Robust asset allocation under model uncertainty

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Introduction and motivation

⋆ Different models, different scenarios, different experts...

$\Rightarrow$ "Model risk".

⋆ Problem of decision making under uncertainty, under ambiguity.

⋆ Idea: to develop a *simple and robust* methodology for decision making, easy to understand and to implement.

⋆ Illustration of this methodology: asset allocation.

Agenda

Objective: Presentation of the general methodology through a particular example.

⋆ Investment problem.

⋆ Classical framework.

⋆ Robust methodology.

⋆ Toy example.

⋆ Empirical results.

⋆ Further research.

The investment problem and the standard approach

Objective and notation

$\star$ An investor wants to allocate her wealth among different assets traded in the capital markets (typically N risky assets and a risk free asset).

Question: Find the proportion of the wealth to invest in each asset.

$\star$ *Choice criterion:* choosing the allocation φ as to maximize the expected utility u of the future wealth X^φ at a given time horizon (subject to some potential constraint).

Remark: There is always the constraint $\sum_{i=0}^N \varphi^i = 1$.

Classical approach (Markovitz - 1952)

The optimization problem of the investor is:

$$\boxed{\max_\varphi \mathbb{E}_\mathbb{P}[u(X^\varphi)]}$$

where $\mathbb{P}$ stands for the prior probability measure of the investor.

Problem

$\star$ Financial investors often develop different models for the financial asset dynamics, none of which they can fully trust.

$\Rightarrow$ Necessity to take into account uncertainty in the decision process.

First step: Subjective expected utility (Savage - 1954)

The investor now considers a set of possible models $\mathcal{Q}$ as well as a distribution μ on these models.

The optimization problem is now:

$$\boxed{\max_\varphi \sum_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_\mathbb{Q}[u(X^\varphi)]\mu(\mathbb{Q})}$$

$\Rightarrow$ The investor has several priors as reference, but there exists *no aversion towards this uncertainty*.

$\Rightarrow$ Aversion to model risk (or ambiguity) prevents the investors from using this classical framework of expected utility maximization to compute optimal allocations.

Taking into account ambiguity

Max-min approach (Gilboa & Schmeidler - 1989)

The investor maximizes her minimum expected utility over the set of models $\mathcal{Q}$.

The optimization problem is:

$$\boxed{\max_{\varphi} \min_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})]}$$

$\Rightarrow$ Too conservative as only considers the worst case scenario.

Penalized max-min approach (Maccheroni, Marinacci & Rustichini - 2006)

The idea is to penalize each model differently, using a penalty function α.

The optimization problem is:

$$\boxed{\max_{\varphi} \min_{\mathbb{Q}\in\mathcal{Q}} \{\mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})] - \alpha(\mathbb{Q})\}}$$

Other references (among many others): Hansen and Sargent (2001...) (entropic penalty function), Garlappi, Uppal and Wang (2007) and Epstein and Schneider (2007) (constrained optimization problem with Lagrangian techniques) + all the references on financial risk measures.

$\Rightarrow$ How to choose the penalty function? Often expressed in terms of a reference model: how to choose the reference?

Integrated approach (Klibanoff, Marinacci and Mukerji (KMM) - 2005)

The idea is to develop a generalized model, by introducing a function Ψ characterizing the ambiguity aversion of the investor through a parameter γ.

The optimization problem is then:

$$\boxed{\max_{\varphi} \sum_{\mathbb{Q} \in \mathcal{Q}} \Psi\left(\mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})], \gamma\right) \mu(\mathbb{Q})}$$

$\Rightarrow$ Almost impossible to implement (calibration, dimension...), especially when some additional constraints are added.

$\Rightarrow$ Our objective: To introduce a *simple* (practical and easily implementable) approach to account for model risk in a *robust* way (i.e. independent of the considered class of models and of the optimization criterion).

Ambiguity robust adjustment

Main idea: some trade-off between optimization and robustness.

⇒ A new approach to model ambiguity that is more flexible, easier to compute, and more tractable than the previous methods.

⇒ Independent of the set of models considered $\mathcal{Q}$, as well as of the choice criterion.

