two methodologically distinct ways. First, we achieve increased correlation during market downturns through the stationary distribution of the multivariate diffusion of state variables that underlines the stock price process. With this 'static' approach we are able to achieve a certain degree of left tail dependence which translates into increased dependence for low levels of the state variables. Second, we allow for dynamic correlation of the state variables, driven by factors that are supposed to capture market volatility and the state of the business cycle. To this end, we choose the Chicago Board Options Exchange Volatility Index (VIX) which measures the implied volatility of S&P index options and thus incorporates market's expectations of near-term volatility. In order to incorporate the effect of the business cycle on the dynamics of correlation, we take the Chicago Fed National Activity Index (CFNAI) that synthesizes information on various macroeconomic factors in a single index. It is a monthly index that aggregates information on overall macroeconomic activity and inflation, as it is a weighted average of 85 indicators of national economic activity, ranging from production, employment, housing and consumption, income, sales, orders and inventories. The methodology behind the CFNAI is based on Stock and Watson (1999), who find a common factor behind various inflation indicators. The

Figure 2.1: Evolution of the VIX index (upper panel) and of CFNAI index (bottom panel) for the period 1986 - 2006.
The VIX is quoted in terms of percentage points and the data is available at the daily frequency. The CFNAI is quoted monthly. A negative value of the CFNAI index indicates a below-average growth of the national economy, whereas a positive value of the index points towards an above-average growth. A zero value means that the economy grows at its historical average rate.



evolution of the VIX and of the CFNAI are given in Figure 2.1.

In order to appreciate the time variation in asset correlations, driven by the chosen indices, we estimate a DCC model with exogenous factors on the asset return series that will be used later in the portfolio application. Data used in this study consists in two stock market indices representing old economy stocks (S&P 500) and new economy stocks (NASDAQ) for the period 1986-2006. This relatively long period includes several market crashes among which the October 1987 crash in the beginning of the sample period, the Asian crisis that triggered the market crash in October 1997, as well as the Dot-com bubble crash in 2000-2002.

The DCC specification, as well as the estimated coefficients are given in Table 2.1, and the correlation dynamics are plotted in Figure 2.2.

All the DCC parameters are significantly estimated which points towards a certain degree of persistence of correlation. Estimated correlation levels range between 0.55 and 0.90 and there can be seen a general tendency of increasing correlation over the years. There are some distinct spikes in conditional correlation, some of which can be linked to specific events (e.g. the late 1987 and 1997 market crashes). There is a distinct period of lower conditional correlations between 1992 and 1997, which is also character-

Table 2.1: Parameter estimates of a DCC model with exgenous factors for SP 500 and NASDAQ returns. The model that we estimate is an extended version of the DCC model of Engle (2002) to include exogenous factors driving the conditional covariance and it has the following specification. Denote by y_t the $d \times 1$ vector of asset returns, and by F_t the $n \times 1$ vector of exogenous variables. Then for the conditional mean equation we have:

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t \\ \varepsilon_t &= H_t^{1/2}\eta_t \;\text{ where }\; \eta_t \sim N(0,1) \;\;\text{ thus } \varepsilon_t \sim N(0, H_t) \end{aligned}$$

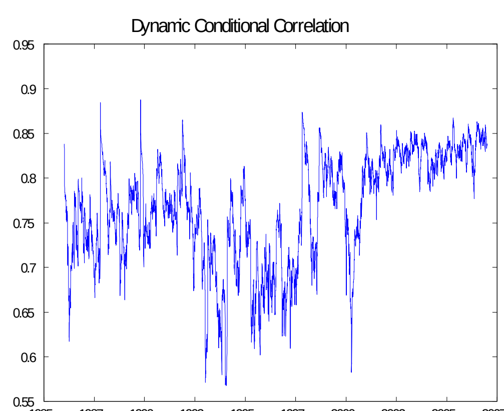
The conditional covariance matrix H_t can be expressed as $H_t = D_t R_t D_t = \left(\rho_{ij,t}\sqrt{h_{ii,t}h_{jj,t}}\right)$, where $\rho_{ij,t}$ are entries of the conditional correlation matrix and $h_{ii,t}$ are entries of the conditional covariance matrix. Further, $R_t = \widetilde{Q}_t^{-1} Q_t \widetilde{Q}_t^{-1}$, where $\widetilde{Q}_t^{-1} = diag\left(\sqrt{q_{ii,t}}\right)$. The dynamics of Q_t are given by:

$$Q_t = \overline{Q}\,(1-\alpha-\beta) + \alpha\widetilde{\varepsilon}_{t-1}\widetilde{\varepsilon}_{t-1} + \beta Q_{t-1} + I p^{\intercal} F_{t-1}$$

where $\widetilde{\varepsilon}_t \sim N(0, R_t)$, and it is a $d \times 1$ vector of standardized residuals $\widetilde{\varepsilon}_t = \frac{\varepsilon_t}{\sqrt{h_{ii,t}}}$, I is the identity matrix and p is an $n \times 1$ vector of parameters pertaining to the exogenous factors F_t.
In our case y_t denotes the returns of S&P 500 and NASDAQ, and F_t are the VIX and the CFNAI indices. Parameter estimates and their corresponding standard errors are given below.

	Parameters	**Standard errors** $\times$ 1000
α	0.0221	(0.1025)
β	0.9744	(0.0063)
$p_1\,(VIX)$	0.0008	(0.0000)
$p_2\,(CFNAI)$	-0.0001	(0.0081)

Figure 2.2: Estimated dynamic conditional correlation for SP 500 and NASDAQ returns from a DCC model with exogenous factors



ized by low market volatility and a generally above average growth trend in the economy. The parameters for the exogenous factors that drive the time-varying conditional covariance have the expected signs: positive for the VIX and negative for the CFNAI, which translates into increasing conditional correlation during hectic periods and recessionary states of the economy.

3 The investment problem

This section describes the problem faced by the investor in allocating her wealth between a set of risky assets and the money market account. It introduces the distributional and utility assumptions we impose and presents the general solution methodology using the Martingale technique following the portfolio decomposition formula of Detemple et al. (2003) and its implementation via Monte Carlo simulations. We consider the case where the investor maximizes expected utility of terminal wealth, so that we do not allow for intermediate consumption.

3.1 The economy

We define a filtered probability space $\left(\mathcal{F}_T^X, \{\mathcal{F}_T^X\}_{t=0}^T, P^Y\right)$over the investment horizon $[0, T]$ where $\mathcal{F}_T^Y$ is the filtration generated by state variables Y_t under the empirical probability measure P^Y. We consider a complete market setup with $d+1$ state variables $Y_{it}, i = 1, ...d$, where uncertainty is driven by $d+1$ Brownian motions $W_{it}, i = 1, ...d+1$. There are $d+2$ securities available for investment: d stocks, a long term pure discount bond, and the risk-free asset. The state variable vector Y_t consists of $d+1$ state variables X_t, each one affecting its corresponding stock price process, and a state variable Y_t^r that governs the dynamics of the short rate r_t, that is $Y_t = (Xt, Y_t^r)^\intercal$.

The investor has at her disposal the following three asset categories. First, she can invest in a risk-free money market account and its value at time t is given by:

$$B_0\left(t\right) = \exp\left\{\int_0^t r\left(s, Y_s^r\right) ds\right\} \tag{3.1}$$

As well, another tradeable asset in the portfolio is a default-free zero-coupon bond with a maturity T. Its price $B\left(t, T\right)$ at time t can be expressed as a conditional expectation under the equivalent martingale measure Q:

$$B\left(t, T\right) = E^Q\left[\exp\left\{-\int_t^T r\left(s, Y_s^r\right) ds\right\} |\mathcal{F}_t^Y\right] \tag{3.2}$$

The rest of the portfolio consists in a collection of stocks whose price process is modeled using the d state variables X_t:

$$S_i(t) = \exp(X_{it} + \varphi(t)) \quad , i = 1, ..., d \tag{3.3}$$

where $\varphi(t)$ is a deterministic function of time. This specification was chosen in order to be as close as possible to the Geometric Brownian motion underlying the Black-Scholes formula for option pricing: if the process for X_{it} is given by $X_{it} = X_{i0} + \sigma_i \int_0^t dW_{it}$, then we are exactly in the Black-Scholes setting where all the assets are independent from each other; if alternatively we apply a stochastic time transformation to the Brownian motion and define the process for X_{it} as $X_{it} = X_{i0} + \int_0^t \sigma(t, X_{it}) dW_{it}$, then we obtain a simple generalization of the Geometric Brownian motion that already departs from the normality assumption. As it will be shown below, we will further introduce a drift to the process for the state variables X_t which will be consistent with a chosen stationary distribution for the process, as well as correlations between the Brownians that will be allowed to be stochastic. This will bring the model closer to the discrete-time alternative of a dynamic conditional correlation model, as the one introduced by Engle (2002).

3.2 The affine setup for the bond price

In what follows, we will restrict the framework for the bond price to the affine class, in that the short interest rate r_t will be an affine function of state variable Y_t^r. This will allow us to express the yield of the bond as an affine function of the state variable as well. Thus, we assume that the short rate can be expressed as:

$$r(t, Y_t^r) = \delta_0 + \delta_1 Y_t^r \tag{3.4}$$

The choice of a one-factor affine model for the short rate may be questionable as there is substantial empirical evidence concerning the shortcomings of affine models[1], and as well using only one factor to capture the dynamics of the term structure may be too restrictive. But as the specification for the bond is marginal for our portfolio application, we proceed with this simple specification which ensures tractable portfolio solutions. As well, Y_t^r has the simple interpretation as a state variable that models the dynamics of the interest rate risk factor which will further determine the hedging terms of the portfolio against changes in the stochastic interest rate.

Following the evidence of time-varying interest rate risk premia on the bond market (e.g. Chan et al. 1992), we allow the state variable Y_t^r to evolve over time according to a square-root process. Its dynamics under the objective measure P^Y are given by:

$$dY_t^r = \kappa_r(\theta^r - Y_t^r) dt + \sigma_r \sqrt{Y_t^r} dW_t^r \tag{3.5}$$

[1] Backus et al. (1998) show that term premiums generated by affine models are too low compared to the observed data; Duffee (2002) finds that this class of models is not flexible enough to replicate temporal patterns in interest rates.

Following Dai and Singleton (2000), we assume a market price of risk of the form $\lambda\sqrt{Y_t^r}$, which ensures that the process for the state variable will be affine under the risk neutral measure as well. Then under the equivalent martingale measure the process will be:

$$dY_t^r = \overline{\kappa_r}\left(\overline{\theta^r} - Y_t^r\right)dt + \sigma_r\sqrt{Y_t^r}dW_t^{*r} \tag{3.6}$$

where $\overline{\kappa_r} = \kappa_r + \sigma_r\lambda$ and $\overline{\theta^r} = \kappa_r\theta^r / (\kappa_r + \sigma_r\lambda)$.

Given the affine term structure parametrization is admissible, we can obtain in closed form the price of the default-free bond:

$$B(t,T) = \exp\{a(T-t) + b(T-t)Y_t^r\} \tag{3.7}$$

where $a(\tau)$ and $b(\tau)$ solve the Ricatti equations:

$$\begin{aligned}\frac{\partial a(\tau)}{\partial\tau} &= \overline{\theta^r}\,\overline{\kappa_r}b(\tau) - \delta_0 \\ \frac{\partial b(\tau)}{\partial\tau} &= -\overline{\kappa_r}b(\tau) + \frac{1}{2}(\sigma_r b(\tau))^2 - 1\end{aligned}$$

Then the process for the bond price can be recovered from (3.6) and (3.7) and the specification of the market price of risk that we adopted. Thus, it can be shown that the bond price follows:

$$\begin{aligned} dB_t &= B_t\left[\mu^B(t,Y_t^r)\,dt + \sigma^B(t,Y_t^r)\,dW_t^r\right] \\ \text{where } \mu^B(t,Y_t^r) &= r(t,Y_t^r) + b(\tau)\sigma_r\lambda Y \\ \text{and } \sigma^B(t,Y_t^r) &= b(\tau)\sigma_r\sqrt{Y_t^r}\end{aligned} \tag{3.8}$$

As a result of the CIR specification of the state variable Y_t^r, the market price of risk defined by $\Theta^B(t,Y_t^r) = \sigma^B(t,Y_t^r)^{-1}\left(\mu^B(t,Y_t^r) - r(t,Y_t^r)\right)$ is stochastic and is given by $\lambda\sqrt{Y_t^r}$. It should be noted that for the bond risk premium to be positive, the market price of risk and thus λ should be negative.

3.3 The copula diffusion for the stock price process with dynamic conditional correlation

In this section we will define the process for the state variables X_t that drive the stock prices. As we are interested in modeling the dependence between extreme realizations of returns, we will adopt the copula diffusion process, introduced in Stefanova (2008) and extend it to a dynamic conditional correlation specification. Thus, we introduce two channels for modeling extremal dependence: one through the properties of the stationary distribution of the process, and the second through the conditional correlation. We will explore two options for modeling the correlation dynamics. A first straightforward way to do so is to allow the conditional correlation to be time-varying by being specified as some known function of the state variables themselves. As there is evidence that correlation increases in volatile states and when returns are low, we propose to model correlation as a function of the volatility and the level of the state variables. Thus, the general form of the state variables X_t is given by:

$$\textit{Case A:} \qquad dX_t = \mu\left(X_t\right)dt + \Lambda\left(X_t\right)dW_t^X \tag{3.9}$$

where Λ is a lower triangular matrix, and W^X is a d-dimensional standard Brownian motion, independent of W^r. If we define a continuously differentiable positive definite matrix $\Sigma = \Lambda\Lambda^\intercal$, then its entries are given by $\nu_{ij}\left(X_t\right) = \Upsilon_{ij}\left(X_t\right)\sigma_i^X\left(X_t\right)\sigma_j^X\left(X_t\right),\ i,j = 1,...,d$, where the conditional correlation coefficients $\Upsilon_{ij}\left(X_t\right)$ and the conditional volatility terms $\sigma_i\left(X_t\right)$ are functions of X_t and thus time varying.

The second way to model dynamic correlation that we explore is by rendering it stochastic in terms of a function of observable factors. Following the empirical evidence, that correlations increase in volatile periods and in bad states of the economy, we introduce two exogenous factors to account for that: the CBOE volatility index (VIX) and the Chicago Fed National Activity Index (CFNAI). Denoting these observable factors as F_t, we propose a second general specification for the state variable process X_t of the form:

$$\textit{Case B:} \qquad dX_t = \widetilde{\mu}\left(X_t, F_t\right)dt + \widetilde{\Lambda}\left(X_t, F_t\right)dW_t^X \tag{3.10}$$

where $\widetilde{\Lambda}$ is a lower triangular matrix, defined as a function of the state variables X_t, as well as the observable factors F_t. The entries of the continuously differentiable positive definite matrix $\widetilde{\Sigma} = \widetilde{\Lambda}\widetilde{\Lambda}'$ are given by $\widetilde{\nu}_{ij}\left(X_t, F_t\right) = \widetilde{\Upsilon}_{ij}\left(X_t, F_t\right)\sigma_i^X\left(X_t\right)\sigma_j^X\left(X_t\right)$, where the conditional correlation coefficient $\widetilde{\Upsilon}_{ij}\left(X_t, F_t\right)$ is stochastic in that it is modeled as a function of the observable factors F_t, as well as the state variables X_t. Note that in this second case we augment the state variable vector Y_t to include also the factors F_t: $Y_t = (X_t, F_t, Y_t^r)^\intercal$.

Using any of the above specifications for X_t and the fact that the stock price is defined following (3.3), we can apply Itô's lemma in order to recover the stock price process:

$$
\begin{aligned}
dS_{it} &= S_{it}\mu_i^S\left(\log S_{it}-\varphi\left(t\right)\right)dt \\
&\quad +S_{it}\sum_{j=1}^{d}\Lambda_{ij}^{I}\left(\log S_{it}-\varphi\left(t\right)\right)dW_{jt}^{X} \\
\text{where } \mu_i^S\left(t,Y_t\right) &= \mu_i^I\left(Y_t\right)+\varphi'\left(t\right)+\frac{1}{2}\sum_{j=1}^{d}\overline{\sigma}_{ij}^2\left(Y_t\right) \quad ,I=1,2 \\
\mu_i^1\left(Y_t\right) &= \mu\left(X_t\right),\ \ \mu_i^2\left(Y_t\right)=\widetilde{\mu}\left(X_t,F_t\right) \\
\text{and } \Lambda_{ij}^I\left(t,Y_t\right) \quad ,I &= 1,2 \text{ are entries of the corresponding matrix:} \\
\Lambda_{ij}^1\left(t,Y_t\right) &= \Lambda\left(X_t\right),\ \ \Lambda_{ij}^2\left(t,Y_t\right)=\widetilde{\Lambda}\left(X_t,F_t\right)
\end{aligned}
\tag{3.11}
$$

It should be noted, that as we need to stay within the complete market setup, the number of sources of risk, generated by the Brownian motions, should be the same as the number of traded assets. Thus, when introducing the observable factors F in the stock price specification, we assume that their dynamics are governed by the same Brownian motions that drive the stock prices themselves.

As the market is complete and we have an invertible matrix $\Lambda^{(I)}$, we can define a market price of risk as $\Theta^S\left(t,Y_t^r\right)=\Lambda^{(I)}\left(t,Y_t\right)^{-1}\left(\mu^S\left(t,Y_t\right)-r\left(t,Y_t^r\right)\iota\right)$, where ι is a d-dimensional vector of ones.

Let us stack the drift and diffusion terms for the bond and the stocks so that to obtain:

$$
\begin{aligned}
M\left(t,Y_t\right) &= \begin{pmatrix} \mu_i^S\left(t,Y_t\right) \\ \mu^B\left(t,Y_t\right) \end{pmatrix} \\
\Xi\left(t,Y_t\right) &= \begin{pmatrix} & & 0 \\ \Lambda^{(I)}\left(t,Y_t\right) & & \vdots \\ & & 0 \\ 0 \ \ldots \ 0 & & \sigma^B\left(t,Y_t^r\right) \end{pmatrix}
\end{aligned}
$$

Then the market price of risk for all the tradeable assets
$\Theta\left(t,Y_t\right)=\left(\Theta\left(t,Y_t\right)_1^S,...,\Theta\left(t,Y_t\right)_d^S,\Theta^B\left(t,Y_t\right)\right)$ is defined as:

$$
\Theta\left(t,Y_t\right)=\Xi\left(t,Y_t\right)^{-1}\left(M\left(t,Y_t\right)-r\left(t,Y_t^r\right)\iota\right)
$$

It is assumed to be continuously differentiable and satisfying the Novikov condition:

$$
E\left[\exp\left(\int_0^T\Theta\left(t,Y_t\right)^{\intercal}\Theta\left(t,Y_t\right)dt\right)\right]<\infty.
$$

The market completeness implies the existence of a unique state price density ξ_t defined as

$$\begin{aligned} \xi_t &\equiv B_0(t)^{-1}\eta_t \exp\left\{-\int_0^t r(s, Y_s^r)\,ds\right\}\times \\ & \exp\left\{-\int_0^t \Theta(s, Y_s)^\intercal\,dW_s - \frac{1}{2}\int_0^t \Theta(s, Y_s)^\intercal\,\Theta(s, Y_s)\,ds\right\} \end{aligned}$$

where η_t is the Radon-Nykodym derivative, $\mathcal{F}_T^Y$-adapted. We can also define the conditional state price density that converts cash flows at time $v \geq t$ into cash flows at time t:

$$\begin{aligned} \xi_{t,v} &\equiv \xi_v / \xi_t \\ &= \exp\left\{\begin{array}{c} -\int_t^v r(s, Y_s^r)\,ds - \int_t^v \Theta(s, Y_s)^\intercal\,dW_s \\ -\frac{1}{2}\int_t^v \Theta(s, Y_s)^\intercal\,\Theta(s, Y_s)\,ds \end{array}\right\} \end{aligned} \tag{3.12}$$

3.3.1 Establishing the diffusion specification for the state variables X that drive the stock price dynamics

Having established two alternative ways to model the conditional correlation dynamics with the aim of answering the stylized fact that asset correlation increases in volatile periods when asset returns are low and the economy is in a downturn, we now turn to the other possibility of accommodating this stylized fact: through the stationary distribution of the state variables, as it has been already explored in Stefanova (2008). This approach models the tail dependence (the asymptotic dependence between tail realizations of the state variables) in a 'static' sense, instead of focusing on the dynamics of a correlation measure (the correlation between state variables changing stochastically through time). By imposing a certain stationary distribution on the state variables' process, one can obtain different degrees of tail dependence in the left or the right tail of the distribution. Thus, for low levels of the state variable, the tail dependence index may be high, while for high levels of the state variable it may be low, reproducing the stylized fact mentioned above.

