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VARIATIONS ON THE HEISENBERG SPHERICAL HARMONICS

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Variations on the Heisenberg spherical harmonics *)

by

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ABSTRACT

This paper collects a large deal of what is presently known about spherical harmonics on the Heisenberg group and the related functions $C_k^{(\alpha,\beta)}$. It contains both new results and new approaches to old results. First, orthogonality properties and generating functions for $C_k^{(\alpha,\beta)}$ are discussed. Next a new approach to Korányi's Kelvin transform on the Heisenberg group is given. After a treatment of Heisenberg harmonics, the Kelvin transform is applied in order to obtain a new proof of Dunkl's expansion of the translate of the fundamental solution for L_γ. Finally it is shown that, if the Dirichlet problem for L_γ on the Heisenberg ball is solvable, then the related functions $C_k^{(\alpha,\beta)}$ form a complete system.

KEY WORDS & PHRASES: *spherical harmonics on the Heisenberg group; functions* $C_k^{(\alpha,\beta)}$*; Kelvin transform on the Heisenberg group; the subelliptic Heisenberg Laplacian* L_γ*; Green's formula for* L_γ*; Dirichlet problem for* L_γ *on the Heisenberg ball; expansion of translate of fundamental solution for* L_γ*; completeness of the functions* $C_k^{(\alpha,\beta)}$.

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0. INTRODUCTION

This article is concerned with the functions $\theta \mapsto C_k^{(\alpha,\beta)}(e^{i\theta})$, $\alpha,\beta \in \mathbb{C}$, $k = 0,1,2,\ldots$ and $0 \le \theta \le \pi$, defined by the generating function

$$(0.1)\qquad (1-re^{-i\theta})^{-\alpha}\,(1-re^{i\theta})^{-\beta} = \sum_{k=0}^{\infty} r^k\, C_k^{(\alpha,\beta)}(e^{i\theta}).$$

The impetus for the study of the $C_k^{(\alpha,\beta)}$-s comes from the Dirichlet problem for a class of second order differential operators, L_γ, on the Heisenberg group H_n. H_n has underlying manifold $\mathbb{C}^n \times \mathbb{R}$ and the non-abelian multiplication

$$(0.2)\qquad (z,t)(z',t') = (z+z',t+t'+2\mathrm{Im}\; z\cdot z'),$$

where $z = (z_1,\ldots,z_n)$ and $z\cdot z' := \sum_{j=1}^n z_j\bar{z}_j'$. With this group law the groups H_n form the simplest class of non-commutative nilpotent Lie groups. Define

$$(0.3)\qquad Z_j := \frac{\partial}{\partial z_j} + i\,\bar{z}_j\,\frac{\partial}{\partial t}, \qquad j = i,\ldots,n.$$

$\{Z_1,\ldots,Z_n,\bar{Z}_1,\ldots,\bar{Z}_n,\frac{\partial}{\partial t}\}$ is a basis for the Lie algebra of left-invariant vector fields on H_n. Set

$$(0.4)\qquad L_\gamma := -\tfrac{1}{2}\sum_{j=1}^{n} (Z_j\bar{Z}_j+\bar{Z}_jZ_j) + i\,\gamma\,\frac{\partial}{\partial t}\,.$$

L_γ is left-invariant with respect to (0.2) and invariant under the natural action of the group $U(n)$ on the z-coordinates. Given $R > 0$ one introduces the dilation R: $(z,t) \mapsto (Rz,R^2t)$. Then L_γ is homogeneous in the sense that $L_\gamma(f\circ R) = R^2(L_\gamma f)\circ R$ for any smooth function f.

L_γ is not elliptic. Nevertheless, FOLLAND [5] (for $\gamma=0$) and FOLLAND & STEIN [6] showed that L_γ has a fundamental solution at any u in H_n:

$$(0.5)\qquad L_\gamma^{(u)}\Phi_\gamma(v^{-1}u) = \delta_{(v)},\quad u,v \in H_n,\quad \pm\gamma \neq n,n+2,\ldots,$$

where

$$(0.6)\qquad \Phi_\gamma((z,t)) := c_\gamma(|z|^2+it)^{-\frac{1}{2}(n-\gamma)}(|z|^2-it)^{-\frac{1}{2}(n+\gamma)}$$

for some constant c_γ.

There is a great deal of similarity between L_γ on H_n and the usual Laplace operator, $\Delta := \sum_{j=1}^{n} \partial^2/\partial x_j^2$ on $\mathbb{R}^n$ (cf. [9]). To deepen the analogy we say that f is H-homogenous of degree k if

$$(0.7) \qquad f(Rz, R^2 t) = R^k f(z,t), \qquad R > 0,$$

and that f is L_γ-harmonic if

$$(0.8) \qquad L_\gamma f = 0.$$

If $\pm\gamma \neq n, n+2, n+4, \ldots$ L_γ-harmonics are real-analytic. This follows from the analyticity of Φ_γ away from the origin. Hence an L_γ-harmonic has a convergent power series expansion near the origin. In analogy with Δ we consider the power series as a sum of H-homogeneous L_γ-harmonic polynomials. Such polynomials were first described in [9] where the discussion was restricted to H_1. DUNKL extended this to H_n in [3]. The space of L_γ-harmonic H-homogeneous polynomials of degree m uniquely splits as a direct sum of irreducible subspaces under the action of U(n). In spherical coordinates adapted to H_n, the functions in these irreducible subspaces factorize and one of the factors is a function $C_k^{(\alpha,\beta)}$. Dunkl also expanded $\Phi_\gamma(v^{-1}u)$ in a series of H-homogeneous L_γ-harmonic polynomials in u whose coefficients are functions of v, which are H-homogeneous L_γ-harmonic functions near infinity, and singular at the origin. This is in complete analogy with such an expansion of the classical Newtonian potential, $|x-y|^{-n+2}$, in a double series of spherical harmonics on $\mathbb{R}^n$, which are Kelvin transforms of each other. It motivated us to introduce an analogue of the Kelvin transform on H_1. Independently, KORÁNYI [17] introduced a Kelvin transform on H_n, guided by group theoretic motivations. Unfortunately, this transform does not operate radially, thus there is no obvious way it can be used to solve Dirichlet's problem for L_γ in the unit Heisenberg ball $\{(z,t) \in H_n \mid |z|^4+t^2 < 1\}$.