General principle

The ambiguity robust adjustment is a two-step procedure, introducing a distinction between two types of ambiguity aversion:

- ⋆ *Absolute ambiguity aversion:* ambiguity aversion for a given model, independently of the other models,
- ⋆ *Relative ambiguity aversion:* ambiguity aversion for a given model, relatively to the other models.

Two-step procedure for the ambiguity robust adjustment:

1. First solve the optimization problem for each model *as if* it was the "true" one. Then adjust the outcome using an *Absolute Ambiguity Robust Adjustment* function ψ.

2. Second, aggregate the adjusted outcomes computed for each model through a *Relative Ambiguity Robust Adjustment* function π.

$$\boxed{\text{simple optimization outcome } x_q \ \Rightarrow\ \psi(x_q)\ \Rightarrow\ \textstyle\sum_q \psi(x_q)\pi(q)}$$

Comments:

⋆ The decision-maker has to solve a series of simple optimization problems (with constraints) instead of a large complex optimization problem.

⋆ The absolute adjustment is made on the outcome of the optimization problem.

$\Rightarrow$ Flexibility to adjust the same model differently according to the purpose.

⋆ Adding a new model simply modifies the second step.

In the case of the portfolio allocation problem:

1. First, for each model $\mathbb{Q} \in \mathcal{Q}$, solve the optimization problem

$$\varphi \; \rightarrow \; \max_{\varphi} \mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})] \qquad \text{and} \qquad \varphi^{\mathbb{Q}} \equiv \text{argmax} \mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})]$$

Then adjust the weights using the function ψ:

$$\psi\left(\varphi_i^{\mathbb{Q}}\right)$$

2. Finally, the ambiguity adjusted weights are obtained as:

$$\boxed{\varphi_i^{ARA} \equiv \sum_{\mathbb{Q} \in \mathcal{Q}} \psi\left(\varphi_i^{\mathbb{Q}}\right) \pi(\mathbb{Q})}$$

Absolute ambiguity robust adjustment

Idea: Scaling down the optimal weights generated by each model.

Axiomatic characterization of the function ψ:

- ⋆ *Universality:* ψ is identical across all models, but parametrized by an ambiguity aversion parameter γ that may vary and be model specific.

- ⋆ *Monotonicity:* ψ preserves the relative order of the optimal weights obtained for any given model $\mathbb{Q}$.
  The relative preference of the investor towards the different risky assets for a given model $\mathbb{Q}$ is preserved through the transformation ψ.

- ⋆ *Convexity:* The function ψ is concave and then convex, so that ψ reduces more the (absolute) largest weights for each model considered (shrinking effect).

Some additional properties:

- *Symmetry:* When short-selling is allowed, it is natural to consider an odd function symmetric around zero for ψ.

  The investor has the same aversion to positive or negative weights of the same absolute value.

- *Invariant point:* $\psi(0) = 0$.

  If the model $\mathbb{Q}$ assigns no weight to a given asset, the transformation ψ should not modify the "‘neutrality"’of the model $\mathbb{Q}$ for this asset.

- *Limit behavior:* When the investor is infinitely averse to ambiguity, all the weights for the risky assets should be zero.

  On the contrary, if the investor is neutral to ambiguity, the function ψ should leave the weights unchanged.

An example for the function ψ is:

$$\psi(x,\gamma) \equiv \begin{cases} \frac{1-\exp^{-\gamma x}}{\gamma}, 0 \leq x \leq 1 \\ \frac{\exp^{\gamma x}-1}{\gamma}, -1 \leq x \leq 0 \end{cases}$$