For the sake of completeness, we will review the construction of a multivariate diffusion with a given invariant distribution, defined in terms of copula functions. It follows Chen et al. (2002) in exploiting the relationship that exists between the density of the stationary distribution, the drift and the diffusion term of the process defined in (3.9) or (3.10):

$$\mu_j = \frac{1}{2} q^{-1} \sum_{i=1}^{d} \frac{\partial(\nu_{ij} q)}{\partial x_i} \tag{3.13}$$

where μ and ν_{ij} denote either $\mu(X_t)$ and $\nu_{ij}(X_t)$ for *Case A* or $\widetilde{\mu}(X_t, F_t)$ and $\widetilde{\nu}_{ij}(X_t, F_t)$ for *Case B*, and q is a strictly positive continuously differentiable multivariate density function that is the stationary density of the Markov process for X. Thus the specification of the drift term μ depends on both the form of the invariant density (which will be modeled to determine the degree of asymmetric tail dependence of the state variables X, that is the 'static' representation of the stylized fact of co-movement asymmetries), and the form of the diffusion term Λ (which will be specified in a way to allow or not for dynamic conditional correlation, dependent or not on observable factors, that is the 'dynamic' representation of the same stylized fact).

In what follows we will establish the alternative assumptions on the form of both the invariant density and the volatility term.

The form of the invariant density. With the choice of the stationary distribution we seek to answer several questions concerning the behavior of asset returns. Our major concern is the ability to allow assets to be dependent when they move towards the tails of the distribution, especially for the left tail. This would ensure our model the ability to replicate the empirical fact that asset returns are increasingly dependent as they jointly move towards the lower quantiles of their distribution, that is during market downturns. As copula functions allow us the flexibility to impose different types of joint behavior on the variables while keeping the marginal distributions unchanged, we build the invariant density q based on the copula density representation following Sklar's theorem:

$$q\left(x_1,...,x_d\right) \equiv \widetilde{c}\left(x_1,...,x_d\right)\prod_{i=1}^{d}\widetilde{f}^i\left(x_i\right) \tag{3.14}$$

where $\widetilde{c}\left(x_1,...,x_d\right) = c\left(F^1\left(x_1\right),...,F^d\left(x_d\right)\right)$ is a copula density defined over the univariate CDFs $F^i\left(x_i\right)$, and $\widetilde{f}^i\left(x_i\right)$ are the corresponding non-normalized univariate densities. We choose the Normal Inverse Gaussian (NIG) distribution[2] to model the univariate behavior because of its proven ability to account for stylized facts of univariate asset return dynamics: autocorrelation of squared returns, semi-heavy tails, possibly asymmetric. Its tail behavior is richly parametrized, nesting tails that vary from an exponential to a power law. As well, NIG is one of the few members of the class of Generalized Hyperbolic (GH) distributions that is closed under convolution, that is if the distribution of log prices is modeled under a NIG law, then the distribution of the increments (asset returns) is also NIG. The univariate NIG diffusion is also an alternative to the widely used NIG Levy process (e.g. Eberlein and Keller 1995, Prause 1999) that allows for an infinite number of jumps in the price process, but that also imposes independence of the increments, which is not the case for its diffusion counterpart.

The most important feature of the copula density representation (3.14) is that it allows us to separate the effect of the marginal behavior from the implications of the dependence structure, modeled using a

[2]See the appendix for details.

copula function. This is important for the portfolio application that we treat in this study, as it allows us to gauge the difference between the different ways to model asset dependence (and thus to reproduce or not the stylized fact of asymmetric asset co-movements) without the impact of the particular assumptions for the univariate stock price processes. Thus we could measure the impact of the 'static' representation of dependence, ranging from Gaussian (no extreme co-movements) to non-negative tail dependence (extreme co-movements, possibly asymmetric) on the optimal portfolio terms.

Let us first remind the definition of the coefficients of upper and lower tail dependence for couples of random variables X and Y: upper tail dependence is defined as the limit probability of the variable Y exceeding the upper quantile as we approach it, conditional upon the fact that the random variable X has exceeded that same quantile:

$$\lambda_U = \lim_{u \to 1} \Pr\left[Y > F_Y^{-1}(u) \,|X > F_X^{-1}(u)\right]$$

Alternatively, we define the coefficient of lower tail dependence as:

$$\lambda_L = \lim_{u \to 0} \Pr\left[Y \leq F_Y^{-1}(u) \,|X \leq F_X^{-1}(u)\right]$$

Both coefficients can be represented in terms of copula functions: $\lambda_U = \lim_{u \to 1} \frac{(1-2u+C(u,u))}{1-u}$ and $\lambda_L = \lim_{u \to 0} \frac{C(u,u)}{u}$. So different copulas will have different degrees of upper and lower tail dependence depending on their parametric specification. Thus, in order to allow for different degrees of tail dependence, we assume several copula specifications for c[3].

Case 1 *Gaussian copula* C^{Ga}*:* $\lambda_U = \lambda_L = 0$

In this case we allow for no dependence between tail realizations of the state variables. The parameter that governs dependence is the correlation coefficient ρ.

Case 2 *Student's t copula* C^t*:* $\lambda_U = \lambda_L = 2t_{\nu+1}\left(-\frac{\sqrt{\nu+1}\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$

where t_ν is the Student's t density for ν degrees of freedom. In this case the copula function allows for symmetric tail dependence, determined by the correlation parameter ρ and the degrees of freedom parameter ν.

Case 3 *A Gaussian - Symmetrized Joe-Clayton (SJC) mixture copula* C^{Ga-SJC}*:* $\lambda_U \neq \lambda_L$

The form of the mixture copula is given by:

$$C^{Ga-SJC} = \omega C^{SJC} + (1-\omega)\, C^{Ga}$$

[3] See the appendix for details on the alternative specifications of the copula functions used in the paper.

where C^{Ga} stands for the Gaussian copula function and C^{SJC} - the Symmetrized Joe-Clayton copula, with a mixing parameter ω that determines the weights of each of the copulas. The symmetrized Joe-Clayton copula models separately upper and lower tail dependence and its form is particularly appealing, as the tail dependence coefficients are themselves the parameters of the copula function. It has been proposed by Patton (2004) as a symmetrized version of the Joe-Clayton copula, in order to overcome the drawback of the latter in that even when the coefficients of upper and lower tail dependence are equal to each other, there still exists some asymmetry in the copula, due to its functional form.

We consider a mixture specification with this copula and the tail independent Gaussian one in order to answer the concerns raised in Poon et al. (2004) that a copula specification whose coefficients explicitly allow for tail dependence may overestimate the dependence in the tail regions. Thus, by the mixture copula we let the data determine whether the dependence structure is closer to one imposing no tail dependence or to one that allows for it.

In the cases considered above dependence is modeled explicitly through the invariant density of the multivariate state variable process. In the following section we will extend this setup and will introduce dynamics in the modeling of dependence through the conditional correlation coefficient.

The conditional correlation dynamics. Before proceeding to the specification of the conditional correlation, we need to define the conditional volatility dynamics. Recall that the diffusion term of X was defined as a lower triangular matrix Λ and the entries of the variance-covariance matrix $\Sigma = \Lambda\Lambda^{\intercal}$ are given by $\nu_{ij}(X_t) = \Upsilon_{ij}(X_t)\,\sigma_i^X(X_t)\,\sigma_j^X(X_t)$. Borrowing the idea of Bibby and Sorensen (2003) for modeling the diffusion term of a univariate GH stationary process, we allow each $\sigma_i^X(X_t)$ to be a function of the state variables X_t:

$$\sigma_i^X(X_t) = \sigma_i \left[\widetilde{f^i}(x_i)\right]^{-\frac{1}{2}\kappa_i} \tag{3.15}$$

where $\widetilde{f^i}(x_i)$ is the non-normalized NIG density for X_i, and we have the following parameter restrictions: $\sigma_i > 0$ and $\kappa_i \in [0,1]$. By expressing the volatility term as the inverse of a power function of the density $\widetilde{f}$ we obtain the familiar U-shape for the volatility, typical for a stationary process. This specification is especially interesting, as it nests the constant conditional volatility as a special case, setting $\kappa_i = 0$. Thus, for the portfolio allocation application, we could easily isolate a volatility hedging component due to stochastic conditional volatility by opposing a model with $\kappa_i \neq 0$ to one that restricts the conditional volatility to be constant ($\kappa_i = 0$).

Earlier in this section we have discussed two possibilities of rendering the conditional correlation coefficient dynamic: through modeling it as a function of the state variables X or by allowing it to be influenced by stochastic factors F. Here we will further elaborate the particular assumptions concerning those two cases.

In both cases the conditional correlation coefficient Υ_{ij} is modeled as a function $h_{ij}(Y_t)$ of the stochastic state variables Y, whether or not augmented with the observable factors. In order to keep the correlation coefficient in $[-1, 1]$, we apply the following logistic transform $\mathcal{A}$ on the function $h(Y_t)$:

$$\Upsilon_{ij}(Y) = \mathcal{A}(h_{ij}(Y)) = \frac{1 - \exp(-h_{ij}(Y))}{1 + \exp(-h_{ij}(Y))}$$

Case A. Dynamic conditional correlation with state variables: $\Upsilon(X_t)$

As our aim is to replicate the stylized fact that correlation between asset returns increases in volatile periods and in extreme market downturns, we model the dynamic conditional correlation coefficient as a function involving the volatility specification considered earlier (3.15), as well as the level of the state variables in terms of their probability integral transforms $F(X_i)$. More specifically, we model the function $h_{ij}(\cdot)$ as:

$$h_{ij}(X_t) = \gamma_{ij,0} + \gamma_{ij,1} \max\left(\sigma_1^X(X_t), ..., \sigma_d^X(X_t)\right) + \gamma_{ij,2} \prod_{i=1}^{d} F(X_{it}) \tag{3.16}$$

where $F(X_{it})$ stands for the corresponding univariate NIG CDF. The second term in this specification involves the conditional volatilities of each univariate series. We expect to obtain a positive coefficient $\gamma_{ij,1}$ to reflect the fact that correlation increases in hectic periods. We define this term as the maximum over all individual volatilities in order to allow high volatility in any of the stocks to trigger increased conditional correlation. This specification was also used in Goorbergh et al. (2003) in order to model the dynamics of a conditional copula through Kendall's tau in an option pricing application. The third term is motivated by the fact that conditional correlation shoots up when stock prices jointly and abruptly decline, thus we expect a negative sign for the coefficient $\gamma_{ij,2}$.

Case B. Dynamic conditional correlation with observed factors and latent variables: $\Upsilon(X_t, F_t)$

Instead of letting the dynamics of the conditional correlation parameter be determined exclusively by the state variables that drive the stock price process, we model it instead with observable factors that are believed to drive conditional correlation: the VIX and the CFNAI macroeconomic index. Thus we aim at replicating the stylized fact that correlation increases in volatile markets when the economy is in a bad state. As the economic cycle does not necessarily coincide with bear/bull financial markets, we leave from the previous specification the term that determines the level of the state variable. More specifically, in this case we model the function $h(\cdot)$ as:

$$h_{ij}(X_t, F_t) = \gamma_{ij,0} + \gamma_{ij,1} F_t^V + \gamma_{ij,2} \prod_{i=1}^{d} F(X_{it}) + \gamma_{ij,3} F_t^M \tag{3.17}$$

where $F_t^V = \log(VIX_t)$ and $F_t^M = CFNAI$. The second term in this expression involves the VIX and thus tries to account for the fact that conditional correlation will rise in periods of increased volatility, so that we expect a positive sign for $\gamma_{ij,1}$. The third term involves the probability integral transforms of the state variables X and is thus meant to capture the fact that correlation increases in market downturns (which entails an expected negative coefficient $\gamma_{ij,2}$). The last term involves the macroeconomic factor and thus aims at capturing the effect of the economic cycle on conditional correlation. As the CFNAI index is designed to take positive values when the economy is in an upturn and negative values otherwise, we expect to obtain a negative sign for $\gamma_{ij,3}$.

Case C. Dynamic conditional correlation with observed factors: $\Upsilon\left(F_t\right)$

If we alternatively believe that correlation is driven by factors that do not affect directly the stock price process, then we may restrict the specification in (3.17) in order to include only observable factors:

$$h_{ij}\left(F_t\right) = \gamma_{ij,0} + \gamma_{ij,1} F_t^V + \gamma_{ij,3} F_t^M \tag{3.18}$$

This specification will prove quite useful in determining the portfolio correlation hedging demands, as we will see in the following sections, as it will allow us to explicitly identify them from the rest of the hedging terms of the portfolio. This is due to the fact that the factors determining conditional correlation do not affect in a direct way the stock price process itself.

We assume the following processes for the two factors: a CIR process for F^V and a Vasicek process for F^M:

$$\begin{aligned} dF_t^V &= \kappa^V\left(\theta^V - F_t^V\right)dt + \sigma^V\sqrt{F_t^V}\,dW_t^X \\ dF_t^M &= \kappa^M\left(\theta^M - F_t^M\right)dt + \sigma^M dW_t^X \end{aligned} \tag{3.19}$$

These processes will greatly facilitate the implementation of the portfolio allocation formula, as the Vasicek specification will allow for a closed-form solution for the Malliavin derivative of the macroeconomic factor F^M, while the CIR diffusion term will make possible a variance-reduction technique for the Monte Carlo simulation of the Malliavin derivative of F^V.

3.4 The investor's objective function

We consider an investor who maximizes utility over terminal wealth, that we denote by $U\left(\omega_T\right)$ by choosing an optimal investment policy $\left\{\alpha_t\right\}_{t\in(0,T)}$ that belongs to an admissible set $\mathcal{A}$ for an investment horizon T:

$$\max_{\alpha\in\mathcal{A}} E\left[U\left(\omega_T\right)\right] \tag{3.20}$$

where the utility function U is strictly increasing, concave and differentiable, and satisfies the conditions $\lim_{x\to\infty} U'(x) = 0$ and $\lim_{x\to 0} U'(x) < \infty$. This standard utility specification includes the case of the Hyperbolic Relative Risk Aversion (HARA) utility function $U(\omega) = \frac{1}{1-\gamma}(\omega + b)^{1-\gamma}$ that we assume for this application. The coefficient of Relative Risk Aversion, defined as $R(\omega) \equiv -\frac{U''(\omega)}{U'(\omega)}\omega$, is equal to $\gamma\frac{\omega}{\omega+b}$ for the HARA case, which boils down to a constant γ for the special case of CRRA utility.

The portfolio policy α is a $(d+1)$-dimensional progressively measurable process that is defined as the proportion of wealth allocated to the risky assets (d stocks and a long term pure discount bond). Thus, the amount invested in the risk-free asset (the money-market account) is $(\omega - \alpha^\intercal 1)$. The portfolio policy generates a wealth process ω whose dynamics are given by:

$$d\omega_t = \omega_t \left\{ r_t dt + \alpha_t^\intercal \left[\left(M(t, Y_t) - r_t \iota \right) dt + S(t, Y_t) dW_t \right] \right\} \tag{3.21}$$

3.5 The complete market solution

The complete market setup that we have adopted allows us to solve for the optimal portfolio using the Martingale solution technique that restates the dynamic budget constraint (3.21) as a static one and first solves for the optimal terminal wealth, and then finds the optimal portfolio policy that finances it. Thus, following Cox and Huang (1989), optimal terminal wealth is given by $\omega_T^* = I(y\xi_T)^+ = \max(I(y\xi_T), 0)$, where $I = [U']^{-1}$ denotes the inverse of the marginal utility function, and y satisfies the static budget constraint $E\left[\xi_T I(y\xi_T)^+\right] = \omega_0$, where ω_0 is the initial wealth.

Following Ocone and Karatzas (1991), and using the portfolio decomposition formula of Detemple et al. (2003), we have the following expression for the optimal portfolio policy, that decomposes the portfolio holdings into a Mean Variance part (α^{MV}), an Interest Rate Hedge (α^{IRH}) and a Market Price of Risk hedge (α^{MPRH}):

$$\begin{aligned}
\alpha_t^* &= \alpha_t^{MV} + \alpha_t^{IRH} + \alpha_t^{MPRH} \\
&\quad \text{where} \\
\alpha_t^{MV} &= \left(\Lambda^\intercal(t, Y_t)\right)^{-1} \frac{1}{R(\omega_T)} \Theta(t, Y_t) E_t\left[\xi_{t,T} \frac{\omega_T}{\omega_t} \frac{R(\omega_t)}{R(\omega_T)} 1_{\omega_T > 0}\right] \\
\left(\alpha_t^{IRH}\right)^\intercal &= -\left(\Lambda^\intercal(t, Y_t)\right)^{-1} E_t\left[\xi_{t,T} \frac{\omega_T}{\omega_t} \left(1 - R(\omega_T)^{-1}\right) I_{\omega_T > 0} H_{t,T}^r\right] \\
\left(\alpha^{MPRH}\right)^\intercal &= -\left(\Lambda^\intercal(t, Y_t)\right)^{-1} E_t\left[\xi_{t,T} \frac{\omega_T}{\omega_t} \left(1 - R(\omega_T)^{-1}\right) 1_{\omega_T > 0} H_{t,T}^\Theta\right]
\end{aligned} \tag{3.22}$$

The terms $H_{t,T}^r$ and $H_{t,T}^\Theta$ involve the sensitivities of the short rate and the market price of risk towards shocks in the Brownian motions that drive uncertainty in the model and are defined as follows:

$$H_{t,T}^{r}=\int_{t}^{T}\mathcal{D}_{t}r_{s}ds=\int_{t}^{T}\partial_{2}r\left(s,Y_{s}\right)\mathcal{D}_{t}Y_{s} \tag{3.23}$$

$$\begin{aligned} H_{t,T}^{\Theta} &= \int_{t}^{T}(dW_{s}+\Theta(s,Y_{s})ds)^{\intercal}\,\mathcal{D}_{t}\Theta(s,Y_{s})ds \\ &= \int_{t}^{T}(dW_{s}+\Theta(s,Y_{s})ds)^{\intercal}\,\partial_{2}\Theta(s,Y_{s})\mathcal{D}_{t}Y_{s}ds \end{aligned} \tag{3.24}$$

where the operator $\mathcal{D}$ is the Malliavin derivative, $\partial_2 f(t,x)$ refers to the derivative with respect of the second argument of $f(t,x)$, and where the second equality was obtained using the chain rule for Malliavin derivatives. For the state variables needed in our application, the Malliavin derivatives are given by:

$$\mathcal{D}_{t}Y_{s}=\begin{pmatrix} \mathcal{D}_{1,t}X_{1,s} & \cdots & \mathcal{D}_{d,t}X_{1,s} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \mathcal{D}_{1,t}X_{d,s} & \cdots & \mathcal{D}_{d,t}X_{d,s} & 0 \\ \mathcal{D}_{1,t}F_{s}^{V} & \cdots & \mathcal{D}_{d,t}F_{s}^{V} & 0 \\ \mathcal{D}_{1,t}F_{s}^{M} & \cdots & \mathcal{D}_{d,t}F_{s}^{M} & 0 \\ 0 & \cdots & 0 & \mathcal{D}_{d+1,t}Y_{s}^{r} \end{pmatrix}=\begin{pmatrix} \mathcal{D}_{t}X_{1,s} \\ \vdots \\ \mathcal{D}_{t}F_{s}^{V} \\ \mathcal{D}_{t}F_{s}^{M} \\ \mathcal{D}_{t}Y_{s}^{r} \end{pmatrix}$$

The implementation of the above formula follows Detemple et al. (2003) and relies on the fact that the Malliavin derivatives, as well as the state variables, follow stochastic differential equations that can be simulated using standard discretization techniques. Given the particular specification of some of the state variables, we can further apply the Doss transformation[4], reducing the stochastic differential equation of the given state variable to one with a constant diffusion term, which ensures that the Malliavin derivative does not involve a stochastic term. Specific solutions for the Malliavin derivative are given in the appendix.