Using probabilistic methods GAVEAU [8] showed that the Dirichlet problem for L_0 has a solution in the Heisenberg ball in H_n. An analytic proof (also for certain L_γ) was later given by JERISON [12], [13]. Heuristically this result suggests that, by restricting the H-homogeneous L_γ-

harmonic polynomials to the surface of the unit Heisenberg sphere one obtains a "complete" system of functions. More precisely, introducing spherical coordinates adapted to H_n, we are interested in the "completeness" of the system $\{\theta \mapsto C_k^{(\alpha,\beta)}(e^{i\theta})\}_{k=0,1,2,\ldots}$ on $(0,\pi)$ (see [9]). We note that that the $C_k^{(\alpha,\beta)}$-s on H_n are the analogues of the Gegenbauer polynomials on $\mathbb{R}^n$.

Finally, a short outline of this article is in order. Section 1 discusses analytic properties of the $C_k^{(\alpha,\beta)}$-s, integral representations, bilinear generating functions, and orthogonality on $[0,2\pi]$, originally found by GASPER (see [7]). Section 2 is devoted to a discussion of the Kelvin transform on H_n (in a way which is even more group theoretical and less computational than in Korányi's approach), while in sections 3 and 4 we calculate the L_γ-harmonic polynomials and discuss Dirichlet's problem. Finally, in chapter 5 we expand $\Phi_\gamma(v^{-1}u)$ in a sum of products of harmonics near zero and of harmonics near infinity by the use of the Kelvin transform. This yields a new proof of Dunkl's expansion. Next, knowing that the Poisson kernel in the Heisenberg ball exists for L_0, we show that its spherical harmonics are dense in the class of continuous functions on the surface of the unit Heisenberg ball.

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1. ANALYTIC PROPERTIES

1.1. Definition of the functions $C_k^{(\alpha,\beta)}$

For complex α,β the functions $C_k^{(\alpha,\beta)}$ $(k = 0,1,2,\ldots)$ are defined by the generating function

$$(1.1.)\qquad (1-z\bar{\zeta})^{-\alpha}(1-z\zeta)^{-\beta} = \sum_{k=0}^{\infty} z^k C_k^{(\alpha,\beta)}(\zeta),\quad z,\zeta \in \mathbb{C},\quad |z| < |\zeta|^{-1}.$$

It follows immediately that

$$(1.2)\qquad C_k^{(\alpha,\beta)}(\zeta) = \sum_{j=0}^{k} \frac{(\alpha)_{k-j}(\beta)_j}{(k-j)!\,j!}\,\bar{\zeta}^{k-j}\zeta^j,\qquad \zeta \in \mathbb{C},$$

$$(1.3) \qquad C_k^{(\alpha,\beta)}(e^{i\phi}) = \sum_{j=0}^{k} \frac{(\alpha)_{k-j}(\beta)_j}{(k-j)!\,j!}\, e^{i(2j-k)\phi}, \quad \phi \in \mathbb{R}.$$

Here we follow GASPER's [7] notation. GREINER [9], who first introduced these functions, denoted them by $H_k^{\alpha,n}$ $(\alpha\in\mathbb{C}, n,k\in\mathbb{Z}, k\geq 0)$. On comparing [9, (8.7)] with (1.1) we find

$$(1.4) \qquad H_k^{(\alpha,n)}(e^{i\phi}) = \begin{cases} C_k^{(-(\alpha-1)/2,\,n+(\alpha-1)/2)}(e^{i\phi}), & n \geq 0, \\ C_k^{(-n-(\alpha-1)/2,\,(\alpha-1)/2)}(e^{i\phi}), & n \leq 0. \end{cases}$$

From (1.3) we obtain:

$$(1.5) \qquad C_k^{(\alpha,\beta)}(-e^{i\phi}) = (-1)^k C_k^{(\alpha,\beta)}(e^{i\phi}),$$

$$(1.6) \qquad C_k^{(\alpha,\beta)}(e^{i\phi}) = C_k^{(\beta,\alpha)}(e^{-i\phi}) = \overline{C_k^{(\bar\beta,\bar\alpha)}(e^{i\phi})} = \overline{C_k^{(\bar\alpha,\bar\beta)}(e^{-i\phi})},$$

$$(1.7) \qquad C_k^{(\alpha,\beta)}(e^{i\phi}) = \frac{(\alpha)_k}{k!}\, e^{-ik\phi}\, {}_2F_1(-k,\beta,1-\alpha-k;e^{2i\phi})\,(\alpha\neq 0,-1,\ldots,-k+1)$$

$$= \frac{(\beta)_k}{k!}\, e^{ik\phi}\, {}_2F_1(-k,\alpha;1-\beta-k;e^{-2i\phi})\,(\beta\neq 0,-1,\ldots,-k+1),$$

$$(1.8) \qquad C_k^{(\alpha,\beta)}(1) = \frac{(\alpha+\beta)_k}{k!}.$$

Special cases are

$$(1.9) \qquad C_k^{(\alpha,\alpha)}(e^{i\phi}) = C_k^{\alpha}(\cos\phi),$$

where C_k^{α} denotes a Gegenbauer polynomial,

$$(1.10) \qquad C_k^{(\alpha,0)}(e^{i\phi}) = \frac{(\alpha)_k}{k!}\, e^{-ik\phi},$$

$$(1.11) \qquad C_k^{(0,\beta)}(e^{i\phi}) = \frac{(\beta)_k}{k!}\, e^{ik\phi}.$$

Finally, by (1.3) and (1.8) we have:

$$(1.12)\qquad |C_k^{(\alpha,\beta)}(e^{i\phi})| \le C_k^{(|\alpha|,|\beta|)}(1) = \frac{(|\alpha|+|\beta|)_k}{k!} = \mathcal{O}(k^{|\alpha|+|\beta|-1}) \text{ as } k \to \infty.$$

1.2. Orthogonality properties

In this subsection we give a new proof of GASPER's [7] orthogonality for the functions $C_k^{(\alpha,\beta)}$ and we show that there is some more freedom in the choice of the weight function. In the special case $\beta = \alpha + 1$ we get an orthogonality which was earlier obtained by ASKEY [2].