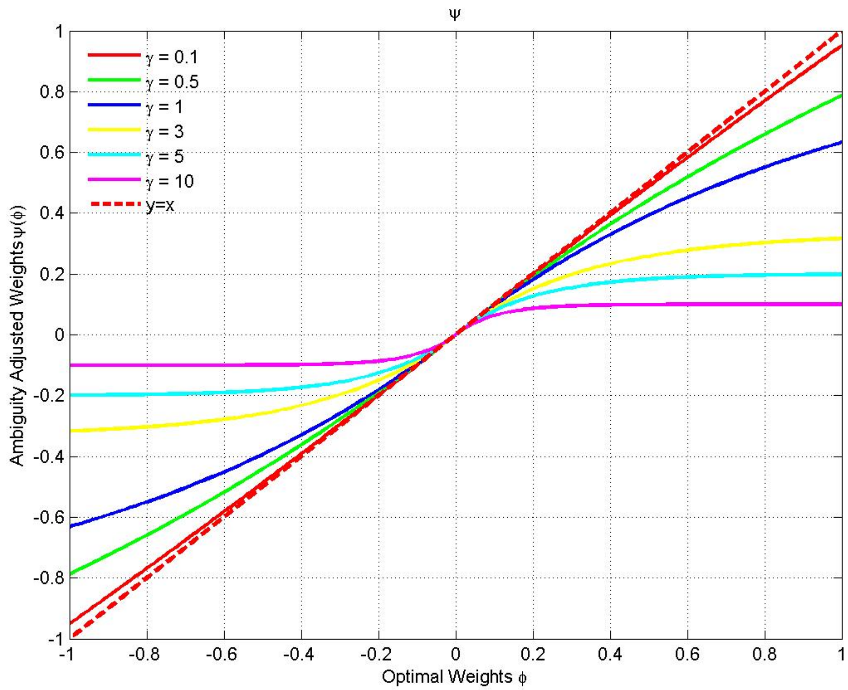


Figure 1: ψ for different values of the ambiguity aversion parameter γ

Relative ambiguity robust adjustment

Characterization of the function π:

- *Second step of the procedure:* after an independent computation and adjustment of the allocation (through ψ), aggregation across all models.
- The relative ambiguity robust adjustment function π represents *how trustworthy each model is* according to the investor's anticipations and *relatively* to the other models of the class $\mathcal{Q}$.
  It measures the *relative ambiguity aversion* the investor displays towards each model $\mathbb{Q}$ among the models in $\mathcal{Q}$.
- The function $\pi : \mathcal{Q} \to [0;1]$ can be seen as a model weighting function.

Comments: π is not necessarily normalized.

$$\forall \mathbb{Q} \in \mathcal{Q}, 0 \leq \pi(\mathbb{Q}) \leq 1 \qquad \text{and} \qquad \sum_{\mathbb{Q} \in \mathcal{Q}} \pi(\mathbb{Q}) \leq 1.$$

If $\sum_{\mathbb{Q} \in \mathcal{Q}} \pi(\mathbb{Q}) < 1$, the investor believes that the set $\mathcal{Q}$ does not give a full understanding of the situation.

Role of the risk-free asset

⋆ The risk-free asset can be seen as a *refuge value*.

The more the investor is averse to ambiguity, the bigger her proportional asset allocation in the risk free asset.

The ambiguity aversion leads the investor to invest less in risky (ambiguous) assets.
⇒ The "desinvested" risky investment value due to the presence of ambiguity is transferred into the risk-free asset.

⋆ After the ambiguity adjustment, the weight of any risky asset is φ_i^{ARA} and the weight of the risk-free asset is:

$$\varphi_0^{ARA} = 1 - \sum_{i=1}^{N} \varphi_i^{ARA}$$

Ambiguity robust adjustment - Calibration

⋆ *Dynamic aversion to ambiguity:* Depending on the considered period, the investor will be more or less confident about the overall set of models she considers (the ambiguity aversion does not necessarily decrease over time).

⋆ *Simple empirical calibration methodology*, taking into account the relative historical performance of the different models to calibrate the functions ψ and π:

- First, we compute historical time series of some performance measure for the different models and the problem we consider.
- The ambiguity aversion parameter γ can be calibrated as the inverse of the performance measure.
- The measure π can then be computed as a weighted average of the performance measure.

A toy example - Comparison with KMM (2005)

The framework

⋆ Three assets with initial value of 1: a risk-free asset $S^{(0)}$, a risky asset $S^{(1)}$ and an ambiguous asset $S^{(2)}$.

⋆ Two states of the universe ω_1 and ω_2 in the future and two different models $\mathbb{Q}_1$ and $\mathbb{Q}_2$.