3.5.1 The long term bond and the interest rate hedging demands

Let us first consider the term $H_{t,T}^{r}$ that involves the sensitivity of the short rate towards shocks in the underlying Brownian motions. Recall that $r(s,Y_s)=\delta_0+\delta_1 Y_t^r$, and that the $(d+3)$-dimensional state variable vector, augmented with the observable factors, is defined as $Y\equiv\left(X_1,...,X_d,F^V,F^M,Y^r\right)^{\intercal}$. Thus $\partial_2 r(s,Y_s)=(0,...,0,\delta_1)$, and using the fact that $\mathcal{D}_{d+1,t}Y_s=(0,...,0,\mathcal{D}_{d+1,t}Y_s^r)$, then:

$$H_{t,T}^{r}=\left(0,...,0,\int_{t}^{T}\delta_{1}\mathcal{D}_{d+1,t}Y_{s}^{r}\right)$$

[4] See Detemple et al. (2003) for further details.

So the long term bond is the sole security in the portfolio that is used to hedge against changes in the short rate.

3.6 Correlation hedging

The above portfolio decomposition formula isolates interntemporal hedging demands due to stochastic changes in the short rate or the market price of risk from the mean-variance demand. As in *Cases B* and *C* we have modeled conditional correlation as a function of certain observable factors, the sensitivities of those factors to shocks in the underlying Brownian motions would give rise to hedging demands that can be related (partially for *Case B*) to correlation hedging. As in *Case A* conditional correlation is modeled as a deterministic function of the state variables, determining as well the drift, volatility, and subsequently the market price of risk dynamics, we cannot isolate correlation hedging from the total intertemporal demands in this case. The only way to judge the importance of dynamic correlation modeling for portfolio allocation in this case is to contrast the hedging demands, obtained under a DCC specification with those obtained from a CCC process. We will consider this possibility in the following sections when we consider a real data application.

3.6.1 Isolating the correlation hedging demands involving observable factors

As the primary objective of this study is to explicitly isolate the correlation hedging demands in the portfolio that arise from stochastic changes in the conditional correlation, let us now consider the second term $H^{\Theta}_{t,T}$ in the portfolio decomposition formula that handles the sensitivity of the market price of risk towards shocks in the underlying state variables. Let us define the vector Ψ in terms of the market price of risk and the state variables:

$$\Psi_t = (dW_t + \Theta(t, Y_t)ds)^{\intercal}\, \partial_2 \Theta(t, Y_t)$$

Note that in *Case B* for the conditional correlation specification, where we have augmented the state variables Y to include observable factors $F = \left(F^V, F^M\right)^{\intercal}$, the vector Ψ will be of dimension $(d+3)$. Then we could represent the $H^{\Theta}_{t,T}$ in terms of Ψ_t and the Malliavin derivatives of the state variables as:

$$H^{\Theta}_{t,T} = \int_t^T \Psi_t \mathcal{D}_t Y_s$$

where $\Psi_t \mathcal{D}_t Y_s$ could be further decomposed as follows:

$$
(\Psi_t \mathcal{D}_t Y_s)^\intercal = \begin{pmatrix} \Psi_{1,t} D_{1,t} X_{1,s} + ... + \Psi_{d,t} D_{1,t} X_{d,s} + \Psi_{d+1,t} D_{1,t} F_s^V + \Psi_{d+2,t} D_{1,t} F_s^M \\ \vdots \\ \Psi_{1,t} D_{d,t} X_{1,s} + ... + \Psi_{d,t} D_{d,t} X_{d,s} + \Psi_{d+1,t} D_{d,t} F_s^V + \Psi_{d+2,t} D_{d,t} F_s^M \\ \Psi_{d+3,t} D_{d+1,t} Y_s^r \end{pmatrix}
$$

Apparently, the term $H^{\Theta}_{t,T,d+1}$ corresponding to the bond, does not involve any other Malliavin derivatives except that of the state variable Y^r driving the short rate. As for the interest rate hedge, Y^r will be the only state variable whose sensitivity with respect to uncertainty shocks will determine the market price of risk hedging terms for the long term bond.

For each one of the d stocks the term $H^{\Theta}_{t,T,i}$ can be expressed as:

$$
\begin{aligned}
H^{\Theta}_{t,T,i} &= \int_t^T \Psi_{1,t} \mathcal{D}_{i,t} X_{1,s} + ... + \int_t^T \Psi_{d,t} \mathcal{D}_{i,t} X_{d,s} \\
&\quad + \int_t^T \Psi_{d+1,t} \mathcal{D}_{i,t} F_s^V + \int_t^T \Psi_{d+2,t} \mathcal{D}_{i,t} F_s^M
\end{aligned}
$$

The last two terms in this expression involve the Malliavin derivatives of the observable factors with respect to the Brownian shocks. As those factors are solely responsible for describing the dynamics of the conditional correlation in the process for asset returns, then the term

$$
\begin{aligned}
C^{\Theta}_{t,T,i} &= \int_t^T \Psi_{d+1,t} \mathcal{D}_{i,t} F_s^V + \int_t^T \Psi_{d+2,t} \mathcal{D}_{i,t} F_s^M \\
&= V^{\Theta}_{t,T,i} + M^{\Theta}_{t,T,i}
\end{aligned} \tag{3.25}
$$

can be considered as defining the correlation hedging demands for the stocks arising from the necessity to hedge against changes in the observable factors F. Thus we can isolate the effect of the market-wide volatility factor on correlation through $V^{\Theta}_{t,T,i} = \int_t^T \Psi_{d+1,t} \mathcal{D}_{i,t} F_s^V$, and the effect of the macroeconomic state variables through $M^{\Theta}_{t,T,i} = \int_t^T \Psi_{d+2,t} \mathcal{D}_{i,t} F_s^M$. However, as we have defined the conditional correlation dynamics in (3.17) as been driven as well by the state variables X through the level of the returns, there will be additional hedging demands, associated with the Malliavin derivatives of X, that cannot be disentangled from the rest of the market price of risk hedging demands. We would have this problem in all cases when conditional correlation is modeled as a function of state variables that are not exclusively 'reserved' for driving its dynamics. If to the contrary we believe that correlation is driven solely by observable factors (eg. by setting $\gamma_{ij,2} = 0$ in (3.17)), or by other latent factors that do not enter the specification for the stock prices (3.3) except through correlation itself, then $C^{\Theta}_{t,T,i}$ alone will be responsible for the correlation hedging in the portfolio.

Note as well that in *Case A*, where conditional correlation was defined in terms of only the state variables X that drive the stock price dynamics, the term $C^{\Theta}_{t,T,i}$ is set to zero, but that does not entail zero correlation hedging. It rather means that the correlation hedging demands cannot be explicitly isolated in this case. Nevertheless, their importance can be judged by comparing the hedging terms that arise from a constant conditional correlation stock price process to those that arise from the dynamic conditional correlation specification.

Let us now get back to the portfolio decomposition formula (3.22). Using (3.25) we can now isolate the Market Price of Risk (MPR) hedging terms that arise from hedging changes in the observable factors that drive correlation, that is, the correlation hedging demands:

$$\left(\alpha^{CORR}\right)^{\intercal} = -\left(\Lambda^{\intercal}(t,Y_t)\right)^{-1} E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\left(1 - R\left(\omega_T\right)^{-1}\right)1_{\omega_T>0}C^{\Theta}_{t,T}\right] \tag{3.26}$$

where $C^{\Theta}_{t,T} = \left(C^{\Theta}_{t,T,1}, ..., C^{\Theta}_{t,T,d}\right)$. This defines the explicitly identifiable correlation hedging demand in our setting. It will amount to the full correlation hedging demand for *Case C* when the factors driving correlation do not affect in a direct way the stock price process.

We can restate the above result in terms of the sensitivity of the cost of optimal wealth to changes in the factors driving the conditional correlation dynamics, as the optimal portfolio policy is indeed obtained as one that finances optimal terminal wealth. Recall that optimal wealth at time t is given by $\omega^*_t = E_t\left[\xi_{t,T}\omega^*_T\right]$, where $\xi_{t,T}\omega^*_T = \xi_{t,T}I\left(y\xi_t\xi_{t,T}\right)^+$ represents its cost. Then for a nonnegative $I\left(y\xi_T\right)$ its sensitivity with respect to fluctuations in the observable factors F is given by:

$$\begin{gathered}\left[I\left(y\xi_t\xi_{t,T}\right) + y\xi_t\xi_{t,T}I'\left(y\xi_t\xi_{t,T}\right)\right]\left(-\xi_{t,T}\right)\times \\ \int_t^T \left(dW_s + \Theta(s,Y_s)ds\right)^{\intercal}\partial_2\Theta(s,Y_s)D_tF_s\end{gathered}$$

where we have used (3.12) and the fact that $I'\left(y\right) = \left(u''\left(I\left(y\right)\right)\right)^{-1}$ which follows from the definition of $I\left(y\right)$ as the inverse of the marginal utility. Thus, the portfolio terms that are responsible for the sensitivity of the cost of optimal terminal wealth to fluctuations in the factors are indeed the correlation hedging demands defined in (3.26).

4 A bivariate application: S&P500 vs. NASDAQ

In order to appreciate the impact of the correlation hedging demands on the optimal portfolio composition in a realistic setting and compare them to the intertemporal hedges that arise due to incorporating tail dependence in the stationary distribution of the process for the state variables, driving asset prices, we offer an application based on real data. We consider a portfolio, formed by a 10-year pure discount bond, as well as two risky funds, represented by old and new economy stocks: S&P 500 and NASDAQ. An application with this choice of a dataset can be found in Detemple et al. (2003). Data is observed at the daily frequency (except for the CFNAI factor, which is observed monthly) and refers to the period 1986-2006.

Without loss of generality, we assume that the coefficients in the short rate specification (3.4) are given by $\delta_0 = 0$ and $\delta_1 = 1$, so that for the short rate we have that $r(t, Y_t^r) = Y_t^r$. Given the fact that both the interest rate and the market price of risk of the long term bond are assumed to be stochastic, the optimal portfolio composition for it will involve both the interest rate and the market price of risk hedging terms. For the CIR specification we have chosen there are no closed-form solutions for the hedging terms. Nevertheless, for the simulations of its Malliavin derivatives we can apply a variance stabilization technique following the Doss transformation that renders a constant the diffusion term of the process for Y^r, as explained in the appendix.

The long term bond is the only risky asset that is responsible for hedging away the source of risk related to the short rate (W^r), as it is the only one exposed to it. The optimal demand for the bond involves a mean-variance component and an intertemporal component used to hedge against fluctuations in the investment opportunity set, induced by W^r:

$$\begin{aligned}
\alpha_{b,t}^* &= \frac{1}{\sigma^B(t, Y_t^r)} \left\{ \begin{array}{c} \frac{1}{R(\omega_T)} \Theta^B(t, Y_t^r) E_t\left[\xi_{t,T} \frac{\omega_T}{\omega_t} \frac{R(\omega_t)}{R(\omega_T)} 1_{\omega_T > 0}\right] \\ -E_t\left[\xi_{t,T} \frac{\omega_T}{\omega_t}\left(1 - R(\omega_T)^{-1}\right) I_{\omega_T > 0} H_{t,T}^r\right] \\ -E_t\left[\xi_{t,T} \frac{\omega_T}{\omega_t}\left(1 - R(\omega_T)^{-1}\right) 1_{\omega_T > 0} H_{b,t,T}^{\Theta}\right] \end{array} \right\} \\
\text{where } \sigma^B(t, Y_t^r) &= b(\tau)\, \sigma_r \sqrt{Y_t^r} \\
\text{and } \Theta^B(t, Y_t^r) &= \lambda \sqrt{Y_t^r}
\end{aligned}$$

In this bivariate application the optimal portfolio parts for the two risky funds have a very intuitive representation. As we have assumed that they are not driven by the Brownian that is responsible for interest rate risk, then the diffusion term of the stock price process is a bivariate triangular matrix:

$$\Lambda^{(I)} = \begin{bmatrix} \sigma_1^X(X_t) & 0 \\ \Upsilon(Y_t)\, \sigma_2^X(X_t) & \sqrt{1 - \Upsilon(Y_t)^2}\sigma_2^X(X_t) \end{bmatrix}$$

where $\sigma_i^X(X_t), i = 1, 2$ is given by (3.15) and the conditional correlation $\Upsilon(Y_t)$ is either a function of the state variables X_t in Case A, a function of both the state variables X_t and the observable factors F_t in Case B, or a function of only the observable factors F_t in Case C. Given this diagonal structure for $\Lambda^{(I)}$, for the two stock prices we obtain:

$$
\begin{aligned}
dS_{1t} &= S_{1t}\left\{\mu_1^S\left(X_t\right)dt+\sigma_1^X\left(X_t\right)dW_{1t}^X\right\}\\
dS_{2t} &= S_{2t}\left\{\mu_2^S\left(X_t\right)dt+\Upsilon\left(Y_t\right)\sigma_2^X\left(X_t\right)dW_{1t}^X+\sqrt{1-\Upsilon\left(Y_t\right)^2}\sigma_2^X\left(X_t\right)dW_{2t}^X\right\}
\end{aligned}
$$

Without loss of generality we have assumed a linear function for $\varphi\left(t\right)$ in the general specification in (3.11) given by $k_it, i=1,2$, where k_i is a deterministic trend. Note that the second fund (NASDAQ in our example) is the only one affected by W_2^X-risk, i.e. it can be thought of as the incremental risk factor that influences 'new-economy' stocks. On the contrary, the W_1^X risk factor affects both funds in our portfolio. This has some implications on the optimal portfolio choice. As we will see below, the demand for the second fund is entirely driven by fluctuations induced by exposure to W_2^X-risk. Following the optimal allocation rule outlined in (3.22), the demand for NASDAQ is given by:

$$
\alpha_{2,t}^*=\frac{1}{\sigma_2^X\left(X_t\right)\sqrt{1-\Upsilon\left(Y_t\right)^2}}\left\{\begin{array}{c}\frac{1}{R(\omega_T)}\Theta_2(t,Y_t)E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\frac{R(\omega_t)}{R(\omega_T)}1_{\omega_T>0}\right]\\-E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\left(1-R\left(\omega_T\right)^{-1}\right)1_{\omega_T>0}H_{2,t,T}^{\Theta}\right]\end{array}\right\}
$$

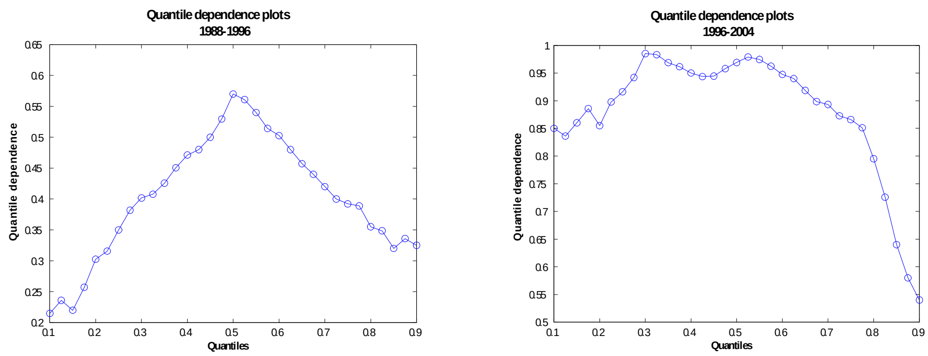
where $\Theta_i(t,Y_t)$ is the market price of risk for the i^{th} fund, and $H_{i,t,T}^{\Theta}$ is the term involving the response to fluctuations in the opportunity set driven by the i^{th} Brownian motion. The absence of the interest rate hedge is due to the fact that the state variable underlying the short rate is not dependent on any of the Brownians driving the risky stocks. The demand for S&P 500 is given by:

$$
\begin{aligned}
\alpha_{1,t}^* &= \frac{1}{\sigma_1^X\left(X_t\right)}\left\{\begin{array}{c}\frac{1}{R(\omega_T)}\Theta_1(t,Y_t)E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\frac{R(\omega_t)}{R(\omega_T)}1_{\omega_T>0}\right]\\-E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\left(1-R\left(\omega_T\right)^{-1}\right)1_{\omega_T>0}H_{1,t,T}^{\Theta}\right]\end{array}\right\}\\
&\quad-\frac{\Upsilon\left(Y_t\right)}{\sigma_1^X\left(X_t\right)\sqrt{1-\Upsilon\left(Y_t\right)^2}}\left\{\begin{array}{c}\frac{1}{R(\omega_T)}\Theta_2(t,Y_t)E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\frac{R(\omega_t)}{R(\omega_T)}1_{\omega_T>0}\right]\\-E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\left(1-R\left(\omega_T\right)^{-1}\right)1_{\omega_T>0}H_{2,t,T}^{\Theta}\right]\end{array}\right\}\\
&= \frac{1}{\sigma_1^X\left(X_t\right)}\left\{\begin{array}{c}\frac{1}{R(\omega_T)}\Theta_1(t,Y_t)E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\frac{R(\omega_t)}{R(\omega_T)}1_{\omega_T>0}\right]\\-E_t\left[\xi_{t,T}\frac{\omega_T}{\omega_t}\left(1-R\left(\omega_T\right)^{-1}\right)1_{\omega_T>0}H_{1,t,T}^{\Theta}\right]\end{array}\right\}-\frac{\Upsilon\left(Y_t\right)\sigma_2^X\left(X_t\right)}{\sigma_1^X\left(X_t\right)}\alpha_{2,t}^*
\end{aligned}
$$

Thus, we can see that for the first fund the optimal portfolio demand has an additional term that involves $\alpha_{2,t}^*$, the optimal holdings of the second fund. It happens because the second fund depends also on W_1^X-risk, so its holding induces also an exposure to it. Consequently, the first fund is used to hedge away this induced exposure, hence the additional term in the optimal portfolio holdings $\alpha_{1,t}^*$. A similar setup with a triangular diffusion term was used in Detemple et al. (2003) in their multiasset application.

Note that the market price of risk hedging demands α^{MPRH} can be decomposed in a similar fashion

Figure 5.1: Quantile dependence plots
Plots of quantile dependence for the de-trended log-prices of S&P 500 vs. NASDAQ for the 1988-1996 and 1996-2004 subperiods.



for the first fund, which will have induced intertemporal hedging demands equal to $-\frac{\Upsilon(Y_t)\sigma_2^X(X_t)}{\sigma_1^X(X_t)}\alpha_{2,t}^{MPRH}$.

5 Numerical Results

Before discussing the estimation results for the various diffusion specifications that we have chosen for the state variables X, let us first look at data itself in order to verify whether the stylized facts that we aim at reproducing are indeed present in the data. In the previous sections we have seen that dynamic conditional correlation, modeled using a DCC model with exogenous factors, is indeed time-varying and we can distinguish periods of relatively high or low correlation, that we were able to attribute to the influence of the macroeconomic or the volatility factor. In a similar fashion, we split the estimation period in two subsamples, one characterized by decreasing and low volatility and improving macroeconomic conditions (1988-1996), and the other characterized by high volatility and declining and relatively low CFNAI index, pointing towards a declining economy (1996-2004). We then construct quantile dependence plots for the de-trended log-prices of both indices for the corresponding subsamples.

As we can see on Figure 5.1, during the first relatively calm period dependence in the extreme quantiles of the joint distribution decreases substantially, even though it does not disappear completely, as one would expect under a Gaussian distributional assumption. As well, a test of tail dependence symmetry, following Hong et al. (2003), does not fail to reject symmetric tails for this particular period, as it can be seen from Table 5.1.

On the other hand, the period of (1996-2004) brings about extremely high dependence in the tail quantiles, especially in the left tail, and the dependence symmetry test indeed rejects symmetric tails for the period. Thus, the unconditional distribution of the two risky funds that we have chosen does possess the features that we try to asses, and namely increased dependence when markets experience extreme

Table 5.1: Test of symmetry in the exceedence correlations

The Hong et al. (2003) test of exceedence correlations symmetry in the lower and upper quartiles for the de-trended log-prices of S&P 500 vs. NASDAQ for the 1988-1996 and 1996-2004 subperiods. The test statistic is given by:

$$J = n\left(\rho^{+} - \rho^{-}\right)\Omega^{-1}\left(\rho^{+} - \rho^{-}\right) \xrightarrow{d} \chi^2_m$$

where ρ^{+} and ρ^{-} are the exceedence correlations calculated at the corresponding quantile levels, n is the sample size and m is the number of quantile levels considered.