LEMMA 1.1. *Let* $\alpha,\beta \in \mathbb{C}$, $\mathrm{Re}(\alpha+\beta) > 0$, $k \in \{0,1,2,\ldots\}$, $\ell \in \{-k-1,-k+1,\ldots,k-1,k+1\}$. *Then*

$$(1.13)\qquad \int_0^\pi C_k^{(\alpha,\beta)}(e^{i\phi})e^{i(\ell+\beta-\alpha)\phi}(\sin\phi)^{\alpha+\beta-1}d\phi =$$

$$= \frac{e^{\frac{1}{2}i(-\alpha+\beta-1)\pi}\pi\Gamma(\alpha+\beta+k)}{2^{\alpha+\beta-1}\Gamma(\beta)\Gamma(\alpha+k+1)}\delta_{\ell,-k-1} + \frac{e^{\frac{1}{2}i(-\alpha+\beta+1)\pi}\pi\Gamma(\alpha+\beta+k)}{2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta+k+1)}\delta_{\ell,k+1}.$$

PROOF. Let I denote the left hand side of (1.13). Then, by (1.7):

$$I = \frac{e^{\frac{1}{2}i\pi(\alpha+\beta-1)}(\alpha)_k}{2^{\alpha+\beta-1}k!}\int_0^\pi {}_2F_1(-k,\beta;\,1-\alpha-k;e^{2i\phi})\,\cdot$$

$$\cdot\; e^{i(\ell-k-2\alpha+1)\phi}(1-e^{2i\phi})^{\alpha+\beta-1}d\phi =$$

$$= \frac{e^{\frac{1}{2}i\pi(\alpha+\beta-2)}(\alpha)_k}{2^{\alpha+\beta}k!}\int_1^{(0+)} {}_2F_1(-k,\beta;1-\alpha-k;z)\,\cdot$$

$$\cdot\; z^{\frac{1}{2}(\ell-k-1)-\alpha}(1-z)^{\alpha+\beta-1}dz \quad (0\le\arg z\le 2\pi).$$

Substitution of the Rodrigues type formula

$${}_2F_1(-k,k+\gamma+\delta+1;\gamma+1;z)\,z^\gamma(1-z)^\delta = \frac{1}{(\gamma+1)_k}\left(\frac{d}{dz}\right)^k[z^{\gamma+k}(1-z)^{\delta+k}]$$

(cf. [4, 10.8(10),10.8(16)]) yields

$$I = \frac{e^{\frac{1}{2}i\pi(\alpha+\beta-2)}(-1)^k}{2^{\alpha+\beta}k!} \int_1^{(0+)} (\frac{d}{dz})^k[z^{-\alpha}(1-z)^{\alpha+\beta+k-1}] . \cdot z^{\frac{1}{2}(k+\ell-1)} dz.$$

Repeated integration by parts yields $I = 0$ if $\ell = -k+1,-k+3,\ldots,k-1$. If $\ell = -k-1$ then

$$I = \frac{e^{\frac{1}{2}i\pi(\alpha+\beta-2)}(-1)^k}{2^{\alpha+\beta}} \int_1^{(0+)} z^{-\alpha-k-1}(1-z)^{\alpha+\beta+k-1}\, dz,$$

which can be evaluated by [4,1.6(9),1.5(5),1.2(6)]. Finally, the case $\ell = k+1$ follows from (1.13) for $\ell = k-1$ by the transformation of integration variable $\phi \to \pi-\phi$ and by (1.5), (1.6). □

PROPOSITION 1.2. *For complex* c_1,c_2,α,β *with* $\mathrm{Re}(\alpha+\beta) > 0$ *let the weight function* w *be defined by*

$$w(\phi) = w(\phi+\pi) := e^{i(\beta-\alpha)\phi}(c_1e^{i\phi}+c_2e^{-i\phi})(\sin\phi)^{\alpha+\beta-1}, \quad 0 < \phi < \pi.$$

Then, for nonnegative integers k,ℓ:

$$(1.14) \qquad \int_0^{2\pi} C_k^{(\alpha,\beta)}(e^{i\phi})C_\ell^{(\alpha,\beta)}(e^{i\phi})w(\phi)\,d\phi = \left(\frac{c_1}{\beta+k} - \frac{c_2}{\alpha+k}\right) \frac{e^{\frac{1}{2}i(-\alpha+\beta+1)\pi}\pi\Gamma(\alpha+\beta+k)}{2^{\alpha+\beta-2}\Gamma(\alpha)\Gamma(\beta)k!}\,\delta_{k,\ell}.$$

PROOF. Because of (1.5) it is sufficient to evaluate the integral at the left hand side from 0 to π for $k-\ell$ even. This can be done by the use of (1.3) and (1.13). □

GASPER [7] showed that

$$\int_0^{\pi} C_k^{(\alpha,\beta)}(e^{i\phi})e^{i(\ell+\beta-\alpha)\phi}(\sin\phi)^{\alpha+\beta}d\phi = 0$$

for $\ell = -k+2,-k+4,\ldots,k-2$, which is implied by our formula (1.13), and next he derived the case $c_1 = -c_2$ of Prop.1.2.