	$\pi(\mathbb{Q}_1) = \frac{1}{2}$		$\pi(\mathbb{Q}_2) = \frac{1}{2}$	
	$\mathbb{Q}_1(\omega_1) = \frac{1}{4}$	$\mathbb{Q}_1(\omega_2) = \frac{3}{4}$	$\mathbb{Q}_2(\omega_1) = \frac{3}{4}$	$\mathbb{Q}_2(\omega_2) = \frac{1}{4}$
S_0	1.15	1.15	1.15	1.15
S_1	3	1	3	1
S_2	2	2	1	1

The two approaches

⋆ *The KMM approach:*

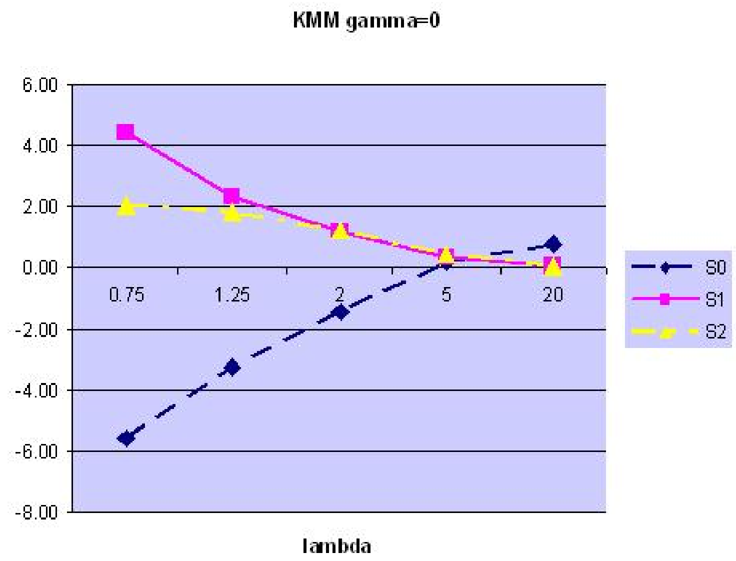
$$\boxed{\max_{\varphi} \sum_{\mathbb{Q}\in\mathcal{Q}} \Psi\left(\mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})], \gamma\right) \mu(\mathbb{Q})}$$

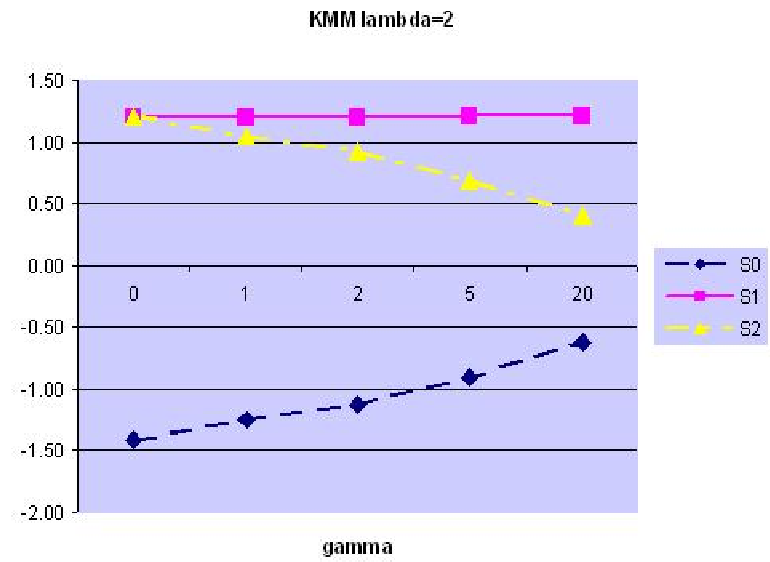
⋆ *The ARA methodology:*

$$\max_{\varphi} \mathbb{E}_{\mathbb{Q}}[u(X^{\varphi})] \qquad \text{and} \qquad \varphi_i^{ARA} \equiv \sum_{\mathbb{Q}\in\mathcal{Q}} \psi\left(\varphi_i^{\mathbb{Q}}\right) \pi(\mathbb{Q})$$

Investment strategies

Using power utility function and exponential-type adjustment (as in KMM), we obtain the weights for the three different assets for various risk aversion λ and ambiguity aversion γ.