	1988-1996	1996-2004
Test statistic (J)	6.9048	21.5517
p-values	(0.4389)	(0.0030)

Table 5.2: Parameter estimates for the observable factors

Estimated parameters for the observable factors VIX and CFNAI that have the following specifications:

$$\begin{aligned} dF_t^V &= \kappa^V\left(\theta^V - F_t^V\right)dt + \sigma^V\sqrt{F_t^V}\,dW_t^X \\ dF_t^M &= \kappa^M\left(\theta^M - F_t^M\right)dt + \sigma^M dW_t^X \end{aligned}$$

where $i = \{V, M\}$.

parameter	$CFNAI$	MC s.e.	SIF	VIX	MC s.e.	SIF
κ^i	2.2521	0.0027	0.8153	1.2094	0.0021	0.8002
θ^i	-0.0457	0.0018	1.7702	2.7800	0.0007	0.8863
$\left(\sigma^i\right)^2$	2.9383	0.0005	0.8631	0.1230	0.0000	1.9260

downturns. Also splitting the sample in two periods with quite distinct characteristics will help us later on to explain the portfolio implications of both conditional correlation and unconditional dependence.

The processes for the observable factors and for the state variables for the risky funds are estimated using Markov Chain Monte Carlo and the Simulation Filter of Golightly and Wilkinson (2006). This estimation methodology is particularly convenient for highly nonlinear multivariate diffusions, as in our case. As well, it allows us to filter out unobservable data points, as is the case of the CFNAI factor, which is observed monthly, whereas the two indices, as well as the VIX factor are observed at the daily frequency. Parameter estimates for the observable factors are given in Table 5.2.

Let us not turn to the estimation results for the whole sample period, as well as the two subsamples for the four conditional correlation specifications (DCC, Cases A through C, and CCC) and the three alternative stationary distribution assumptions (no tail dependent Gaussian, symmetric tail dependent Student's t, and asymmetric tail dependent Gaussian-SJC diffusions). As in this application we aim at

Table 5.3: Univariate parameter estimates

Parameter estimates from the univariate Normal Inverse Gaussian (NIG) diffusions with density $f_{NIG}(x;\theta)$, where $\theta = (\alpha, \beta, \delta, \mu)$ is the vector of NIG parameters that satisfy the restrictions, given in the Appendix. The diffusion for each of the state variables X_{it} has the following specification:

$$
\begin{aligned}
dX_{it} &= b(X_{it};\theta_i)\,dt + v(X_{it};\theta_i)\,dW_{it} \\
\text{where } b(x;\theta) &= \frac{1}{2} v(x;\theta) \frac{d}{dx} \ln\left[v(x;\theta) f_{NIG}(x;\theta)\right] \\
v(x;\theta) &= \sigma^2 f_{NIG}(x;\theta)^{-\kappa}, \qquad \sigma^2 > 0, \kappa \in [0,1]
\end{aligned}
$$

Monte Carlo standard errors, obtained using the batch-mean approach (multiplied by a factor of 1000) and the simulation inefficiency factor (SIF) are reported for each parameter estimate.

parameter	X_1 ($S\&P500$)	MC s.e.	SIF	X_2 ($NASDAQ$)	MC s.e.	SIF
α	5.6431	0.0601	1.0262	4.2938	0.2138	0.8070
β	-0.6272	0.3091	1.1979	-0.7072	0.4151	0.6343
δ^2	0.0471	0.0016	0.7755	0.0549	0.0026	0.8782
μ	4.6342	0.0083	1.0129	5.1191	0.0146	0.6724
σ^2	0.0268	0.0006	0.8375	0.0222	0.0003	0.2821
κ	0.5776	0.0128	1.0339	0.5349	0.0356	1.2291

determining the impact of the stationary distribution and hence tail dependence on the optimal portfolio holdings, regardless of the univariate marginals, we do not proceed to a full-scale optimization of all model parameters, as would be otherwise preferred, but rather undertake a two-step estimation procedure. In a first step, we assume that the two price processes are independent from each other, imposing the independence (or product) copula on their stationary distribution, as well as zero conditional correlation. Thus we are able to estimate them separately, and further use the same marginal distribution parameters for all alternative processes that we consider. In this manner, differences in portfolio demands between the alternative specifications will not depend on the particular parameter choice of the univariate marginals. Parameter estimates are reported in Table 5.3. The trend parameters k_i for each of the state variables X_i are estimated separately as a linear trend. Their values are 0.1014 for S&P 500 and 0.1100 for NASDAQ.

In a second step, we assume the marginal parameters as known and we proceed to the estimation of the multivariate processes by assuming all the alternative specifications for the stationary distribution of the conditional correlation. Results are reported in Table 5.4.

Note that the conditional correlation parameters that pertain to volatility (γ_1) (either observed through the VIX factor or modeled through the state variables X) are generally positive through all the specifications, pointing towards an increase in conditional correlation when there is rise in market-wide volatility. An exception to this is the 1996-2004 period, during which the VIX coefficient is negatively

Table 5.4: Parameter estimates from the multivariate diffusion specifications (1986-2006)

Estimates for the parameters of the stationary density, defined in terms of copula functions, and the parameters governing the correlation dynamics for a bivariate diffusion, defined as:

$$dX_t = \mu(X_t)\,dt + \Lambda(X_t)\,dW_t^X$$

$$\text{where } \Lambda = \begin{bmatrix} \sigma_1 \left[\widetilde{f}^1(x1)\right]^{-\frac{1}{2}\kappa_1} & 0 \\ \Upsilon_{12}(X_t)\,\sigma_2 \left[\widetilde{f}^2(x_2)\right]^{-\frac{1}{2}\kappa_2} & \sqrt{1-\Upsilon_{12}^2(X_t)}\sigma_2 \left[\widetilde{f}^2(x_2)\right]^{-\frac{1}{2}\kappa_2} \end{bmatrix}$$

$$\mu_j = \frac{1}{2} q^{-1} \sum_{i=1}^{2} \frac{\partial(\nu_{ij} q)}{\partial x_i}, \quad j = 1, 2$$

$$\text{and } q(x_1, ..., x_d) \equiv \widetilde{c}(x_1, ..., x_d) \prod_{i=1}^{d} \widetilde{f}^i(x_i)$$

where ν_{ij} are entries of the matrix $\Sigma = \Lambda\Lambda^{\intercal}$, and $q(x_1, ..., x_d)$ is the stationary density of the diffusion, defined in terms of a copula function $\widetilde{c}$ and the NIG marginal densities $\widetilde{f}^i$. Parameter estimates are given for three cases of copulas: Ga refers to the Gaussian copula, $Ga-SJC$ - to the mixture Gaussian-Symmetrized Joe-Clayton copula, and T - to the Student's t copula. The copula parameters are as follows: ρ is the correlation parameter for the Gaussian or the Student's t copula, ν stands for the degrees of freedom of the Student's t copula, τ_U and τ_L are the upper and lower tail dependence parameters of the Symmetrized Joe-Clayton copula, and ω is the weighting parameter in the Symmetrized Joe-Clayton copula. The parameters that describe the correlation dynamics are $\gamma_i, i = 0, ..., 3$, consistent with the specification in (3.16) for Case A, with (3.17) for Case B and with (3.18) for Case C. The Constant Conditional Correlation model in Panel 4 assumes that all correlation parameters are zero but γ_0.

Panel 1. Dynamic conditional correlation (Case A)

param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.4612	0.3126	0.9440	0.4686	0.2022	0.1966	0.4433	0.6026	1.8164
ν	-	-	-	-	-	-	6.4394	2.0178	0.7087
τ_U	-	-	-	0.5179	0.6057	1.2630	-	-	-
τ_L	-	-	-	0.5003	0.5589	1.2407	-	-	-
ω	-	-	-	0.5599	0.7806	1.6945	-	-	-
γ_0	2.0695	0.0126	0.1636	2.0475	0.0292	0.8041	1.9795	0.0454	1.0962
γ_1	0.4430	1.6643	2.4494	0.6850	0.7402	0.4886	1.3272	0.9481	1.3758
γ_2	-1.4731	0.0422	0.5547	-1.2649	0.0721	0.9250	-0.8214	0.0987	1.3498

Panel 2. Dynamic conditional correlation (Case B)

param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.4036	0.3654	0.9608	0.4596	0.7086	1.6656	0.3652	0.2750	1.6598
ν	-	-	-	-	-	-	6.6976	9.2680	1.2306
τ_U	-	-	-	0.4669	0.3453	0.6012	-	-	-
τ_L	-	-	-	0.5178	0.3165	1.1565	-	-	-
ω	-	-	-	0.5513	0.7156	0.7900	-	-	-
γ_0	1.7273	0.0166	0.6051	1.7401	0.0252	0.6578	1.7589	0.0381	1.2715
γ_1	0.0060	0.0126	0.9784	0.0034	0.0062	0.3958	-0.0020	0.0145	0.7090
γ_2	-0.2873	0.0642	0.9762	-0.2745	0.0484	0.6097	-0.4227	0.0806	1.1133
γ_3	-0.3086	0.0263	1.0807	-0.3487	0.0240	0.9340	-0.2944	0.0252	1.3209

Panel 3. Dynamic conditional correlation (Case C)

param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.3734	0.3210	0.3841	0.4984	0.5621	0.9206	0.4146	0.9349	1.8356
ν	-	-	-	-	-	-	6.0105	2.2653	0.4660
τ_U	-	-	-	0.5619	0.3398	0.6210	-	-	-
τ_L	-	-	-	0.4805	0.2818	0.8046	-	-	-
ω	-	-	-	0.4690	0.2544	0.2023	-	-	-
γ_0	1.6288	0.0237	1.0122	1.5920	0.0303	1.7129	1.6122	0.0190	1.1783
γ_1	0.0085	0.0102	0.9935	0.0089	0.0108	0.7495	0.0090	0.0112	2.7264
γ_2	-	-	-	-	-	-	-	-	-
γ_3	-0.2628	0.0394	1.5915	-0.3540	0.0269	0.7109	-0.2510	0.0183	0.6519

Panel 4. Constant conditional correlation

param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.4565	0.2337	1.2678	0.4918	0.3299	1.1136	0.4052	0.2187	0.5737
ν	-	-	-	-	-	-	4.3149	2.5652	1.6841
τ_U	-	-	-	0.5012	0.6331	2.3965	-	-	-
τ_L	-	-	-	0.5801	0.4020	1.6656	-	-	-
ω	-	-	-	0.3816	0.6329	1.4994	-	-	-
γ_0	1.9955	0.0139	1.8733	2.0374	0.0128	0.9472	2.0470	0.0090	0.7893

Table 5.4 (A). Parameter estimates from the multivariate diffusion specifications (1988-1996)

Panel 1. Dynamic conditional correlation (Case A)									
param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.4130	0.5533	0.8635	0.3971	0.6171	0.8538	0.3951	0.5808	0.6071
ν	-	-	-	-	-	-	5.8728	8.5498	1.0732
τ_U	-	-	-	0.4479	0.4510	0.5219	-	-	-
τ_L	-	-	-	0.4685	0.6630	1.2260	-	-	-
ω	-	-	-	0.5147	0.8890	1.6429	-	-	-
γ_0	1.8897	0.0680	0.9762	1.8835	0.0812	0.9175	1.9037	0.1103	1.1216
γ_1	1.7028	3.9466	2.3019	2.4512	5.0493	2.0684	3.4598	5.1660	1.5876
γ_2	-1.7556	0.6689	1.6051	-1.7040	0.3697	0.5851	-1.3860	0.4324	0.9937

Panel 2. Dynamic conditional correlation (Case B)									
param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.4011	0.3787	0.4744	0.3705	0.8203	1.3778	0.4590	0.9433	1.4121
ν	-	-	-	-	-	-	6.0486	6.6124	0.3657
τ_U	-	-	-	0.5159	0.9509	0.9491	-	-	-
τ_L	-	-	-	0.5466	0.7998	0.7297	-	-	-
ω	-	-	-	0.5258	1.4451	1.5272	-	-	-
γ_0	2.1724	0.0426	0.4523	2.1661	0.0716	1.0785	2.1827	0.0425	0.5952
γ_1	0.0102	0.0207	1.1832	0.0079	0.0185	0.5042	0.0112	0.0209	0.8377
γ_2	-0.7282	0.3580	1.2605	-0.9716	0.2619	0.5328	-0.7620	0.2754	1.0575
γ_3	-0.2691	0.1471	1.2750	-0.2887	0.1229	0.7116	-0.2734	0.1109	0.9831

Panel 3. Dynamic conditional correlation (Case C)									
param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.4111	0.5567	0.5478	0.3633	1.4273	1.2948	0.3155	0.5125	0.5295
ν	-	-	-	-	-	-	5.3833	6.6008	1.1860
τ_U	-	-	-	0.6179	0.4655	0.3730	-	-	-
τ_L	-	-	-	0.4446	1.2408	1.4057	-	-	-
ω	-	-	-	0.5042	0.8104	0.6855	-	-	-
γ_0	2.1615	0.0277	0.4684	2.1441	0.0629	1.7323	2.1550	0.0431	1.2044
γ_1	0.0046	0.0294	2.3139	0.0127	0.0192	1.2285	0.0154	0.0159	1.1747
γ_2	-	-	-	-	-	-	-	-	-
γ_3	-0.3223	0.1225	1.9323	-0.2987	0.0682	0.5600	-0.3104	0.1126	2.1642

Panel 4. Constant conditional correlation									
param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.3348	0.5310	0.7963	0.4497	0.5185	0.7457	0.3677	0.8407	1.5087
ν	-	-	-	-	-	-	5.5060	8.8090	1.9514
τ_U	-	-	-	0.5447	1.0077	1.2661	-	-	-
τ_L	-	-	-	0.5016	1.1308	1.7278	-	-	-
ω	-	-	-	0.5765	0.8678	0.9065	-	-	-
γ_0	1.7174	0.0585	1.8229	1.6532	0.0460	0.9813	1.6437	0.0397	1.0072

Table 5.4 (B). Parameter estimates from the multivariate diffusion specifications (1996-2004)

Panel 1. Dynamic conditional correlation (Case A)

param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.5637	0.7203	1.2560	0.5274	0.7029	0.5754	0.3722	0.7644	0.5408
ν	-	-	-	-	-	-	4.5172	4.7443	0.6594
τ_U	-	-	-	0.5158	1.1290	1.1144	-	-	-
τ_L	-	-	-	0.4926	0.6007	0.3596	-	-	-
ω	-	-	-	0.4565	0.7126	1.2339	-	-	-
γ_0	1.4097	0.0483	0.6112	1.3723	0.0702	1.0511	1.3127	0.0621	0.6209
γ_1	2.3400	1.0788	1.0603	2.6907	1.2589	0.8808	2.6206	0.6113	0.4612
γ_2	-0.2872	0.1821	0.8152	-0.3649	0.1190	0.3422	-0.1736	0.1280	1.7100

Panel 2. Dynamic conditional correlation (Case B)

param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.5380	0.6157	0.5776	0.5383	1.0569	0.6704	0.3368	0.7195	0.8666
ν	-	-	-	-	-	-	4.4252	8.1150	1.5499
τ_U	-	-	-	0.5093	0.4792	0.2987	-	-	-
τ_L	-	-	-	0.5322	0.9009	1.5219	-	-	-
ω	-	-	-	0.5023	0.7294	1.1890	-	-	-
γ_0	1.9191	0.1318	2.0517	1.9198	0.0576	0.4783	1.7604	0.0689	0.6817
γ_1	-0.0134	0.0221	0.5537	-0.0034	0.0157	0.4258	-0.0083	0.0232	0.5074
γ_2	-0.7266	0.1758	0.8608	-0.7292	0.2284	1.6010	-0.6427	0.1392	0.5934
γ_3	-0.0825	0.0983	0.5140	-0.1403	0.0834	0.6450	-0.0741	0.0710	0.4916

Panel 3. Dynamic conditional correlation (Case C)

param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.5892	0.8027	1.4944	0.4499	0.7358	0.7108	0.3368	0.7195	0.8666
ν	-	-	-	-	-	-	4.4252	8.1150	1.5499
τ_U	-	-	-	0.5475	0.5474	0.7226	-	-	-
τ_L	-	-	-	0.4939	0.6894	0.7287	-	-	-
ω	-	-	-	0.6078	0.7941	0.6022	-	-	-
γ_0	1.7373	0.1156	0.9672	1.7783	0.0269	0.4362	1.7604	0.0689	0.6817
γ_1	-0.0341	0.0211	0.5716	-0.0215	0.0074	0.2996	-0.0083	0.0232	0.5074
γ_2	-	-	-	-	-	-	-0.6427	0.1392	0.5934
γ_3	-0.2906	0.1588	1.5428	-0.3344	0.0732	0.9997	-0.0741	0.0710	0.4916

Panel 4. Constant conditional correlation

param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF	T	MC s.e.	SIF
ρ	0.3533	0.5154	0.7736	0.3853	1.4276	0.6995	0.3981	0.5216	0.3485
ν	-	-	-	-	-	-	6.0479	3.9435	0.2350
τ_U	-	-	-	0.5242	0.7559	0.9003	-	-	-
τ_L	-	-	-	0.5091	0.7778	0.7893	-	-	-
ω	-	-	-	0.5142	0.9299	0.7847	-	-	-
γ_0	1.1262	0.0668	1.1726	1.1751	0.0473	0.6244	1.1200	0.0567	0.9024

estimated for all stationary distributional assumptions. However, γ_1 has the expected positive sign for the conditional correlation specification with no observable factors. On the other hand, the parameter pertaining to the macroeconomic factor (γ_3) is always negatively estimated, pointing towards a decrease in conditional correlation when there is an improvement in macroeconomic conditions, and vice versa.

5.1 Correlation hedging demands along realized paths of the state variables

In order to examine the evolution of the portfolio hedging demands for the estimation period, we proceed to a market timing exercise that consists in simulating ahead the Malliavin derivatives of the state variables, the state price density, as well as the portfolio terms involving hedging against changes in the interest rate (3.23) and the market price of risk (3.24), while keeping the state variables (the latent variables and the observable factors) at their observed values throughout the period[5]. First, we obtain the optimal portfolio terms for the whole period between 1986-2006 for an investor with a constant, moving-window horizon of 4 years. With this we aim at studying differences between the optimal portfolio parts for the alternative specifications considered above for modeling unconditional or conditional dependence, without any influence of the time horizon. Next, we consider an investor who keeps her investment horizon fixed at the end of the period, thus investigating the horizon effect on the optimal portfolio shares.

As during this relatively long 20 year horizon one can distinguish hectic periods, associated with high volatility, negative CFNAI, pointing towards a slow-down in the economy, and subsequently rising conditional correlation, as well as relatively calm periods with low volatility, mostly positive levels of the CFNAI index and thus low conditional correlation, we proceed to a second market timing experiment, considering instead two subperiods of 8 years. The first one spans between 1988 and 1996 and is characterized by increased volatility and a recession in the US economy in the beginning of the period (between July 1990 and March 1991, as determined by NBER), followed by improving macroeconomic conditions (positive and rising CFNAI), as well as relatively low and declining volatility. As it can be seen on Figure 2.2, this period is characterized by falling dynamic conditional correlation. On the other hand, the second period, spanning between 1996 and 2004 is characterized by increased volatility for the whole period, a recession towards the end of the period (March 2001 marks the end of a 10-year expansion period, according to NBER, and there is a trough in business activity in November 2001). Figure 2.2 shows a rising trend in the dynamic conditional correlations for the period. For both subperiods we consider an investor who has a fixed investment horizon at the end of each period.

[5] As the CFNAI index is observed at a monthly frequency, we filter the unobservable data points at the daily frequency using the MCMC sequential filter.

5.1.1 Correlation hedging for the whole estimation horizon

For the first market timing experiment that involves a 20-year investment horizon fixed at the end of the sample, we consider the three cases of modeling the unconditional distribution of the state variables underlying the price processes (non-tail dependent Gaussian, symmetric tail dependent Student's t and asymmetric tail dependent Gaussian-SJC mixture distribution), as well as the three ways to account for dynamically changing conditional correlation with or without observable factors driving it. The same experiment is repeated, but with a moving-window horizon of 4 years. Thus we are able to distinguish the horizon effect in the evolution of the optimal portfolio hedging demands from the effect of the dynamically changing investment opportunity set.