PROPOSITION 1.3. *If* $\mathrm{Re}(\alpha+\beta) > 0$ *then*

$$(1.15)\qquad \int_0^\pi e^{ik\phi} C_k^{(\alpha,\beta)}(e^{i\phi}) e^{-i\ell\phi} C_\ell^{(\beta-1,\alpha+1)}(e^{-i\phi}) e^{i\phi(\beta-\alpha-1)} (\sin\phi)^{\alpha+\beta-1} d\phi = = \frac{e^{\frac12 i\pi(-\alpha+\beta-1)}\pi\Gamma(\alpha+\beta+k)}{2^{\alpha+\beta-1}\Gamma(\alpha+1)\Gamma(\beta)k!}\,\delta_{k,\ell}\,.$$

PROOF. If $k \ge \ell$ then substitute (1.3) for $C_\ell^{(\beta-1,\alpha+1)}(e^{-i\phi})$ and apply (1.13). If $k < \ell$ then make the change of integration variable $\phi \mapsto \pi-\phi$ in (1.15), substitute (1.3) for C_k and again apply (1.13). □

COROLLARY 1.4. *If* $\alpha > -\frac12$ *then*

$$(1.16)\qquad \int_0^\pi e^{ik\phi} C_k^{(\alpha,\alpha+1)}(e^{i\phi}) \overline{e^{i\ell\phi} C_\ell^{(\alpha,\alpha+1)}(e^{i\phi})} (\sin\phi)^{2\alpha}\, d\phi = \frac{\pi\Gamma(2\alpha+k+1)}{2^{2\alpha}\Gamma(\alpha+1)\Gamma(\alpha+1)k!}\,\delta_{k,\ell}.$$

PROOF. (1.15) with $\beta = \alpha+1$. □

Formulas (1.15), (1.16) were given by ASKEY in [2] and [1], respectively. Note that (1.15) and (1.16) give a biorthogonality respectively orthogonality for the functions $\phi \mapsto e^{ik\phi} C_k^{(\alpha,\beta)}(e^{i\phi})$ on $(0,\pi)$ and that (1.14) gives an orthogonality for the functions $\phi \mapsto C_k^{(\alpha,\beta)}(e^{i\phi})$ on $(0,2\pi)$. However, what would be needed for the applications we have in mind and what is unfortunately unknown is a (bi)orthogonality for the latter functions on $(0,\pi)$.

Formula (1.13) implies yet another orthogonality:

PROPOSITION 1.5. *Let* $\mathrm{Re}(\alpha+\beta) > 1$. *Then, for* $\ell, m \in \{0,1,\ldots,k\}$:

$$(1.17)\qquad \int_0^\pi (\sin\phi)^\ell C_{k-\ell}^{(\alpha+\ell,\beta+\ell)}(e^{i\phi}) (\sin\phi)^m C_{k-m}^{(\alpha+m,\beta+m)}(e^{i\phi})\,\cdot \cdot\,(\sin\phi)^{\alpha+\beta-2} e^{i(\beta-\alpha)\phi}\, d\phi = = \frac{e^{\frac12 i\pi(\beta-\alpha)}(\alpha+\beta+2\ell)_{k-\ell}\,\pi\Gamma(\alpha+\beta+2\ell-1)}{2^{\alpha+\beta+2\ell-2}(k-\ell)!\,\Gamma(\alpha+\ell)\Gamma(\beta+\ell)}\,\delta_{\ell,m}\,.$$

PROOF. In the case $\ell \neq m$ apply (1.13). In the case $\ell = m$ (1.13) can also be used in order to rewrite the left hand side of (1.17) as

$$C_{k-\ell}^{(\alpha+\ell,\beta+\ell)}(1) \int_0^\pi C_{k-\ell}^{(\alpha+\ell,\beta+\ell)}(e^{i\phi})(\sin\phi)^{\alpha+\beta+2\ell-2} e^{i(\beta-\alpha-k+\ell)\phi} d\phi \; .$$

By (1.7), (1.8) and [4,1.5(29)] this becomes

$$\frac{e^{\frac{1}{2}i\pi(\beta-\alpha)}(\alpha+\beta+2\ell)_{k-\ell}(\beta+\ell)_{k-\ell}\pi\Gamma(\alpha+\beta+2\ell-1)}{2^{\alpha+\beta+2\ell-2}(k-\ell)!\,(k-\ell)!\,\Gamma(\alpha+\ell)\Gamma(\beta+\ell)} \cdot$$

$$\cdot\ {}_2F_1(-k+\ell,-\beta-\ell+1;-\beta-k+1;1)\,.$$

Finally apply [4,2.8(46)]. □

The above proposition tells us that the functions $\phi \mapsto (\sin\phi)^\ell C_{k-\ell}^{(\alpha+\ell,\beta+\ell)}(e^{i\phi})$ ($\ell = 0,1,\ldots,k$) form an orthogonal basis on $[0,\pi]$ for the space of trigonometric polynomials f of degree $\leq k$ satisfying $f(\phi+\pi) = (-1)^k f(\phi)$.

1.3. Integral representations

ISMAIL [11] derived the following Laplace type integral representation:

$$(1.18) \qquad \frac{C_k^{(\alpha,\beta)}(e^{i\phi})}{C_k^{(\alpha,\beta)}(1)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\pi (\cos\phi+i\sin\phi\cos\psi)^k \cdot$$

$$\cdot\ (\sin\tfrac{1}{2}\psi)^{2\alpha-1}(\cos\tfrac{1}{2}\psi)^{2\beta-1}d\psi, \qquad \mathrm{Re}\,\alpha > 0,\ \mathrm{Re}\,\beta > 0.$$

For the proof note that

$$(\cos\phi+i\sin\phi\cos\psi)^k = (e^{i\phi}\cos^2\tfrac{1}{2}\psi+e^{-i\phi}\sin^2\tfrac{1}{2}\psi)^k,$$

write down the binomial expansion of the right hand side, use the beta integral and apply (1.3).

More generally we have

$$(1.19) \qquad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\pi (\cos\phi+i\sin\phi\cos\psi)^k\, P_\ell^{(\alpha-1,\beta-1)}(\cos\psi) \cdot$$

$$\cdot\ (\sin\tfrac{1}{2}\psi)^{2\alpha-1}(\cos\tfrac{1}{2}\psi)^{2\beta-1}d\psi =$$

$$= \frac{k!(\alpha)_\ell(\beta)_\ell}{\ell!(\alpha+\beta)_{k+\ell}} (2i\sin\phi)^\ell \, C_{k-\ell}^{(\alpha+\ell,\beta+\ell)}(e^{i\phi}), \quad \mathrm{Re}\,\alpha > 0, \ \mathrm{Re}\,\beta > 0,$$

where $P_\ell^{(\alpha-1,\beta-1)}$ is a Jacobi polynomial. For the proof substitute the Rodrigues type formula for the Jacobi polynomial into the left hand side of (1.19), perform integration by parts and reduce to (1.18).