KMM gamma=0
6.00
4.00
2.00
0.00
-2.00
-4.00
-6.00
-8.00
0.75
1.25
2
5
20
lambda
S0
S1
S2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1.00
0.50
0.00
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0
1
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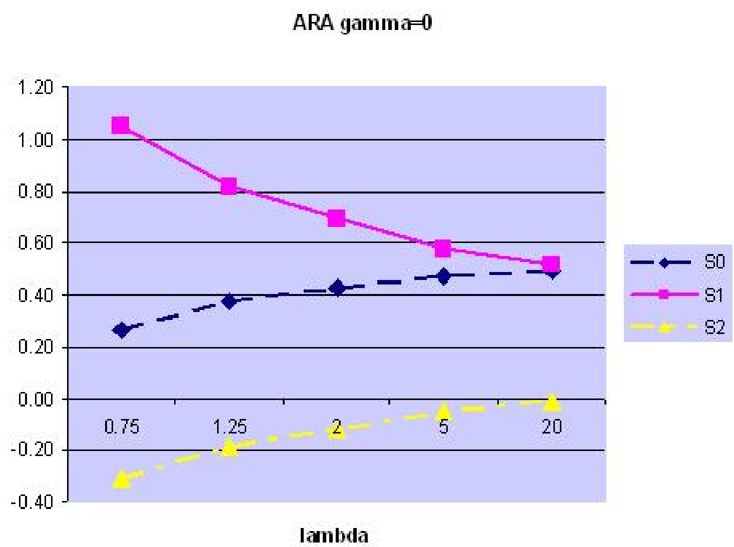
20
gamma
S0
S1
S2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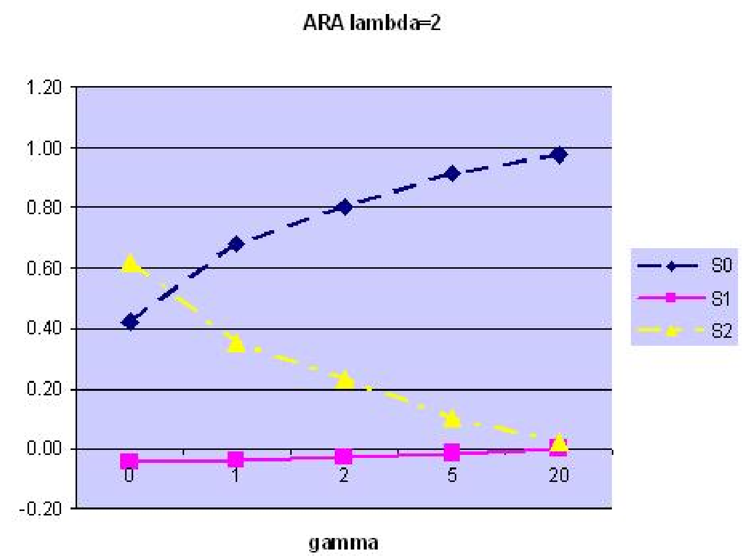
0.80
0.60
0.40
0.20
0.00
-0.20
-0.40
0.75
1.25
2
5
20
lambda
S0
S1
S2


ARA lambda=2
1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
0
1
2
5
20
gamma
S0
S1
S2

$\star$ *For different values of* λ *and constant ambiguity aversion,*

ARA tends to better discriminate the risky and ambiguous assets than KMM.

The weights are more extreme in KMM than in ARA.

$\star$ *For different values of* γ *and constant risk aversion,*

Both approaches display the same behaviour as the proportion invested in the ambiguity asset decreases: more wealth is invested in the risk-free asset and the weight of the risky asset remains constant.

$\Rightarrow$ Similar patterns in terms of asset allocation for both approaches.

$\Rightarrow$ The ARA methodology tends to be more balanced between the different assets in the market and less sensitive to the parametrization of the risk and ambiguity aversion coefficients.

Empirical study

Objective: To evaluate the performance of the ARA portfolios on European stock data.

Data and assumptions

⋆ Daily close prices of Eurostoxx 600 constituents from Jan 2000 to March 2008.

⋆ Assumptions:

◇ The risk free rate is assumed to be zero (daily data),

◇ The transactions costs, fees and slippage are taken as 3 basis point of the daily turnover of the strategy.

⋆ We run a back test on historical data when at each date the investor re-balances her portfolio by re-estimating the different models over a training window of 120 days.

Different models

Note that here we work directly on the outcome $\varphi^{\mathbb{Q}}$ of each model.