In order to get an impression of the magnitude and the variability of the hedging demands for the risky assets in the portfolio, let us first consider the results displayed on Figure 5.2 for a HARA investor with varying degrees of relative risk aversion[6]. The intertemporal hedging demands are a sizeable component of the total portfolio, and they are responsible for a larger portion of the portfolio demands if we increase the level of relative risk aversion of the investor. As well, the hedging demands are larger for longer horizons: an investor with a horizon fixed at the end of the 20-year sample period would have higher hedging demands at each period of time than an investor who has a short rolling-window horizon (4 years in our case). Also the fixed horizon would cause the hedging demands to shrink as we approach it (it is visible during the last 4 years on the left column of Figure 5.2), so that the Mean-Variance component would be increasingly more important in the total portfolio holdings. The results there are based on a Gaussian-SJC diffusion with dynamic correlation driven by observed factors (Case B), but the relative importance of the hedging demands for the other cases is qualitatively the same.

Let us now turn to the results for the differences in the hedging demands of the two risky stocks in the portfolio due to the unconditional dependence structure (through the stationary distribution of the process for the state variables underlying stock prices) and due to the dynamics of conditional correlation. On Figure 5.3, Panel A we have plotted the correlation hedging demands due to observable factors (CFNAI and VIX) that we have isolated following (3.26) for an investor with a fixed horizon at the end of the sample period (left column) and an investor with a rolling-window horizon (right column). On Figure 5.3, Panel B we can see the relative importance of the correlation hedging terms due to each one of the factors for the same 20-year investment horizon. The hedge due to the macroeconomic factor is generally negative, reducing the total portfolio demand, while the hedging term due to volatility is positive but very small in absolute value, compared to the CFNAI hedge.

[6] As the long term bond is the only security in the investor's portfolio that is responsible for hedging interest rate risk and as the Brownian motion driving the short rate is independent of the Brownian motions driving the rest of the state variables, and that the short rate does not enter the stock price dynamics, the hedging terms for the risky assets consist solely of market price of risk hedges. Due to the chosen specification of the market price of risk of the long term bond, it has a negative market price of risk hedging term, and a positive interest rate hedge.

Figure 5.2: Total portfolio holdings and intertemporal hedging demands for the two risky stocks over the entire sample

The figure displays the holdings of the two risky stocks in the portfolio for the entire sample period 1886-2006. The total holdings are contrasted with the intertemporal hedging demands, which for the stocks are entirely given by the market price of risk hedges. The figure on the left represents the portfolio holdings for a fixed investment horizon at the end of the 20-year sample. The figure on the right represents the holdings for a moving-window 4-year horizon. The two top figures concern a HARA investor with relative risk aversion of 5, while the bottom two - a HARA investor with relative risk aversion of 10. The data generating process is a Gaussian-SJC diffusion with dynamic correlation (Case B).

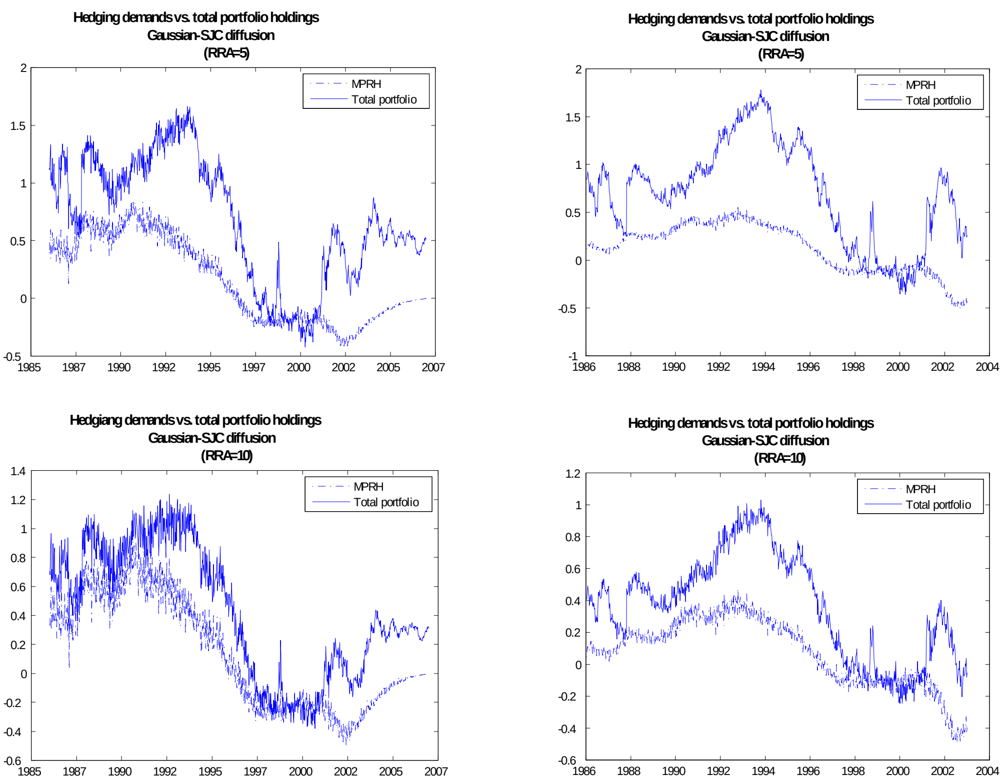


Figure 5.3: Correlation hedging demands due to observed factors

Panel A. The figure displays the sum of the hedging demands due to observed factors (CFNAI and VIX) driving conditional correlation for the two risky stocks in the portfolio for the entire sample period 1886-2006. The figure on the left represents the correlation hedging demands for a fixed investment horizon at the end of the 20-year sample. The figure on the right represents the correlation hedging demands for a moving-window 4-year horizon. HARA investor with relative risk aversion of 5. The data generating process is a Gaussian-SJC diffusion with dynamic correlation (Case C).

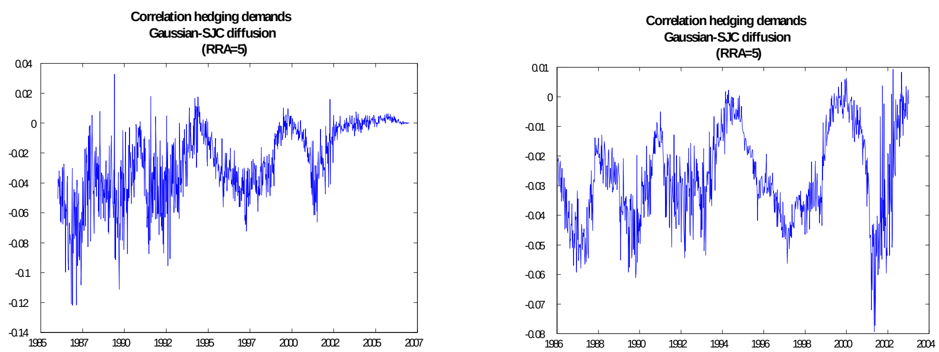
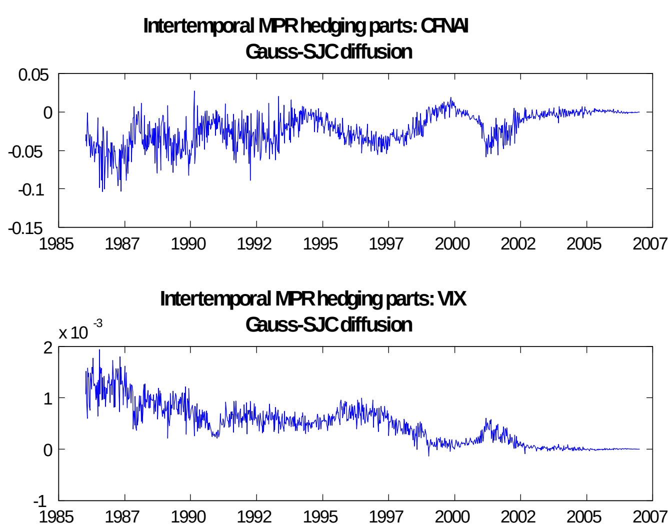


Figure 5.3. Panel B. The figure displays the hedging demands due to observed factors driving conditional correlation for the two risky stocks in the portfolio for the entire sample period 1886-2006. The top figure represents the demands due to hedging changes factor that proxies the macroeconomic conditions (CFNAI), while the bottom figure represents the correlation hedging demands due to the factor that proxies market volatility (VIX). HARA investor with relative risk aversion of 5. The data generating process is a Gaussian-SJC diffusion with dynamic correlation (Case B).



The magnitude of these correlation hedging components is quite small compared to the total hedging demands on Figure 5.2. They are negative in sign, pointing towards a reduction in the total portfolio holdings. One can as well distinguish periods with peaks in the absolute value of the correlation hedging demands, that can be attributable to some market events (e.g. the market crashes in 1987, 1990-1992, 2001). Those demands are also higher for longer investment horizon, which can be seen by comparing the holdings of the investor with a fixed vs. rolling-window shorter horizon, and they decline to zero as we approach the investment horizon. The results are obtained for the dynamic correlation specification following Case C, that is the case when only the VIX and the CFNAI indices drive conditional correlation. Results for the Case B, as well as Gaussian or the Student's t diffusion are qualitatively the same and are not reported for brevity.

Those hedging demands arise in order to hedge against stochastic changes in the observable factors that proxy volatility or the macroeconomic conditions, and they constitute the total correlation demands in Case C, where the dynamics of conditional correlation are not driven by other state variables. However, as we consider the case of conditional correlation being dependent as well on the level of the state variables X (Case B), then there would be another component in the correlation hedging demands apart from the influence of the factors that is not directly identifiable. In order to gauge its importance, we compare the intertemporal market price of risk hedging parts for a process with dynamic vs. constant conditional correlation. Figure 5.4 reports the results for an underlying Gaussian and a Gaussian-SJC diffusion for a fixed investment horizon at the end of the sample period.

The presence of dynamically varying conditional correlation asks for an increase in the intertemporal hedging demands for the Gaussian diffusion, which is mainly driven by NASDAQ, while the hedging demands for S&P 500 are virtually unchanged. At first sight these results are surprising given the evidence that correlation hedging demands due to observable factors for both fixed and rolling window horizon are negative throughout the period, so that we would expect a reduction in the total intertemporal hedging terms for the dynamic conditional correlation case compared to the terms under constant conditional correlation. However, the influence of dynamic correlation does not show up in the correlation hedging term (3.25) only through the Malliavin derivatives of the factors. It influences as well the market price of risk $\Theta(t, Y_t)$, which determines the total market price of risk hedging demands. So while the portfolio term that is due to the need to hedge against stochastic changes in the observable factors driving correlation is indeed correlation hedging demand, the difference in the level of the market price of risk hedge terms between dynamic and constant conditional correlation is not entirely explained by this demand. Hence the possible disparity, even in sign, between the correlation hedging demands and the difference in the level of market price of risk hedges between constant and dynamic conditional correlation diffusions.

It is also of interest to contrast the differences in hedging demands due to dynamic correlation to those due to dynamic volatility, so we have reported on the right column of Figure 5.4, Panel A the

Figure 5.4: Hedging demands along realized paths for the risky stocks for the 20-year fixed investment horizon (Case B)

Plotted are the intertemporal demands (separately for each risky fund and their sum) along realized paths of the state variables for the whole sample period for the risky stocks for a fixed investment horizon at the end of the period The left column plots the intertemporal hedging demands obtained under a DCC specification vs. those under CCC; the right column contrasts hedging terms under constant and time-varying volatility.

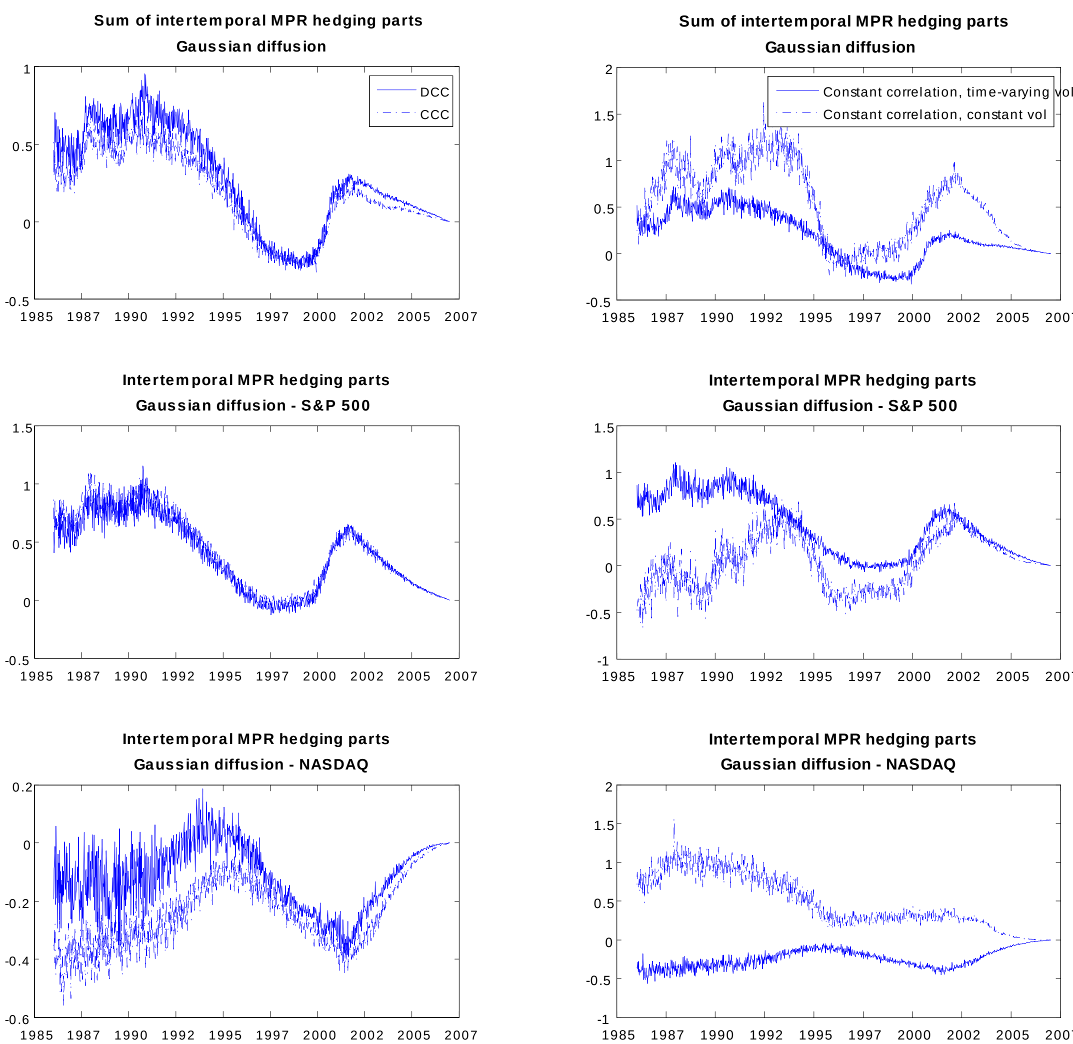
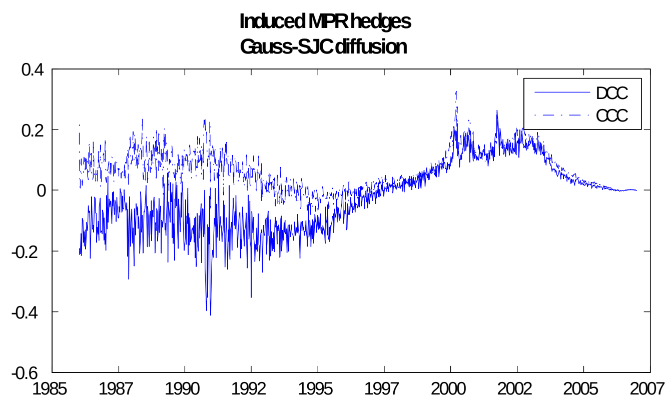


Figure 5.4. Panel B. Induced hedging demands (Case B)



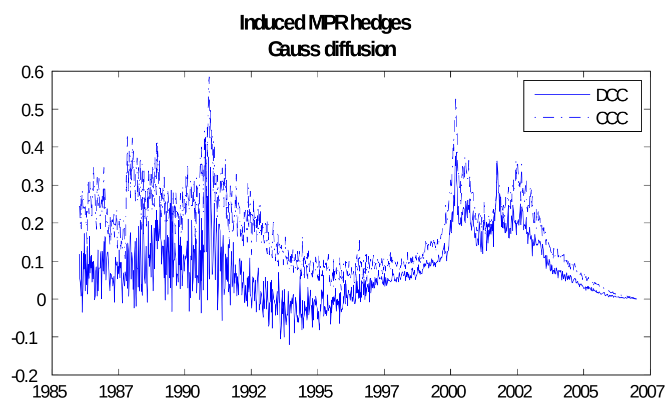
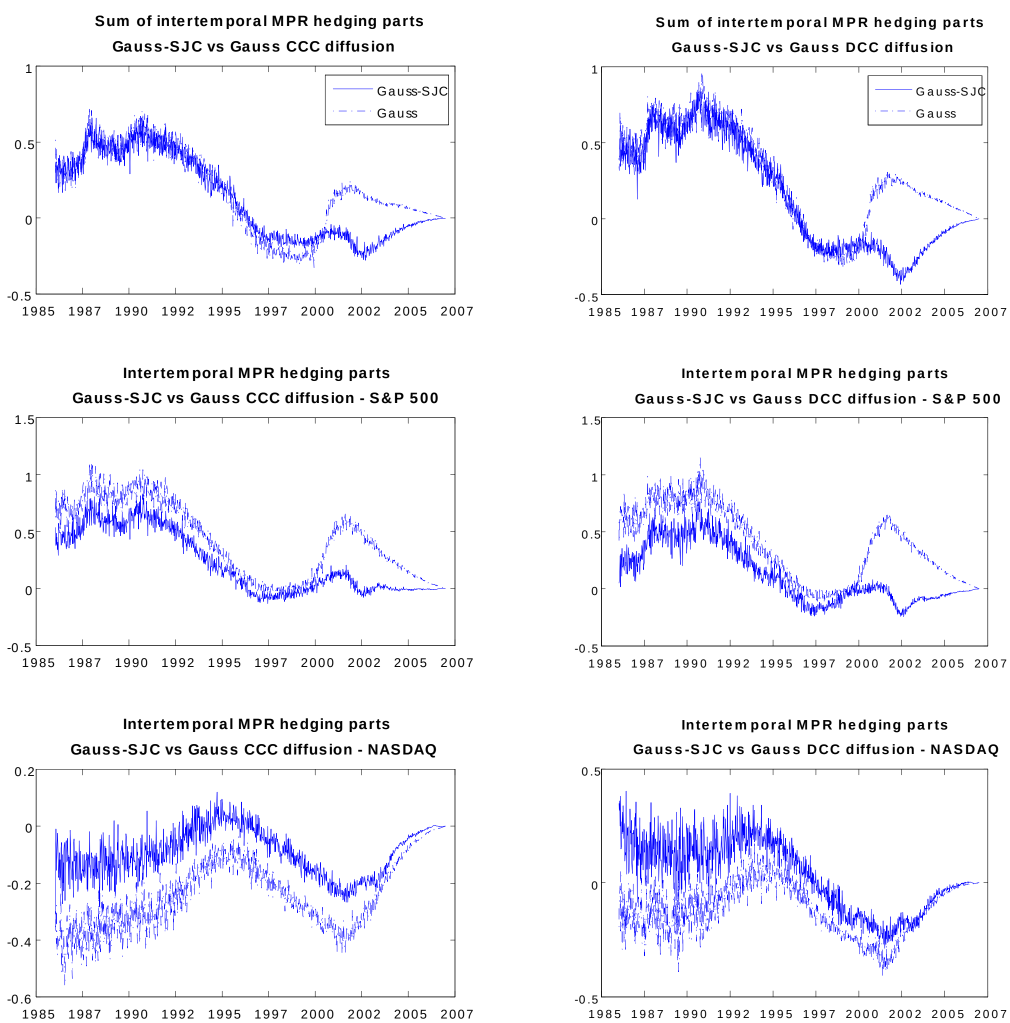


Figure 5.4. Panel C. Hedging demands due to differences in the unconditional distribution (tail dependence vs. no tail dependence) for the risky stocks for the 20-year fixed investment horizon for a CCC diffusion (left column) and a DCC diffusion (right column) (Case B)



results for a process for which we have assumed constant volatility and correlation (note that constant volatility is nested in the specification given in (3.15) and is achieved by setting the parameter κ to zero). Throughout the sample period the hedging demands for the constant volatility model are significantly higher than those with time-varying volatility, rendering the volatility effect much more pronounced than the effect of conditional correlation. The effect is qualitatively the same for a fixed and a rolling-window investment horizon. Unlike the correlation hedging demands, the hedging parts for the S&P 500 are increased when we allow for variations in volatility, while those of NASDAQ are significantly reduced for the whole period.