From (1.19) we obtain the Jacobi series expansion

$$(1.20) \qquad (\cos\phi + i\sin\phi\cos\psi)^k =$$

$$= \sum_{\ell=0}^{k} \frac{(2\ell+\alpha+\beta-1)k!(\alpha+\beta)_\ell}{(\ell+\alpha+\beta-1)(\alpha+\beta)_{k+\ell}} (2i\sin\phi)^\ell C_{k-\ell}^{(\alpha+\ell,\beta+\ell)}(e^{i\phi}) P_\ell^{(\alpha-1,\beta-1)}(\cos\psi).$$

For $\alpha = \beta$ these three formulas reduce to well-known formulas for Gegenbauer polynomials.

Because of Prop.1.5, the right hand side of formula (1.20) can be viewed as a double orthogonal expansion of the left hand side, with respect to the measure $(\sin\phi)^{\alpha+\beta-2} e^{i(\beta-\alpha)\phi} d\phi$ on $(0,\pi)$ in the ϕ-variable and with respect to the measure $(\sin\frac{1}{2}\psi)^{2\alpha-1}(\cos\frac{1}{2}\psi)^{2\beta-1} d\psi$ on $(0,\pi)$ in the ψ-variable. Hence, by (1.17) the following formula is also an integrated form of (1.20):

$$(1.21) \qquad \frac{2^{\alpha+\beta-2}\Gamma(\alpha)\Gamma(\beta)}{e^{\frac{1}{2}i\pi(\beta-\alpha)}\pi\Gamma(\alpha+\beta-1)} \int_0^\pi (\cos\phi + i\sin\phi\cos\psi)^k .$$

$$\cdot\, C_{k-\ell}^{(\alpha+\ell,\beta+\ell)}(e^{i\phi})(\sin\phi)^{\alpha+\beta+\ell-2} e^{i(\beta-\alpha)\phi} d\phi =$$

$$= \frac{(-k)_\ell(\alpha+\beta-1)_\ell}{(2i)^\ell(\alpha)_\ell(\beta)_\ell} P_\ell^{(\alpha-1,\beta-1)}(\cos\psi).$$

A Mehler-Dirichlet type integral representation

$$(1.22) \qquad e^{\frac{1}{2}i(\beta-\alpha)\phi} \frac{(\sin\phi)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \frac{C_k^{(\alpha,\beta)}(e^{i\phi})}{C_k^{(\alpha,\beta)}(1)} =$$

$$= \int_0^\phi \frac{(\sin(\phi-\theta))^{\alpha-1}}{\Gamma(\alpha)} \frac{(\sin\theta)^{\beta-1}}{\Gamma(\beta)} e^{i(k+\frac{1}{2}(\alpha+\beta))(2\theta-\phi)} d\theta,$$

$$0 < \phi < \pi, \quad \mathrm{Re}\,\alpha > 0, \ \mathrm{Re}\,\beta > 0,$$

can be derived from (1.18) as follows. First make the substitution

$z = \cos\phi + i\sin\phi\cos\psi$ in (1.18), next deform the contour to an arc from $e^{i\phi}$ to $e^{-i\phi}$ and finally put $z = e^{-i\phi}e^{2i\theta}$. Note that, in a sense, (1.22) is dual to (1.3): the rôle of k and ϕ is interchanged. Reduction to the case $\alpha = \beta$ again gives a familiar formula for Gegenbauer pulynomials.

1.4. Bilinear generating functions

Appell's hypergeometric function F_1 is defined by the double power series

$$(1.23)\qquad F_1(\alpha,\beta,\beta',\gamma;x,y) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}\, m!\, n!} x^m y^n, \quad \gamma \neq 0,-1,-2,\ldots,$$

which converges for $|x|,|y| < 1$. By the integral representation [4,5.8(5)], valid for Re $\alpha > 0$, $\mathrm{Re}(\gamma-\alpha) > 0$, the function $F_1(\alpha,\beta,\beta',\gamma,\cdot,\cdot)$ has an analytic continuation to a one-valued function on $\{(x,y) \in \mathbb{C}^2 \mid x,y \notin [1,\infty)\}$.

LEMMA 1.6. *If* Re $\gamma > 0$, $z \in \mathbb{C}$, $|z| < 1$ *then*

$$(1.24)\qquad \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{(\gamma+\frac{1}{2})_k} z^{2k} C_{2k}^{(\alpha,\beta)}(e^{i\phi}) = (1+ze^{i\phi})^{-\beta}(1+ze^{-i\phi})^{-\alpha}\cdot$$

$$\cdot F_1\left(\gamma,\alpha,\beta,2\gamma;\ \frac{2ze^{-i\phi}}{1+ze^{-i\phi}},\ \frac{2ze^{i\phi}}{1+ze^{i\phi}}\right).$$

PROOF. We prove (1.24) for $z = r$ with $0 < r < 1$. Then the general case $|z| < 1$ follows by analytic continuation in view of (1.11). An easy calculation yields

$$(1.25)\qquad \frac{(\frac{1}{2})_k}{(\gamma+\frac{1}{2})_k} r^{2k} = \frac{\Gamma(\gamma+\frac{1}{2})}{\Gamma(\gamma)\Gamma(\frac{1}{2})} r^{1-2\gamma} \int_{-r}^{r} (r^2-\rho^2)^{\gamma-1}\rho^{2k}d\rho.$$

Thus, again in view of (1.11), the left hand side of (1.24) equals

$$\frac{\Gamma(\gamma+\frac{1}{2})r^{1-2\gamma}}{\Gamma(\gamma)\Gamma(\frac{1}{2})} \int_{-r}^{r} (r^2-\rho^2)^{\gamma-1} \sum_{k=0}^{\infty} \rho^{2k} C_{2k}^{(\alpha,\beta)}(e^{i\phi})d\rho,$$

which, by the use of (1.1), can be written as

$$\frac{\Gamma(\gamma+\frac{1}{2})r^{1-2\gamma}}{\Gamma(\gamma)\Gamma(\frac{1}{2})} \int_{-r}^{r} (r^2-\rho^2)^{\gamma-1}(1+\rho e^{-i\phi})^{-\alpha}(1+\rho e^{i\phi})^{-\beta}d\rho.$$