- The equally weighted portfolio.
- The minimum variance portfolio.
- The CAPM portfolio.
- The "CAPM Uncertain" portfolio, with an adjusment for the variance of the CAPM residuals.
- The Price Earning portfolio, based upon price earning ratio of each asset.
- The Cash Flow portfolio, based upon the relative cash flow ratio of each asset.
- The Price to Book portfolio, based upon the relative price to book ratio of each asset.

Performance measures

- Sharpe ratio: ratio of the mean return of a portfolio over its standard deviation.
- Sortino Price ratio: ratio of the mean return of a portfolio over the standard deviation of its negative returns.
- Gain Loss ratio: which is the ratio of total positive returns over total negative returns.
- Winner Loser ratio: ratio of the number of total positive returns over the number of total negative returns.
- Certainty equivalent ratio: equivalent risk-free return ($=$ portfolio return adjusted for its risk).
- Turnover: change in portfolio weights from one re-balancing period to the next.

Raw portfolio performances

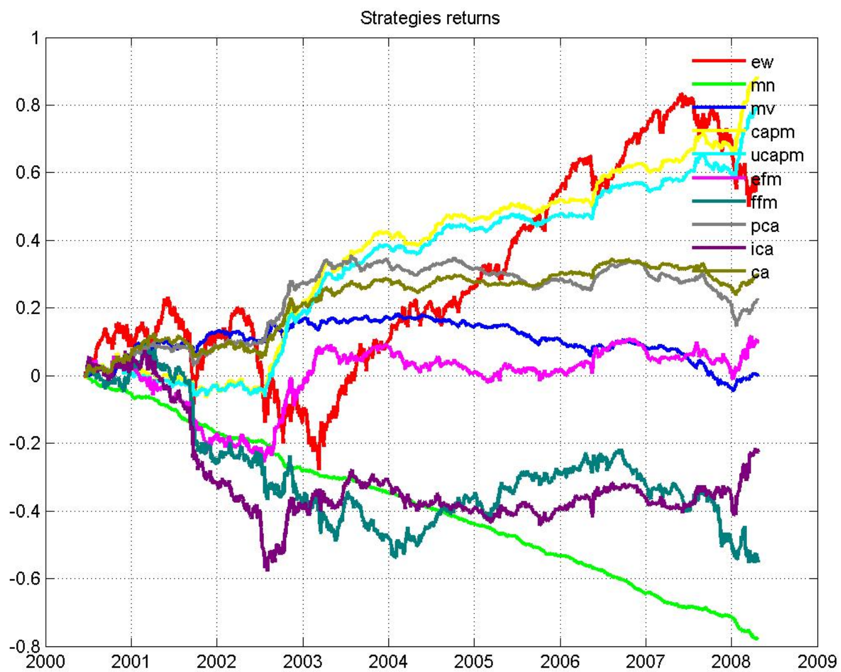


Figure 2: Strategies Returns

	ew	mn	mv	ca	uc	pe	cf	pb
$\mu(\%)$	69.74	35.03	10.68	13.95	8.71	11.04	5.42	1.89
$\overline{\mu}$(Bps)	3.41	1.71	0.52	0.68	0.43	0.54	0.26	0.09
$\sigma(\%)$	15.26	6.77	4.75	4.71	3.95	4.38	5.06	4.93
max(μ)(Bps)	470.80	199.81	133.63	182.75	210.82	193.83	225.39	234.74
min(μ)(Bps)	−540.82	−317.57	−219.94	−162.10	−139.12	−217.76	−205.43	−234.40
Sharpe	0.56	0.63	0.27	0.36	0.27	0.31	0.13	0.05
Sortino	0.72	0.75	0.32	0.52	0.40	0.44	0.19	0.07
GainLoss(%)	110.48	112.05	104.98	106.77	104.94	105.88	102.48	100.88
WinLose(%)	119.16	123.96	122.98	101.00	100.60	102.52	99.11	98.72
CER(Bps)	2.94	1.62	0.48	0.64	0.39	0.50	0.21	0.04
T/O(%)	2.89	21.85	28.60	135.31	135.01	141.51	140.83	141.59