An alternative way to illustrate the importance of dynamically changing correlation on intertemporal hedging demands is to look at the induced portfolio holdings of S&P 500 from the position in NASDAQ, as explained in the previous section. On Panel B of Figure 5.4 we have plotted the induced MPR hedging demands for S&P 500 for a HARA investor with a 20-year investment horizon. We contrast the induced hedges for a DCC vs. a CCC model under two alternative unconditional distribution assumptions (Gaussian and Gaussian-SJC)[7]. Regardless of the form of the stationary density that we suppose, the induced hedging demands are lower for the DCC case then for the CCC one, pointing towards a reduction in the total portfolio holdings when dynamics of conditional correlation are explicitly accounted for.

Until now we have discussed the magnitude and sign of the hedging demands that arise due to stochastic changes of the state variables driving conditional correlation which increases in down markets, volatile periods or bad states of the economy. An important question is whether there would be a similar shift in portfolio composition when the unconditional dependence structure is changed, that is the same stylized fact is reproduced through the stationary distribution of the process for the state variables X through a Gaussian copula (no tail dependence) or Gaussian-SJC copula (asymmetric tail dependence). On Figure 5.4, panel C we have reported the hedging demands of a Gaussian vs. a Gaussian-SJC diffusion under a CCC assumption, and the hedging demands of a Gaussian vs. a Gaussian-SJC diffusion under a DCC assumption for an investment horizon fixed at the end of the 20-year period. The presence of tail dependence changes the composition of the portfolio by reducing the absolute value of the intertemporal hedging terms. The latter are generally positive for S&P 500 and generally negative for NASDAQ, so tail dependence reduces in absolute value the holdings of both assets, driving them closer to zero. This result is maintained throughout the investment horizon, regardless of the way conditional correlation is modeled. Thus, for portfolio allocation, the impact of tail dependence through the unconditional distribution cannot be swept away by allowing conditional correlation to vary through time, rising in down markets.

The effect of tail dependence is somewhat subdued for the sum of the intertemporal hedges for both assets for the first half of the sample period, while towards the end of the period, mainly after 2000, the

[7] Here we have reported results for DCC following Case B. All alternative cases of DCC were considered against the CCC model, and they all yield qualitatively similar results.

effect is more pronounced in the sense that the total intertemporal hedging demands are reduced for the case where we allow for tail dependence. It appears that for different subperiods of this relatively long sample hedging demands may have qualitatively different behavior. In order to gather more insight into the reasons behind differences in those demands, we concentrate our attention on two 8-year subperiods: one relatively calm in the sense of diminishing volatility, economy on the rise, low conditional correlation (1988-1996), and another period characterized by more hectic behavior in terms of high volatility, declining economic indicators and increased conditional correlation (1996-2004).

5.1.2 Correlation hedging for the two subperiods

Comparing the intertemporal hedging demands on Figure 5.5 and 5.6 for each one of the two subperiods, regardless of the assumptions we have made on the conditional correlation or the unconditional distribution, we see that those demands are generally positive throughout the first relatively calm period of economy on the rise and generally negative for the second hectic period of slowing down economy. There is just one exception to this rule that deserves attention - the hedging demands turn positive towards the second half of the 1996-2004 period for the Gaussian diffusion for both constant and dynamic specifications for the conditional correlation. Thus, failing to account for tail dependence increases the demand for the two risky funds and the fact that we allow for dynamically varying conditional correlation does not change this. It appears, following this preliminary observation, that unconditional dependence has a portfolio impact beyond the one induced by correlation hedging.

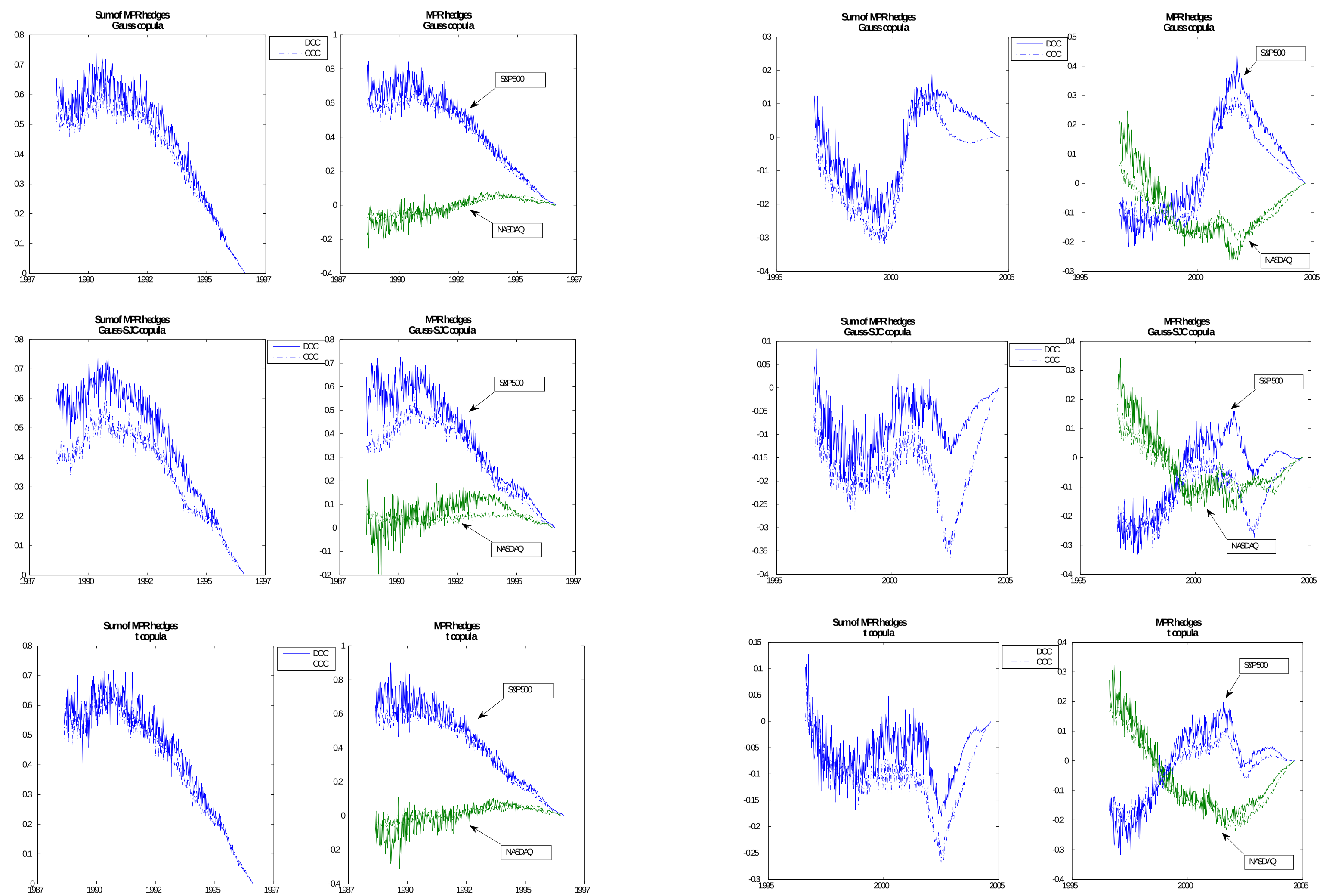
We now turn to a more detailed analysis of the portfolio implications of modeling conditional or unconditional dependence. The first comparison that we consider for the two chosen subperiods is one that is aimed at bringing forward the importance of correlation hedging through contrasting the intertemporal demands for the risky funds under a constant vs. a dynamic conditional correlation specification (for any of the three cases considered). To this end, we have plotted on Figure 5.5 the evolution of the hedges for a Gaussian, Gaussian-SJC and a t-diffusion for a HARA investor with a coefficient of relative risk aversion of 5.

For any of the unconditional distribution assumptions during the 1988-1996 period the presence of dynamically varying conditional correlation brings about increased hedging demands. When looking at the individual demands for any of the risky funds, we find that under the DCC assumption those demands are larger in absolute value, generally positive for S&P 500 and generally negative for NASDAQ. During the 1996-2004 period dynamic conditional correlation also leads to higher demands in absolute value for both funds, but the effect on the total hedging demands is more pronounced in the case when conditional correlation depends both on observable factors F and the level of the state variables X (Case B). In this case dynamic correlation leads to an increase in the total hedging demands. Results for conditional correlation specifications under Case A and C are qualitatively the same and are not reported for brevity.

Figure 5.5: Market price of risk hedging terms for the two subperiods - effect of the conditional correlation

Panel A. (Case B).

The figure displays the intertemporal hedging demands of a HARA investor with RRA coefficient of 5 and an investment horizon fixed at the end of the period. The results of a DCC specification (Case B) are plotted against those coming from a CCC specification. All alternative unconditional distributions are considered (Gaussian, Gaussian-SJC, Student's t). The two left columns plot the results for the 1988-1996 period, the two right ones - the results for the 1996-2004 period.



Second, we consider the effect of the unconditional distribution on the hedging demands by comparing the results under the assumption of Gaussianity with those under the two alternatives of allowing for tail dependence - a Gaussian-SJC or a Student's t distribution. With this we aim to determine whether there is any portfolio effect induced by different assumptions on modeling tail dependence beyond the one incurred by dynamic conditional correlation.

On Panel A of Figure 5.6 we have plotted the hedging demands of a HARA investor with a relative risk aversion coefficient of 5 who models the stock price process using a Gaussian vs. a Gaussian-SJC diffusion (the effect of disregarding tail dependence) or alternatively a Student's t vs. a Gaussian-SJC diffusion (the effect of disregarding asymmetric tail dependence). In all cases we have constant conditional correlation. Contrary to the results on Figure 5.5 which tried to gauge the importance of modeling conditional correlation, here we have the opposite impact of the presence of tail dependence: it leads to smaller hedging demands in absolute value for both risky funds which reduces the total intertemporal demands for the risky assets. Those differences are more pronounced during the 1996-2004 period, and they are quite significant when the investor disregards tail dependence by assuming a Gaussian diffusion (in this case hedging demands grow to be positive in the second half of the period, whereas accounting for tail dependence both through the Gaussian-SJC and the t-diffusions leads to negative hedges).

However, when we allow for dynamically varying correlation some interesting results follow. Looking at Panel B on Figure 5.6, the large difference between the alternative unconditional distribution assumptions seems to vanish for the first subperiod. Allowing or not for tail dependence leads to virtually the same hedging demands. So, for this relatively calm period of improving economic conditions towards its end the presence of tail dependence does not lead to any significant change in the portfolio composition beyond the impact of correlation hedging. Still, the picture for the second highly volatile period is quite different. Accounting for tail dependence still leads to a decrease in absolute terms of the hedging components for both risky funds which generally leads to a decrease in the total hedging demand, especially for the Gaussian case. Thus, for a volatile period of deteriorating economic conditions tail dependence has a significant impact on the portfolio composition, even when dynamic conditional correlation has been accounted for.

5.2 Simulations

Having examined the distinct ways that dynamic conditional correlation or tail dependence influence the optimal portfolio decisions for a particular period and for realized paths of the state variables, we now turn to a simulations experiment that determines optimal portfolio shares for varying investment horizons while simulating ahead all the state variables involved. With this we aim to determine whether for the estimated parameters of the corresponding processes the relative importance of conditional and unconditional dependence on portfolio hedging demands will remain qualitatively the same as with the

Figure 5.6: Market price of risk hedging terms for the two subperiods - effect of the unconditional distribution

Panel A. (CCC).

The figure displays the intertemporal hedging demands of a HARA investor with RRA coefficient of 5 and an investment horizon fixed at the end of the period. The results of a Gaussian-SJC diffusion are plotted against those coming from a Gaussian diffusion (first row), or from a Student's t diffusion (second row). All alternative conditional correlation specifications come from the CCC model according to Case A. The two left columns plot the results for the 1988-1996 period, the two right ones - the results for the 1996-2004 period.

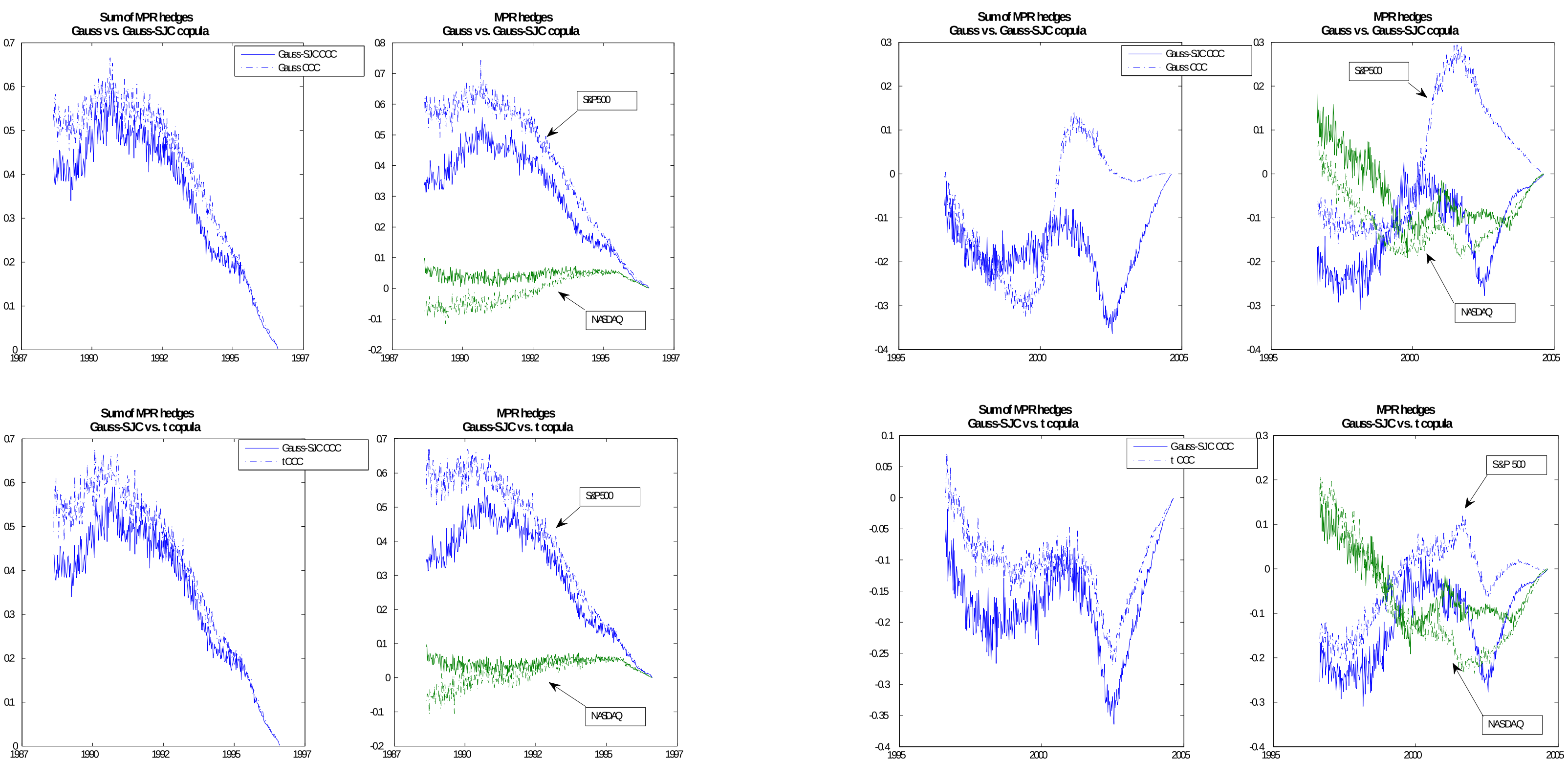
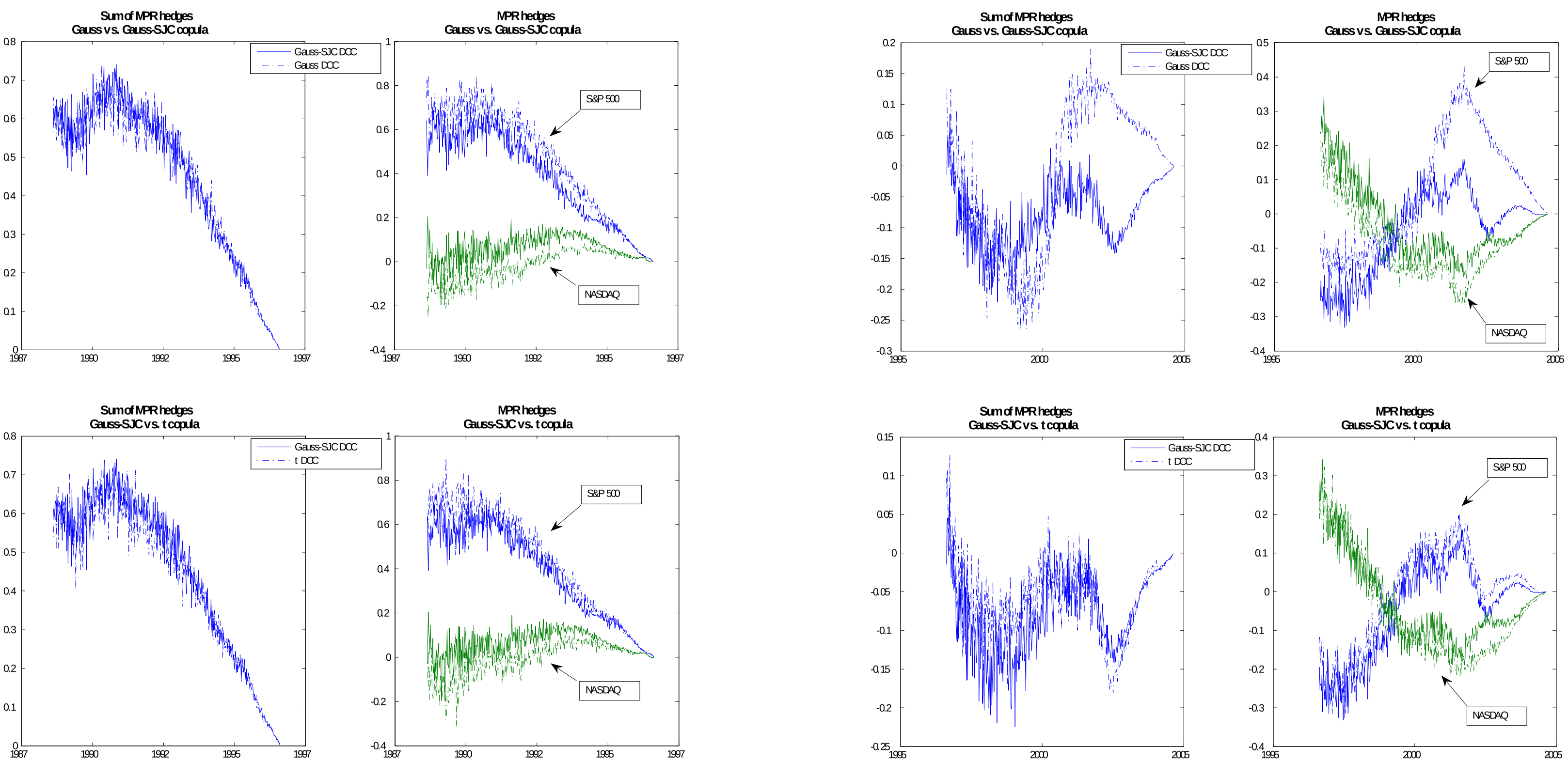


Figure 5.6. Panel B. (DCC, Case B)

The figure displays the intertemporal hedging demands of a HARA investor with RRA coefficient of 5 and an investment horizon fixed at the end of the period. The results of a Gaussian-SJC diffusion are plotted against those coming from a Gaussian diffusion (first row), or from a Student's t diffusion (second row). All alternative conditional correlation specifications come from the DCC model according to Case B. The two left columns plot the results for the 1988-1996 period, the two right ones - the results for the 1996-2004 period.



historical data considered.