By making the change of integration variable $t = \frac{r-\rho}{2r}$ this equals

$$\frac{2^{2\gamma-1}\Gamma(\gamma+\frac{1}{2})}{\Gamma(\gamma)\Gamma(\frac{1}{2})}(1+re^{-i\phi})^{-\alpha}(1+re^{i\phi})^{-\beta}\,\cdot$$

$$\cdot\int_0^1 (t(1-t))^{\gamma-1}\left(1-t\,\frac{2re^{-i\phi}}{1+re^{-i\phi}}\right)^{-\alpha}\left(1-t\,\frac{2re^{i\phi}}{1+re^{i\phi}}\right)^{-\beta}dt.$$

Now (1.24) follows by the use of [4,5.8(5)] and [4,1.2(15)]. □

Because of [4,5.10(1)], formula (1.24) can be simplified in the case $\gamma = \frac{1}{2}(\alpha+\beta)$:

$$(1.26)\qquad \sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{(\frac{1}{2}(\alpha+\beta+1))_k}\,z^{2k}C_{2k}^{(\alpha,\beta)}(e^{i\phi}) =$$

$$= (1+ze^{-i\phi})^{-\alpha}(1+ze^{i\phi})^{\frac{1}{2}(\alpha-\beta)}(1-ze^{i\phi})^{-\frac{1}{2}(\alpha+\beta)}\,\cdot$$

$$\cdot\, {}_2F_1\left(\tfrac{1}{2}(\alpha+\beta),\alpha;\alpha+\beta;\,\frac{2z(e^{-i\phi}-e^{i\phi})}{(1-ze^{i\phi})(1+ze^{-i\phi})}\right),$$

$$\mathrm{Re}(\alpha+\beta) > 0,\quad z\in\mathbb{C},\quad |z|<1.$$

Next we derive a bilinear generating function involving $C_k^{(\alpha,\beta)}$ and the Gegenbauer polynomial C_k^{γ}.

<u>THEOREM 1.7</u>. *If* $\mathrm{Re}\,\gamma > 0$, $z\in\mathbb{C}$, $|z|<1$ *then*

$$(1.27)\qquad \sum_{k=0}^{\infty} z^k\, C_k^{(\alpha,\beta)}(e^{i\theta})\,\frac{C_k^{\gamma}(\cos\phi)}{C_k^{\gamma}(1)} = F_1\Big(\gamma,\alpha,\beta,2\gamma;$$

$$;\,\frac{2ize^{-i\theta}\sin\phi}{1-ze^{-i(\theta+\phi)}},\ \frac{2ize^{i\theta}\sin\phi}{1-ze^{i(\theta-\phi)}}\Big)(1-ze^{-i(\theta+\phi)})^{-\alpha}(1-ze^{i(\theta-\phi)})^{-\beta}.$$

<u>PROOF</u>. We prove (1.27) for $|z| < \frac{1}{2}$. Then the more general case $|z| < 1$ will follow by analytic continuation in view of (1.12). We will start the proof with an additional parameter δ ($\mathrm{Re}\,\delta>0$) and, at a certain stage, we will put $\delta = \gamma$. It follows from (1.18), (1.12) and (1.1) that

$$\sum_{k=0}^{\infty} z^k C_k^{(\alpha,\beta)}(e^{i\theta}) \frac{C_k^{(\gamma,\delta)}(e^{i\phi})}{C_k^{(\gamma,\delta)}(1)} = \sum_{k=0}^{\infty} z^k C_k^{(\alpha,\beta)}(e^{i\phi}) \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} \cdot$$

$$\cdot \int_0^{\pi} (\cos\phi + i\sin\phi\cos\psi)^k (\sin\tfrac{1}{2}\psi)^{2\gamma-1} (\cos\tfrac{1}{2}\psi)^{2\delta-1}\, d\psi =$$

$$= \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} \int_0^{\pi} (1-ze^{-i\theta}(\cos\phi+i\sin\phi\cos\psi))^{-\alpha} \cdot$$

$$\cdot (1-ze^{i\theta}(\cos\phi+i\sin\phi\cos\psi))^{-\beta} (\sin\tfrac{1}{2}\psi)^{2\gamma-1} (\cos\tfrac{1}{2}\psi)^{2\delta-1}\, d\psi =$$

$$= \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} (1-ze^{-i\theta}\cos\phi)^{-\alpha} (1-ze^{i\theta}\cos\phi)^{-\beta} \cdot$$

$$\cdot \int_0^{\pi} \left(1- \frac{ize^{-i\theta}\sin\phi\cos\psi}{1-ze^{-i\theta}\cos\phi}\right)^{-\alpha} \left(1- \frac{ize^{i\theta}\sin\phi\cos\psi}{1-ze^{i\theta}\cos\phi}\right)^{-\beta} \cdot$$

$$\cdot (\sin\tfrac{1}{2}\psi)^{2\gamma-1} (\cos\tfrac{1}{2}\psi)^{2\delta-1}\, d\psi =$$

$$= \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} (1-ze^{-i\theta}\cos\phi)^{-\alpha} (1-ze^{i\theta}\cos\phi)^{-\beta} \cdot$$

$$\cdot \sum_{k,\ell=0}^{\infty} \frac{(\alpha)_k (\beta)_\ell}{k!\,\ell!} \left(\frac{ize^{-i\theta}\sin\phi}{1-ze^{-i\theta}\cos\phi}\right)^k \left(\frac{ize^{i\theta}\sin\phi}{1-ze^{i\theta}\cos\phi}\right)^\ell \cdot$$

$$\cdot \int_0^{\pi} (\cos\psi)^{k+\ell} (\sin\tfrac{1}{2}\psi)^{2\gamma-1} (\cos\tfrac{1}{2}\psi)^{2\delta-1}\, d\psi .$$