Table 1: Strategies Performances SXXP Index

SEU portfolios performances

	sharpe	sortino	gainloss	winlose
Total return(%)	77.59	75.26	55.72	59.58
Mean daily return (Bps)	3.79	3.68	2.72	2.91
Volatility (%)	7.06	6.92	6.89	6.95
Max return (Bps)	326.58	322.47	310.73	321.65
Min return (Bps)	−411.38	−412.40	−275.82	−270.02
Sharpe	1.34	1.33	0.99	1.05
Sortino	1.52	1.50	1.25	1.33
GainLoss(%)	128.85	128.59	119.60	120.95
WinLose(%)	135.45	135.09	121.06	123.23
Certain Equivalent Ratio (Bps)	3.69	3.58	2.63	2.81
Turnover(%)	76.33	78.26	112.96	112.65

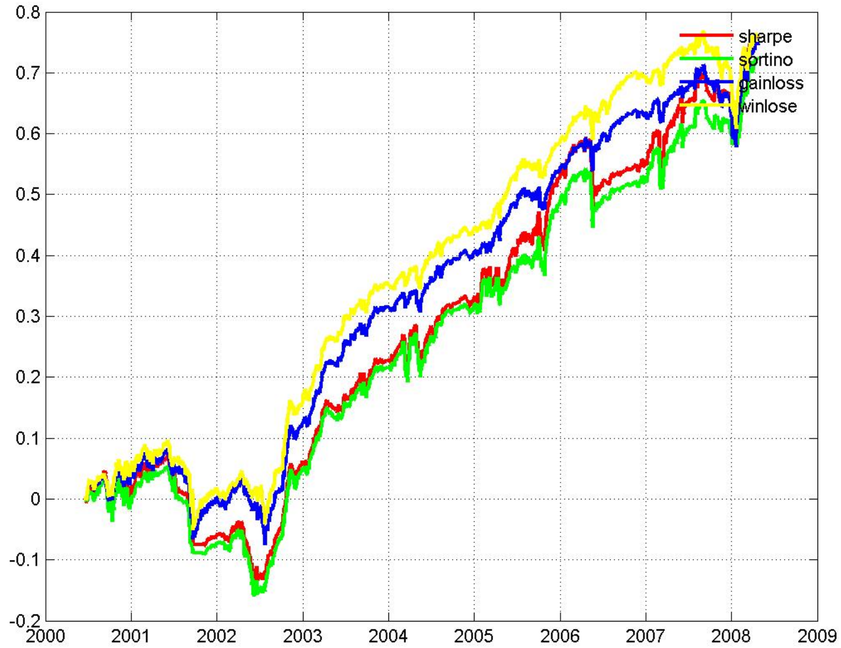
Table 2: SEU Strategies Performances

ARA portfolio performances

	sharpe	sortino	gainloss	winlose
Total return (%)	84.79	80.28	68.18	76.39
Mean daily return (Bps)	4.14	3.92	3.33	3.73
Volatility (%)	6.97	6.78	6.79	7.06
Max return (Bps)	351.97	344.72	300.82	321.42
Min return (Bps)	−423.97	−422.49	−276.14	−282.74
Sharpe	1.49	1.45	1.23	1.32
Sortino	1.64	1.58	1.51	1.65
GainLoss(%)	133.65	132.56	124.66	126.87
WinLose(%)	137.44	137.94	128.97	130.00
Certain Equivalent Ratio (Bps)	4.05	3.83	3.24	3.63
Turnover (%)	67.21	70.41	98.71	98.49

Table 3: Ambiguity Robust Strategies Performances

Figure 3: Robust Ambiguity Strategies Returns



Ambiguity robust adjustment

⋆ **General comments:**

◇ Two-step methodology to account for ambiguity in decision making.

◇ *Simple and robust* with respect to the choice criterion and the class of models.

◇ *Disentanglement of absolute and relative ambiguity* aversion towards a model.

⋆ **Asset allocation problem:**

◇ *Simple and easy to implement* methodology for practitioners who want to allocate their asset portfolio.

◇ *Encouraging results* with a simple parametrization for ψ and π: the ambiguity robust portfolio is *more stable* (lower turnover) and *less risky* (higher certain equivalent ratio) than unadjusted portfolios as well as SEU portfolios.

⋆ **Work in progress:**

- ⋄ Methods to calibrate the adjustment functions.
- ⋄ Relative ambiguity robust adjustment (function π).
- ⋄ Application to other decision-making problems.

Some references (among many others...)

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