Thus, we set up a first simulations exercise that aims at determining the importance of correlation hedging demands for a HARA investor who already believes that the process underlying stock prices has asymmetric tail dependence, incorporated through the Gaussian-SJC diffusion. Then we alternate the way to model conditional correlation by letting it be either constant or dynamic. In this way we can analyze the correlation hedging demands that arise beyond those that could be attributed to tail dependence through the unconditional distribution. We use the parameters estimated from a Gaussian-SJC process with DCC following Case B for the whole estimation period as a benchmark. Then, in order to obtain a CCC model, we set all parameters, driving conditional correlation, to zero, except for γ_0. We calibrate this parameter in order to reflect the same average correlation throughout the estimation period as the one implied by the benchmark process. In order to gauge the relative importance of adding each one of the observable factors to the dynamic correlation specification, we alternatively set either γ_1 (the VIX coefficient) or γ_3 (the CFNAI coefficient) to its corresponding value from the benchmark process, while setting all the other parameters to zero except γ_0 that is again calibrated in order to reflect the same average correlation. We then simulate ahead all the state variables involved in each of the four alternative processes, as well as their Malliavin derivatives, in order to obtain the Monte Carlo estimates of their conditional expectations in (3.22) and thus the intertemporal hedging demands. Results for investment horizons of 1 and 5 years are reported in Table 5.5, Panels A through C and Panel F.

The major conclusion that we may draw from those results is that for all investment horizons considered, as well as for all degrees of relative risk aversion, the market price of risk hedge for the DCC model is the lowest. If we add only the macroeconomic factor to render conditional correlation dynamic, we get results that are quite close to the benchmark model. So for this application the macroeconomic factor seems to be the major driving force to determine the optimal portfolio composition. However, adding only the VIX factor does not change in any substantial way the portfolio holdings and they remain virtually unchanged with respect to the CCC model. As in the portfolio allocation example along realized paths of the state variables, here we also observe a larger spread between the holdings of S&P 500 and NASDAQ for the DCC case with respect to CCC. These results are confirmed for a CRRA as well as HARA investor and are valid for all investment horizons considered, as well as levels of risk aversion. Increasing the level of risk aversion invariably leads to a decrease in the intertemporal hedging demands in absolute terms. It also happens for a HARA investor with a certain subsistence level b below which she is unwilling to fall as compared to a CRRA investor.

A second simulations experiment that we consider aims at determining the importance of the stationary distribution and hence tail dependence for an investor who has already accounted for dynamically varying conditional correlation. We pick again the Gaussian-SJC diffusion with DCC according to Case B as the benchmark case and compare its implied hedging demands with those from a Gaussian or a

Table 5.5: Portfolio hedging terms through simulation

Intertemporal hedging demands for the 2 funds with different specifications for the conditional dependence through the conditional correlation (Gaussian-SJC diffusion with DCC modeled following Case B (observed factors and latent variables). For each investment horizon of 1 and 5 years the table displays the sum of market price of risk hedges (MPRH), the separate demands for S&P 500 and NASDAQ, as well as the two correlation hedges (CorrH), corresponding to the macroeconomic factor F^M and the volatility factor F^V (the latter multiplied by 100).

Panel A. Gaussian-SJC CCC diffusion ($\gamma_1 = \gamma_2 = \gamma_3 = 0$)										
Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.1056	0.0614	0.0442	-	-	0.8623	0.6588	0.2035	-	-
CRRA, γ=10	0.0643	0.0370	0.0273	-	-	0.7341	0.5807	0.1534	-	-
HARA, γ=5,b=-0.2	0.0873	0.0505	0.0368	-	-	0.8201	0.6315	0.1885	-	-
HARA, γ=10,b=-0.2	0.0539	0.0307	0.0232	-	-	0.7113	0.5661	0.1451	-	-
Panel B. Gaussian-SJC DCC diffusion with only VIX driving conditional correlation ($\gamma_2 = \gamma_3 = 0$)										
Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.1060	0.0613	0.0446	-	0.4368	0.8617	0.6575	0.2041	-	0.4103
CRRA, γ=10	0.0646	0.0370	0.0276	-	0.2815	0.7348	0.5809	0.1539	-	0.1492
HARA, γ=5,b=-0.2	0.0878	0.0505	0.0372	-	0.3693	0.8203	0.6314	0.1890	-	0.3392
HARA, γ=10,b=-0.2	0.0541	0.0307	0.0234	-	0.2429	0.7112	0.5657	0.1455	-	0.1078
Panel C. Gaussian-SJC DCC diffusion with only CFNAI driving conditional correlation ($\gamma_1 = \gamma_2 = 0$)										
Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.0353	0.0620	-0.0267	-0.0673	-	0.8030	0.7259	0.0771	-0.0521	-
CRRA, γ=10	0.0204	0.0379	-0.0174	-0.0403	-	0.7356	0.6523	0.0833	0.0024	-
HARA, γ=5,b=-0.2	0.0285	0.0512	-0.0227	-0.0558	-	0.7798	0.7021	0.0777	-0.0373	-
HARA, γ=10,b=-0.2	0.0160	0.0313	-0.0153	-0.0338	-	0.7215	0.6371	0.0844	0.0101	-

Table 5.5 (cont.). Intertemporal hedging demands for the 2 stocks with different specifications for the unconditional dependence (Gaussian diffusion with no tail dependence, Student's t diffusion with symmetric tail dependence, and Gaussian-SJC diffusion with asymmetric tail dependence). Dynamic conditional correlation with observed factors (Case B). For each investment horizon of 1 and 5 years the table displays the sum of market price of risk hedges (MPRH), the separate demands for S&P 500 and NASDAQ, as well as the two correlation hedges (CorrH), corresponding to the macroeconomic factor F^M and the volatility factor F^V (the latter multiplied by 100).

Panel D. Gaussian DCC diffusion										
Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.0557	0.0652	-0.0094	-0.0469	0.5440	0.9605	0.7419	0.2187	-0.0127	0.1512
CRRA, γ=10	0.0374	0.0431	-0.0057	-0.0277	0.3209	0.8905	0.6694	0.2211	0.0237	-0.2819
HARA, γ=5,b=-0.2	0.0476	0.0555	-0.0078	-0.0387	0.4491	0.9353	0.7167	0.2186	-0.0026	0.0358
HARA, γ=10,b=-0.2	0.0321	0.0372	-0.0052	-0.0232	0.2673	0.8752	0.6559	0.2193	0.0292	-0.3400

Panel E. Student's t DCC diffusion										
Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.0442	0.0676	-0.0235	-0.0487	-0.1939	1.0418	0.7998	0.2419	-0.0670	-0.2690
CRRA, γ=10	0.0280	0.0424	-0.0145	-0.0283	-0.1128	0.9319	0.7114	0.2205	-0.0245	-0.1000
HARA, γ=5,b=-0.2	0.0364	0.0562	-0.0198	-0.0400	-0.1596	1.0045	0.7699	0.2346	-0.0538	-0.2182
HARA, γ=10,b=-0.2	0.0237	0.0362	-0.0125	-0.0236	-0.0940	0.9082	0.6915	0.2167	-0.0187	-0.0771

Panel F. Gaussian-SJC DCC diffusion										
Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.0268	0.0594	-0.0326	-0.0652	0.3782	0.8419	0.7265	0.1154	-0.0245	0.1429
CRRA, γ=10	0.0140	0.0356	-0.0216	-0.0386	0.2243	0.7708	0.6500	0.1208	0.0226	-0.1317
HARA, γ=5,b=-0.2	0.0206	0.0484	-0.0277	-0.0538	0.3129	0.8166	0.7007	0.1159	-0.0110	0.0664
HARA, γ=10,b=-0.2	0.0107	0.0297	-0.0190	-0.0323	0.1885	0.7552	0.6350	0.1202	0.0303	-0.1761

Student's t alternative. Results are presented on Panels D through F of Table 5.5. As in the portfolio example over realized paths of the state variables, the stationary distribution still plays a role in determining the hedging demands, rendering them smaller in the presence of tail dependence. For shorter horizons its effect is smaller than the effect of disregarding conditional correlation, but at the 5-year horizon the Gaussian diffusion renders the highest hedging demands, even higher than the CCC case, which confirms our findings of the market timing exercise.

The above results may be sensitive to the level of conditional correlation that we impose. Thus, we repeat the simulations experiment with a Gaussian-SJC diffusion and DCC following Case B for varying values of the γ_0 parameter for the conditional correlation. For levels of γ_0 of 1, 2 and 3 obtain conditional correlation levels (averaged over the estimation period) of 0.45, 0.75 and 0.90. For each one of those DCC cases we find the appropriate CCC calibration for the conditional correlation parameters, keeping the same average correlation levels. Results are plotted on Figure 5.7.

Regardless of the investment horizon, for relatively low correlation levels (0.45) the DCC model implies significantly lower intertemporal hedging demands, compared to a CCC specification, even after tail dependence has been accounted for through the Gaussian-SJC stationary distribution. For extremely high correlation levels (the case of $\gamma_0 = 3$) the roles of DCC and CCC change and now it is the latter that implies lower hedging demands. Depending on the investment horizon, we may have higher or lower hedge levels for a mean conditional correlation of 0.75. This behavior can thus explain the higher hedging demands implied by the DCC specification over a realized path of the state variables that we encountered earlier.

5.3 Certainty equivalent cost of ignoring correlation hedging

We follow the common approach in literature on portfolio choice and study the effect of ignoring correlation hedging on the wealth of the investor using the utility loss, or the certainty equivalent cost (see Liu et al. 2003). The approach consists in computing the additional amount of wealth that would be needed for an investor to consider a suboptimal allocation strategy (that results from ignoring correlation hedging) instead of the optimal one (that takes into account the dynamics of conditional correlation), in order to achieve the same expected utility of terminal wealth. In other words, we are looking to determine the amount ceq such that:

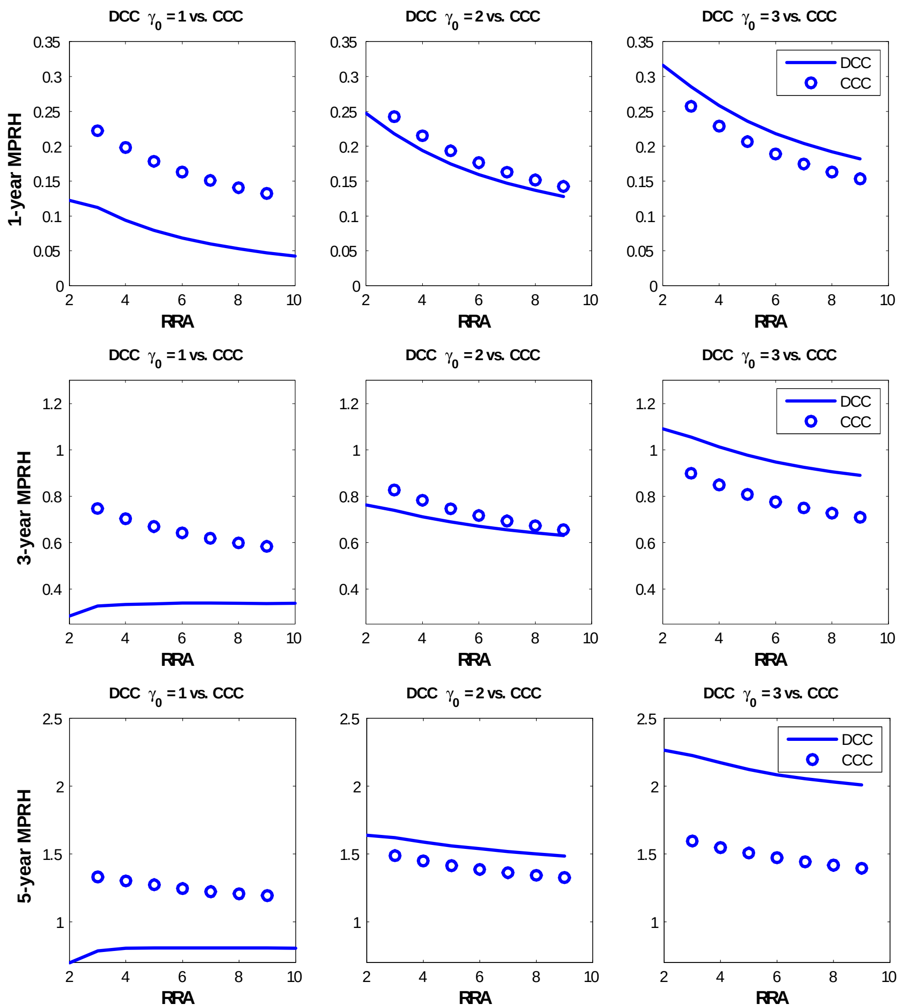
$$E\left[U\left(\omega_T^* \mid \omega_0 = 1\right)\right] = E\left[U\left(\omega_T \mid \omega_0 = 1 + ceq\right)\right]$$

where ω_T^* is the terminal wealth achieved under the optimal investment strategy and ω_T is the terminal wealth under the suboptimal one.

The first question that we address, in accordance with the simulation exercise above, is whether the

Figure 5.7: Dynamic correlation-induced portfolio hedging terms through simulation: the influence of correlation level

Intertemporal hedging demands for a benchmark Gaussian-SJC diffusion with DCC (Case B) vs. a CCC specification with parameter calibrated to match the mean conditional correlation of the corresponding DCC model. Varying average values of conditional correlation through the parameter γ_o. HARA investor with $b = -0.2$ and varying degrees of relative risk aversion, and investment horizon of 1, 3 and 5 years.



investor would lose anything if she disregards the dynamics of conditional correlation, modeled using observable factors, given the fact that tail dependence in the unconditional distribution has already been accounted for. Thus, we choose as a benchmark process the Gaussian-SJC diffusion with DCC according to Case B. Then we alternate between setting all conditional correlation parameters to zero except for γ_0 (CCC alternative), letting only γ_3 be zero (conditional correlation being driven by the VIX factor), or letting γ_1 be nonzero (conditional correlation being driven by the macroeconomic factor). In those alternative models the γ_0 parameter is calibrated in order to reflect the same average correlation as the DCC benchmark over the estimation horizon. We consider again a HARA investor with varying degrees of relative risk aversion and a parameter b in the utility function equal to -0.2, 0 or 0.2. The case of $b = 0$ corresponds to a CRRA investor, while if $b < 0$ relative risk aversion is decreasing and convex in wealth, in which case the investor is intolerant towards wealth falling below a certain subsistence level $-b$, and alternatively, if $b > 0$, then relative risk aversion is increasing and concave. Table 5.6 summarizes the results on the certainty equivalent cost in each case, calculated in cents per dollar.

The cost of disregarding the dynamics of conditional correlation is comparable to the cost of disregarding the presence of the macroeconomic factor driving its dynamics, so we may conclude that the CFNAI factor is the major player in the present setting in terms of utility loss. The cost decreases with rising levels of the risk aversion coefficient, and is highest for a HARA investor with relative risk aversion that is increasing and concave in wealth. However, the impact of disregarding the VIX factor is almost insignificant.

We next address the alternative problem of finding the utility cost for an investor who disregards the fact that extreme realizations of the assets in her portfolio may be dependent, as modeled through the stationary distribution of X. Results are summarized in Table 5.7, where we take as a benchmark process either the DCC Gaussian-SJC diffusion (left column), or the CCC one (right column) against the two Elliptic counterparts. In order to isolate only the impact of the tail dependence through the stationary distribution, conditional correlation parameters for all processes are taken from the Gaussian-SJC type with DCC (Case B).

The main conclusion that we can draw from comparing the wealth loss across the alternative specifications is that the investor loses more from disregarding tail dependence if she has not taken into account the dynamics in conditional correlation. It is an anticipated result, as both ways of modeling dependence through the dynamics of the conditional correlation or through the stationary distribution aim at reproducing the same stylized fact of increased dependence in down markets. Thus if at least one of them is taken into account when making portfolio decisions, the impact of disregarding the other in terms of wealth loss will be subdued.

As we saw in the above simulations exercise, the portfolio composition changes considerably for varying levels of the mean conditional correlation, modeled through the parameter γ_0. In order to determine

Table 5.6: Certainty equivalent cost of ignoring dynamic conditional correlation, modeled with observable factors

The benchmark process is a Gaussian-SJC diffusion with DCC according to Case B. All of the alternative processes have a Gaussian-SJC stationary distribution, but their conditional correlation specifications vary from CCC to DCC with no VIX ($\gamma_1 = 0$), and DCC with no CFNAI factor ($\gamma_2 = 0$).All parameters of the stationary distribution are from the Gaussian-SJC type with DCC (Case B), the conditional correlation parameters of the alternative processes were calibrated in order to reflect the same mean conditional correlation as the benchmark process. The Certainty Equivalent Cost is given in cents per dollar. Investment horizon is 5 years.

Panel A. The cost of disregarding DCC (CCC alternative)			
	HARA, $b = -0.2$	CRRA	HARA, $b = 0.2$
$\gamma = 2$	2.3054	2.4039	2.5024
$\gamma = 4$	1.8987	1.9369	1.9751
$\gamma = 6$	1.7983	1.8216	1.8449
$\gamma = 8$	1.7538	1.7706	1.7873
$\gamma = 10$	1.7289	1.7419	1.7549
Panel B. The cost of disregarding the CFNAI factor (DCC with $\gamma_2 = 0$ alternative)			
	HARA, $b = -0.2$	CRRA	HARA, $b = 0.2$
$\gamma = 2$	2.4273	2.5533	2.6792
$\gamma = 4$	1.9309	1.9832	2.0355
$\gamma = 6$	1.7988	1.8315	1.8643
$\gamma = 8$	1.7384	1.7622	1.7860
$\gamma = 10$	1.7039	1.7226	1.7413
Panel C. The cost of disregarding the VIX factor (DCC with $\gamma_1 = 0$ alternative)			
	HARA, $b = -0.2$	CRRA	HARA, $b = 0.2$
$\gamma = 2$	0.0000	0.0000	0.0000
$\gamma = 4$	0.0000	0.0000	0.0000
$\gamma = 6$	0.0000	0.0000	0.0000
$\gamma = 8$	0.0000	0.0000	0.0000
$\gamma = 10$	0.0000	0.0000	0.0000

Table 5.7: Certainty equivalent cost of ignoring tail dependence

The benchmark process is a Gaussian-SJC diffusion with DCC according to Case B. The alternative processes have either a DCC specification (left figures) or a CCC specification (right column), and their unconditional distribution varies from Gaussian to Student's t. All parameters of the conditional correlation specification are from the Gaussian-SJC type with DCC (Case B) (left column) and from Gaussian-SJC type with CCC (right column). The Certainty Equivalent Cost is given in cents per dollar. Investment horizon is 5 years.

Panel A. The cost of disregarding tail dependence

	(Gaussian alternative, DCC)			(Gaussian alternative, CCC)		
	HARA	CRRA	HARA	HARA	CRRA	HARA
	b=-0.2	b=0	b=0.2	b=-0.2	b=0	b=0.2
$\gamma = 2$	1.3153	1.5158	1.7162	3.2467	3.8692	4.4916
$\gamma = 4$	0.6384	0.7438	0.8492	1.1366	1.4361	1.7357
$\gamma = 6$	0.3912	0.4619	0.5326	0.4602	0.6562	0.8523
$\gamma = 8$	0.2658	0.3189	0.3719	0.1301	0.2757	0.4212
$\gamma = 10$	0.1902	0.2327	0.2751	0.0000	0.0507	0.1664

Panel B. The cost of disregarding asymmetric tail dependence

	(Student's t alternative, DCC)			(Student's t alternative, CCC)		
	HARA	CRRA	HARA	HARA	CRRA	HARA
	b=-0.2	b=0	b=0.2	b=-0.2	b=0	b=0.2
$\gamma = 2$	0.1886	0.1696	0.1506	0.5891	0.6486	0.7081
$\gamma = 4$	0.4271	0.4416	0.4561	0.4755	0.5176	0.5597
$\gamma = 6$	0.4259	0.4403	0.4546	0.3960	0.4260	0.4559
$\gamma = 8$	0.4121	0.4245	0.4369	0.3509	0.3740	0.3970
$\gamma = 10$	0.3999	0.4106	0.4213	0.3224	0.3411	0.3598

the economic significance of this finding, we determine the certainty equivalent cost for disregarding correlation dynamics for any of the three cases that we considered at the end of the previous section. Results are summarized on Panel A of Figure 5.8.