Now assume $\gamma = \delta$. Then

$$\int_0^{\pi} (\cos\psi)^{k+\ell} (\sin\tfrac{1}{2}\psi)^{2\gamma-1} (\cos\tfrac{1}{2}\psi)^{2\gamma-1} d\psi =$$

$$= \begin{cases} \dfrac{2^{1-2\gamma}\Gamma(\frac{1}{2}(k+\ell+1))\Gamma(\gamma)}{\Gamma(\frac{1}{2}(k+\ell+1)+\gamma)} & \text{if } k+\ell \text{ is even,} \\ 0 & \text{if } k+\ell \text{ is odd.} \end{cases}$$

Hence, by [4,1.2(15)], the left hand side of (1.27) equals

$$(1-ze^{-i\theta}\cos\phi)^{-\alpha}(1-ze^{i\theta}\cos\phi)^{-\beta}.$$

$$\cdot \sum_{\substack{k,\ell=0\\k+\ell\text{ even}}}^{\infty} \frac{(\alpha)_k}{k!}\frac{(\beta)_\ell}{\ell!}\frac{(\frac12)_{\frac12(k+\ell)}}{(\gamma+\frac12)_{\frac12(k+\ell)}}\left(\frac{ize^{-i\theta}\sin\phi}{1-ze^{-i\theta}\cos\phi}\right)^k\left(\frac{ize^{i\theta}\sin\phi}{1-ze^{i\theta}\cos\phi}\right)^\ell =$$

$$= (1-ze^{-i\theta}\cos\phi)^{-\alpha}(1-ze^{i\theta}\cos\phi)^{-\beta}\ .$$

$$\cdot \sum_{p=0}^{\infty}\sum_{\ell=0}^{2p}\frac{(\beta)_\ell}{\ell!}\frac{(\alpha)_{2p-\ell}}{(2p-\ell)!}\frac{(\frac12)_p}{(\gamma+\frac12)_p}\left(\frac{-z^2\sin^2\phi}{1-2z\cos\phi\cos\theta+z^2\cos^2\phi}\right)^p\ .$$

$$\cdot\left(e^{i\theta}\left(\frac{1-ze^{-i\theta}\cos\phi}{1-ze^{i\theta}\cos\phi}\right)^{\frac12}\right)^{2\ell-2p}.$$

By substitution of (1.3) this equals

$$(1-ze^{-i\theta}\cos\phi)^{-\alpha}(1-ze^{i\theta}\cos\phi)^{-\beta}\sum_{p=0}^{\infty}\frac{(\frac12)_p}{(\gamma+\frac12)_p}\left(\frac{-z^2\sin^2\phi}{1-2z\cos\phi\cos\theta+z^2\cos^2\phi}\right)^p.$$

$$\cdot\ C_{2p}^{(\alpha,\beta)}\left(e^{i\theta}\left(\frac{1-ze^{-i\theta}\cos\phi}{1-ze^{i\theta}\cos\phi}\right)^{\frac12}\right).$$

Finally, substitution of (1.24) leads to the right hand side of (1.27). □

<u>COROLLARY 1.8</u>. *If* $\mathrm{Re}(\alpha+\beta) > 0$, $z \in \mathbb{C}$, $|z| < 1$ *then*

$$\sum_{k=0}^{\infty} z^k C_k^{(\alpha,\beta)}(e^{i\theta})\,\frac{C_k^{\frac12(\alpha+\beta)}(\cos\phi)}{C_k^{\frac12(\alpha+\beta)}(1)} = \tag{1.28}$$

$$= \left(\frac{1-ze^{i\theta}e^{-i\phi}}{1-ze^{-i\theta}e^{-i\phi}}\right)^{\frac12(\alpha-\beta)}(1-2z\cos(\phi+\theta)+z^2)^{-\frac12(\alpha+\beta)}\ .$$

$$\cdot\ {}_2F_1\left(\tfrac12(\alpha+\beta),\alpha;\alpha+\beta;\ \frac{4z\sin\phi\sin\theta}{1-2z\cos(\phi+\theta)+z^2}\right).$$

<u>PROOF</u>. Use [4,5.10(1)]. □

REMARK 1.9. Formula (1.28) has the following significance. As will be apparent later in this paper, the main obstruction to finding the Poisson kernel for the Dirichlet problem on the Heisenberg ball is the fact that an explicit kernel for the transform sending $\sum_{k=0}^{\infty} c_k C_k^{(\alpha,\beta)}(e^{i\theta})$ to

$$\sum_{k=0}^{\infty} c_k z^k C_k^{(\alpha,\beta)}(e^{i\phi})\quad(|z|<1)$$

is not available. Formula (1.28) gives an answer to a related question. Due to the orthogonality property of the Gegenbauer polynomials it provides the kernel for the transform which sends $\sum_{k=0}^{\infty} c_k C_k^{\frac{1}{2}(\alpha+\beta)}(\cos\phi)$ to

$$\sum_{k=0}^{\infty} c_k(\tfrac{1}{2}(\alpha+\beta)+k)^{-1} z^k C_k^{(\alpha,\beta)}(e^{i\phi}).$$

REMARK 1.10. C.F. Dunkl (personal communication, unpublished) obtained a dual formula to (1.26):

$$\int_0^{\pi} C_m^{(\alpha,\beta)}(e^{i\theta})C_n^{(\gamma,\delta)}(e^{i\theta})e^{i\delta(\theta-\pi/2)}\sin^{\lambda-1}\theta\, d\theta,$$

with $\alpha+\beta = \gamma+\delta = \lambda$, expressed in terms of a balanced ${}_4F_3$ of unit argument.