The certainty equivalent cost is lower for the lowest levels of correlation considered ($\gamma_0 = 1$ or average correlation of 0.45 over the estimation horizon) and increases significantly for higher correlation levels. It also increases with the investment horizon. Results are consistent over the utility specifications considered (CRRA and 2 types of HARA utility).

For the above cases we have considered the Case B DCC specification as a benchmark, that is the case when dynamic conditional correlation is driven by both the observable factors F and the state variables X. In order to gauge the economic importance of any of the other DCC specifications, we calculate the wealth loss of an investor who believes that conditional correlation is either driven exclusively by observed factors (Case C) or they do not enter correlation dynamics (Case A), instead of the benchmark Case B. Results for an investment horizon of 5 years are summarized on Panel B on Figure 5.8. We find that the difference in terms of wealth loss between cases B and C is negligible, that is the investor does not lose much by just considering the observed factors for the dynamics of conditional correlation. The loss for an investor who totally disregards observed factors is higher, especially for low levels of risk aversion. But for extremely risk averse investors there is virtually no cost for considering any of the alternative DCC models instead of the benchmark one.

Being consistent with the simulations experiment, we consider also the economic loss for disregarding tail dependence, given that the dynamics of conditional correlation have been accounted for. We compute it by comparing the benchmark Gaussian-SJC diffusion with DCC according to Case B with a corresponding Gaussian diffusion with the same correlation dynamics. We do so for varying weights ω of the mixture copula $C^{Ga-SJC} = \omega C^{SJC} + (1-\omega)\, C^{Ga}$. Parameters are taken from the benchmark model over the whole estimation horizon, and the Gaussian correlation parameter is set so that the Kendall's tau implied by the Gaussian copula is equal to the one implied by the SJC copula, so varying the composition of the Gaussian-SJC copula will not change the Kendall's tau, but only the relative importance of tail dependence. Results are presented on Panel C on Figure 5.8. Even if dynamic conditional correlation has already been accounted for, there are substantial economic costs for disregarding tail dependence, reaching over ten cents per dollar for a 5-year investment horizon. They increase with increasing the weight of the SJC copula in the benchmark model (and hence the importance of tail dependence in the data generating process), and are higher for investors with lower levels of risk aversion.

6 Conclusion

In this chapter we address the issue of determining the impact of dynamic correlation modeled through observable factors on the portfolio hedging demands. The solution methodology that we apply allows us

Figure 5.8: Certainty Equivalent Cost

Panel. A. Certainty Equivalent Cost of ignoring dynamic conditional correlation, modeled with observable factors for varying mean levels of conditional correlation

The certainty equivalent cost of disregarding dynamic conditional correlation for a benchmark Gaussian-SJC diffusion with DCC (Case B) vs. a Gaussian diffusion with CCC with parameter calibrated to match the mean conditional correlation of the corresponding DCC model. Varying average values of conditional correlation through the parameter γ_o. HARA investor with $b = -0.2$ and varying degrees of relative risk aversion, and investment horizon of 1, 3 and 5 years.

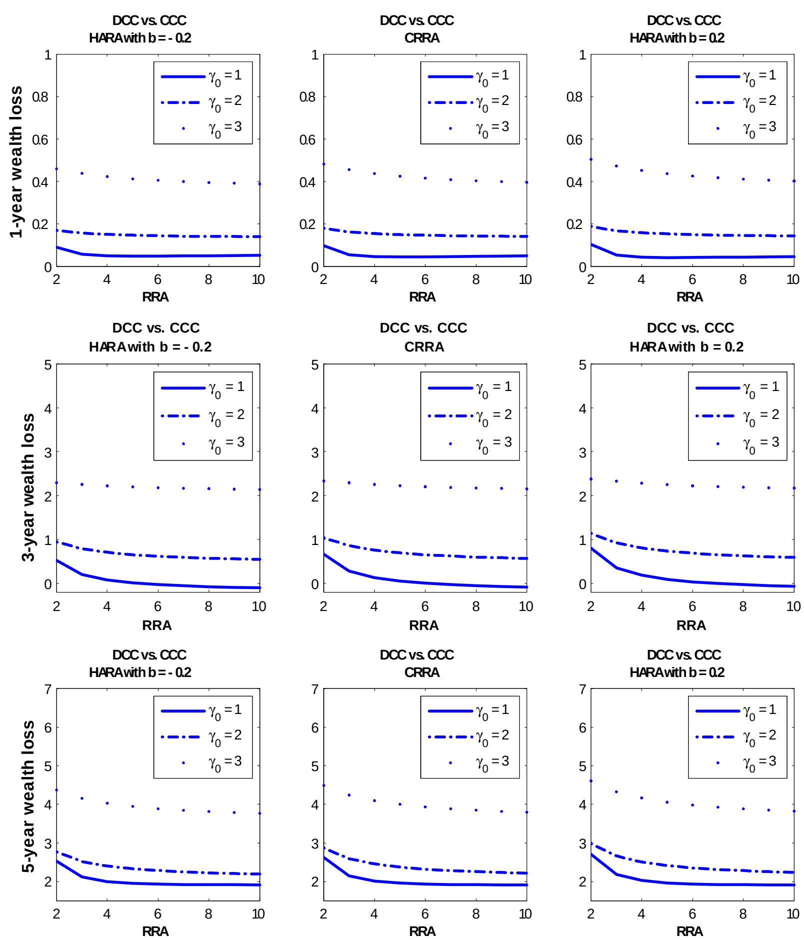
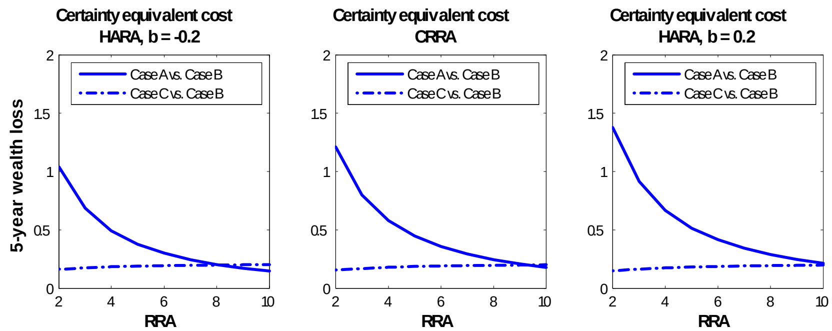


Table 5.8. Panel. B. Certainty Equivalent Cost of using alternative DCC specifications
The certainty equivalent cost of modeling DCC following Case A or C vs. the benchmark case B for a Gaussian-SJC diffusion. 5-year investment horizon. Parameters for cases A and C are calibrated so as to reflect the same average conditional correlation over the estimation period as that implied by the benchmark case.

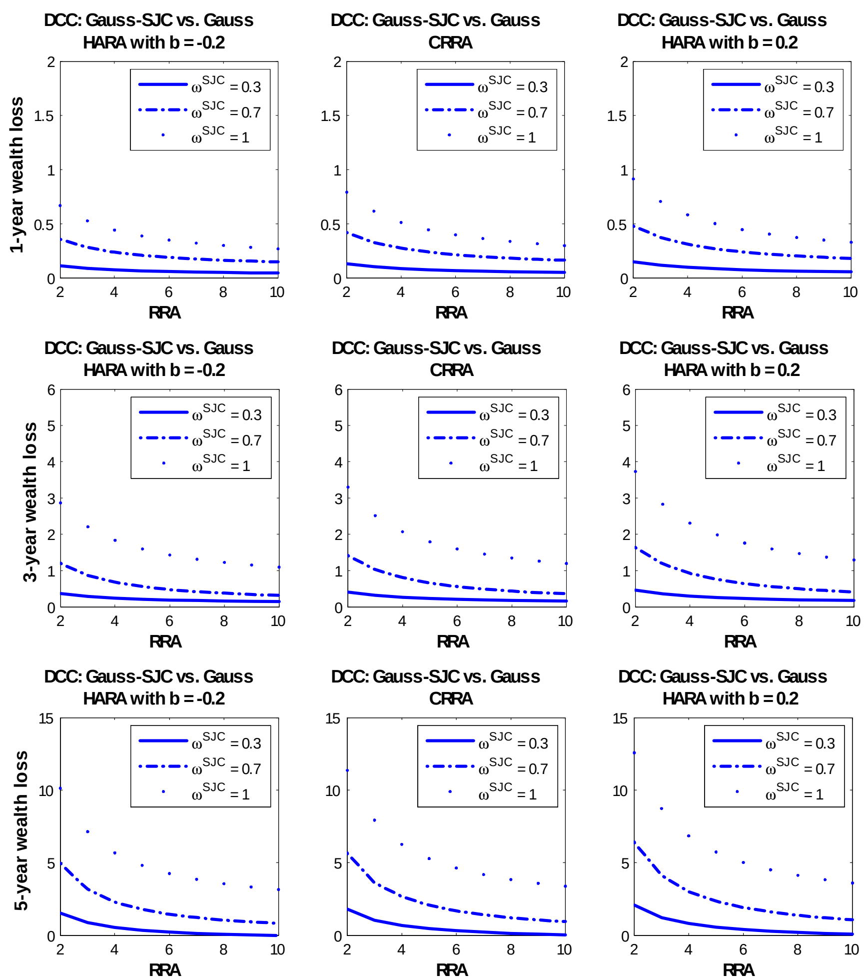


to disentangle the intertemporal demands due to the need to hedge against stochastic changes in those factors from the rest of the market price of risk hedging terms. We also account for tail dependence that manifests itself through increased co-movements between risky stocks during sharp market downfalls. We find that demands for correlation hedging and intertemporal demands due to high tail dependence have a distinct impact on the optimal portfolio behavior both in terms of optimal portfolio composition and of loss of wealth criterion.

There are a number of ways in which the present study could be extended. First, we could test the sensitivity of the results to an increased number of assets in the portfolio, as we would expect that hedging demands should increase as a result of the higher level of uncertainty linked to both the conditional correlation structure and the dependence through the stationary distribution. Second, it would be of interest to extend the dynamic treatment to the dependence structure modeled by the copula, assumed to be fixed in the present setup, in the spirit of dynamic copula models as in Patton (2004) . By letting observable factors affect the evolution of tail dependence we may find similar hedging demands as those implied by dynamic correlation. As well, we have seen that the dependence structure changes dramatically from relatively calm periods of low volatility and rising economic conditions, when it is not far from Gaussian to highly volatile periods marked with recessionary states, when dependence exhibits asymmetries and high tail coefficients. This could motivate us to consider a specification where the copula composition changes from normal to extreme value dependent one through varying weights of the copula.

Finally, for the sake of simplicity, we have assumed so far that the bond and stock dynamics are independent from each other. As there is compelling evidence of co-movement between bond and stock returns that could be linked to common exposure to macroeconomic factors (e.g. Li 2002), it would be of interest to incorporate this finding in the present portfolio solution setup.

Table 5.8. Panel. C. Certainty Equivalent Cost of disregarding tail dependence
The certainty equivalent cost of disregarding tail dependence by considering a Gaussian DCC diffusion instead of the benchmark data generating process of a Gaussian-SJC DCC diffusion for varying levels of the ω^{SJC} parameter determining the weight of the SJC copula in the mixture distribution. DCC specification follows Case B. Parameters are taken from estimating the benchmark case over the whole estimation horizon, while the correlation parameter of the Gaussian copula is calibrated so that to reflect the same Kendall's tau as the one implied by the SJC copula with the estimated parameters.



A Copula Functions

The following d-dimensional copula functions are used in the chapter.

- **Gaussian copula**

$$\begin{aligned} & C^{Ga}\left(u_1, u_2, ..., u_d \mid R_{Ga}\right) \\ = & \int_{-\infty}^{\Phi^{-1}(u_1)} ... \int_{-\infty}^{\Phi^{-1}(u_d)} \frac{1}{2\pi \left|R_{Ga}\right|^{1/2}} \exp\left\{-\frac{1}{2} x^{\intercal} R_{Ga}^{-1/2} x\right\} dx_1 ... dx_d \end{aligned}$$

where R_{Ga} denotes the correlation matrix, and $\Phi^{-1}\left(u_i\right)$ is the inverse of the univariate standard normal CDF.

- **Student's t copula**

$$\begin{aligned} & C^{t}\left(u_1, u_2, ..., u_d \mid R_t, \nu\right) \\ = & \int_{-\infty}^{t_\nu^{-1}(u_1)} ... \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma\left(\frac{\nu+d}{2}\right) \left|R_t\right|^{1/2}}{\Gamma\left(\frac{\nu}{2}\right) \left(\nu\pi\right)^{d/2}} \left(1 + \frac{1}{\nu} x^{\intercal} R_t^{-1} x\right)^{-\frac{\nu+d}{2}} dx_1 ... dx_d \end{aligned} \tag{A.1}$$

where R_t denotes the correlation matrix, ν is the degrees of freedom parameter, and $t^{-1}\left(u_i\right)$ is the inverse CDF of the univariate Student's t distribution with ν degrees of freedom.

- **Symmetrized Joe-Clayton copula**

This copula function was introduced by Patton (2004) and is based on the bivariate Joe-Clayton copula, that is a two-parameter copula function with parameters $\tau_L \in (0,1)$ and $\tau_U \in (0,1)$ that are a measure of the lower and upper tail dependence. The Joe-Clayton copula has the following form:

$$\begin{aligned} & C^{JC}\left(u_1, u_2 \mid \tau_L, \tau_U\right) \\ = & 1 - \left\{1 - \left[\left(1 - \left(1 - u_1\right)^{\kappa}\right)^{-\gamma} + \left(1 - \left(1 - u_2\right)^{\kappa}\right)^{-\gamma} - 1\right]^{-\frac{1}{\gamma}}\right\}^{\frac{1}{\kappa}} \\ \text{where } \kappa = & \frac{1}{\log_2\left(2 - \tau_U\right)} \\ \gamma = & -\frac{1}{\log_2\left(2 - \tau_L\right)} \end{aligned}$$

The symmetrized version of the copula, designed to render it completely symmetric for equal values of the lower and upper tail dependence parameters has the following form:

$$\begin{aligned} & C^{SJC}\left(u_1, u_2 \mid \tau_L, \tau_U\right) \\ = & \frac{1}{2}\left[C^{JC}\left(u_1, u_2 \mid \tau_L, \tau_U\right) + C^{JC}\left(1 - u_1, 1 - u_2 \mid \tau_U, \tau_L\right) + u_1 + u_2 - 1\right] \end{aligned}$$

B Malliavin Derivatives of the State Variables

Recall that the Malliavin derivatives of the state variables $Y \equiv \left(X_1, X_2, F^V, F^M, Y^r\right)$ can be represented as the solutions to a linear stochastic differential equation[8]:

$$D_t Y_s = \sigma^Y(t, Y_t) \exp\left\{\int_t^s dL_v\right\}$$

where $\sigma^Y(t, Y_t)$ is the 5×5 matrix of diffusion terms of the state variables, and dL_t is defined by:

$$dL_t \equiv \left(\partial_2 \mu^Y(t, Y_t) - \frac{1}{2}\sum_{j=1}^{5} \partial_2 \sigma^Y_{\cdot j}(t, Y_t)\, \partial \sigma^Y_{\cdot j}(t, Y_t)^\intercal\right) dt + \sum_{j=1}^{5} \partial_2 \sigma^Y_{\cdot j}(t, Y_t)\, dW_{jt}$$

where $\partial_2 \mu^Y(t, Y_t)$ and $\partial_2 \sigma^Y_j(t, Y_t)$ denote the derivatives of $\mu^Y(t, Y_t)$ and $\sigma^Y_{\cdot j}(t, Y_t)$ with respect to Y_t, and $\sigma^Y_{\cdot j}(t, Y_t)$ denotes the j^{th} column of the matrix $\sigma^Y(t, Y_t)$. The particular forms of the drift $\mu^Y(t, Y_t)$ and the diffusion term $\sigma^Y(t, Y_t)$ of the state variables are given by:

$$\mu^Y(t, Y) = \begin{pmatrix} \mu^X_1\left(t, X_t, F^V, F^M\right) \\ \mu^X_2\left(t, X_t, F^V, F^M\right) \\ \mu^{F^V}\left(t, F^V\right) \\ \mu^{F^M}\left(t, F^M\right) \\ \mu^{Y^r}(t, Y^r) \end{pmatrix}$$

where $\mu^X_i\left(t, X_t, F^V, F^M\right), i = 1, 2$ are given by (3.13), $\mu^{F^V}\left(t, F^V\right) = \kappa^V\left(\theta^V - F^V\right)$, $\mu^{F^M}\left(t, F^M\right) = \kappa^M\left(\theta^M - F^M\right)$, $\mu^{Y^r}(t, Y^r) = \kappa_r(\theta^r - Y^r_t)$.

$$\sigma^Y(t, Y) = \begin{pmatrix} \sigma^X_{11}(t, X) & \sigma^X_{12}(t, X) & 0 \\ \sigma^X_{21}(t, X) & \sigma^X_{22}(t, X) & 0 \\ \sigma^{F^V}\left(t, F^V\right) & \sigma^{F^V}\left(t, F^V\right) & 0 \\ \sigma^{F^M}\left(t, F^M\right) & \sigma^{F^M}\left(t, F^M\right) & 0 \\ 0 & 0 & \sigma^{Y^r}(t, Y^r) \end{pmatrix}$$

where $\sigma^X(t, X)$ is given by (**??**), $\sigma^{F^V}\left(t, F^V\right) = \sigma^V\sqrt{F^V}$, $\sigma^{F^M}\left(t, F^M\right) = \sigma^M$, and $\sigma^{Y^r}(t, Y^r) = \sigma_r\sqrt{Y^r_t}$.

Given the chosen specifications for the state variables, we can solve separately for the Malliavin derivatives of state variable driving the short rate, as well as for the Malliavin derivatives of the two factors. The processes that we have assumed for the observable factors (F^V for the VIX and F^M for CFNAI), as well as for the state variable Y^r, allow for either closed form solutions for the Malliavin derivatives (in the case of a Vasicek process) or for significant variance reduction in their simulation following the Doss transformation[9] that eliminates the stochastic term in the Malliavin derivative (for a CIR process).

In the Vasicek case, the Malliavin derivative of F^M simplifies significantly to:

$$D_{i,t} F^M_s = \sigma^M \exp\left\{-\kappa^M(s - t)\right\}, i = 1, 2$$

[8] See Theorem 1 in Detemple et al. (2003)

[9] See Detemple et al. (2003)

For the other two state variables, Y^r and F^V, we have assumed a CIR process, that can be reduced to have constant diffusion term through a suitable change of variable technique, which then eliminates the stochastic terms for the simulation of the corresponding Malliavin derivatives. For a univariate diffusion, this variance stabilizing transformation is described in detail in Proposition 2 of Detemple et al. (2003) and we reproduce it here for completeness.

Consider a state variable Y satisfying a stochastic differential equation

$$dY_t = \mu(t, Y_t)\, dt + \sigma(t, Y_t)\, dW_t$$

We can replace it with a new state variable $Z_t = F(t, Y_t)$ where the function $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ is such that $\partial_2 F = \frac{1}{\sigma^Y}$. Then for a continuously differentiable drift μ, twice continuously differentiable diffusion term σ, that also satisfy the growth conditions that $\mu(t, 0)$ and $\sigma(t, 0)$ are bounded for all $t \in [0, T]$, then we have for $t \leq s$:

$$\begin{aligned} D_t Y_s &= \sigma(t, Y_t)\, D_t Z_s \\ \text{where } D_t Z_s &= \exp\left\{\int_t^s \partial_2 m(v, Z_v)\, dv\right\} \\ m(t, Z) &\equiv \left[\frac{\mu}{\sigma} - \frac{1}{2}\partial_2 \sigma + \partial_1 F\right](t, Y) \end{aligned}$$

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