2. THE KELVIN TRANSFORM ON THE HEISENBERG GROUP

W. Thomson (Lord Kelvin) proved in 1847 the following fact: If U is harmonic on $\mathbb{R}^3$ then the function V defined by

$$V(x,y,z) := ar^{-1}U(a^2r^{-2}x,a^2r^{-2}y,a^2r^{-2}z)$$

($r := \sqrt{x^2+y^2+z^2}, a>0$) is harmonic on $\mathbb{R}^3\setminus\{0\}$ (cf. KELLOGG [14,p.232]). For this reason the transformation $U \mapsto V$ is called the *Kelvin transform*. In looking at DUNKL's [3, Theorem 1.6] expansion of the translated fundamental solution for L_γ on the Heisenberg group the authors of the present paper conjectured, by analogy to the corresponding case for the Laplace operator, that this double expansion involves, beside harmoncis on the Heisenberg group, certain Kelvin type transforms of these harmonics, which should also be L_γ-harmonic. Indeed, in the case of L_0 on H_1 we were able to give the formula for the Kelvin transform explicitly. Independently, KORÁNYI [17] obtained the Kelvin transform for general L_γ on a Heisenberg group H_n of arbitrary dimension. He was guided by considering H_n as the nilpotent factor in the Iwasawa decomposition of a noncompact semisimple Lie group G and by looking at the action of the Weyl group of G on H_n. Thus he could guess the general form of the formula for the Kelvin transform and by a calculation he could next prove it.

In the following we will present a proof of the Kelvin transform which is even more conceptual and less computational than Koranyi's proof. We will consider H_n as a boundary of a symmetric space and obtain L_γ as a limit case of a Laplace-Beltrami type operator on this symmetric space. Thus L_γ will inherit the symmetries of the Laplace-Beltrami type operator.

As a side result we now have a canonical way of introducing L_γ on H_n for all γ, rather than only for $\gamma = n, n-2, \ldots, -n+2, -n$ by an interpretation using $\Box_b$ (cf. FOLLAND & STEIN [6,§5]).

For $n = 1,2,\ldots$ consider the group

$$G := \{T \in SL(n+2,\mathbb{C}) \mid T^* J T = J\},$$

$$\text{where } J := \begin{pmatrix} 0 & 0 & -\frac{1}{2}i \\ 0 & I_n & 0 \\ \frac{1}{2}i & 0 & 0 \end{pmatrix} \text{ and } T^* \text{ means}$$

adjoint of T. Then G is a noncompact connected semisimple Lie group isomorphic to $SU(n+1,1)$. The group G acting on $\mathbb{C}^{n+2}$ with coordinates $(w_0, w_1, \ldots, w_{n+1})$ leaves the form $|w_1|^2 + \ldots + |w_n|^2 - \mathrm{Im}(w_0 \bar{w}_{n+1})$ invariant. The differential operator Δ on $\mathbb{C}^{n+2}$ defined by

$$\Delta := -\sum_{j=1}^{n} \frac{\partial^2}{\partial w_j \partial \bar{w}_j} + 2i \frac{\partial^2}{\partial w_0 \partial \bar{w}_{n+1}} - 2i \frac{\partial^2}{\partial \bar{w}_0 \partial w_{n+1}} \tag{2.1}$$

is G-invariant.

We now consider some structural facts about G (cf. HELGASON [10,Ch.6, 9] for general structure theory). Let

$$K := \{T \in G \mid T(i,0,\ldots,0,1) = e^{i\phi}(i,0,\ldots,0,1) \text{ for some real } \phi\} \simeq S(U(n+1)\times U(1)),$$

$$A := \{a_s = \mathrm{diag}(e^s, 1, \ldots, 1, e^{-s}) \mid s \in \mathbb{R}\} \simeq \mathbb{R},$$

$$N := \left\{ n_{z,t} = \begin{pmatrix} 1 & 2i\bar{z}_1 & \cdots & 2i\bar{z}_n & t+i|z|^2 \\ & 1 & & \phi & 2z_1 \\ & & \ddots & & \vdots \\ & \phi & & 1 & 2z_n \\ & & & & 1 \end{pmatrix} \middle| (z,t) \in \mathbb{C}^n \times \mathbb{R} \right\}$$

Then G = KAN *is an Iwasawa decomposition* of G. Note that

$$n_{z,t}n_{z',t'} = n_{z'',t''} \quad \text{with}$$

$$(z'',t'') = (z+z', t+t'+2\mathrm{Im}\, z.z'), \tag{2.2}$$

where $z.z' := \sum_{j=1}^{n} z_j \overline{z_j'}$. Thus N is isomorphic to H_n, the Heisenberg group of real dimension 2n+1.

Let M and M' be the centralizer and normalizer, respectively, of A in G. Then

$$M = \left\{ m_T = \begin{pmatrix} (\det T)^{-\frac{1}{2}} & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & (\det T)^{-\frac{1}{2}} \end{pmatrix} \;\middle|\; T \in U(n) \right\}$$

(note that $(\det T)^{-\frac{1}{2}}$, and hence m_T, can assume two different values). Furthermore, M is a normal subgroup of M', the *Weyl group* W := M'/M has order two and $M' = M \cup m_w M$ with

$$m_w = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ +1 & 0 & 0 \end{pmatrix} .$$

Now $G = MAN \cup MAN\, m_w\, MAN$ (disjoint union), a *Bruhat decomposition* of G. Hence the action of G as a transformation group is completely determined by the actions of M,A,N and m_w.

Let $\bar{N} := m_w^{-1} N m_w$. A fact related to the Bruhat decomposition is that $NMA\bar{N}$ is open and dense in G. If N is considered as $NMA\bar{N}/MA\bar{N}$, open and dense in the *flag manifold* $G/MA\bar{N}$, then G acts locally on N. Similarly, if NA is considered as NAK/K = G/K then G acts on NA. We will construct a G-space which includes the G-space NA as an open G-orbit and the G-space N as the boundary of NA.

Introduce new coordinates $\zeta_0, \ldots, \zeta_{n+1}$ on $\mathbb{C}^{n+2}$:

$$\zeta_j = w_j/w_{n+1} \ (j = 0,1,\ldots,n), \quad \zeta_{n+1} = w_{n+1}.$$

Write $\zeta := (\zeta_0, \zeta_1, \ldots, \zeta_n)$. Then the action of G on $\mathbb{C}^{n+2}$, expressed in terms of the new coordinates, takes the form

$$(2.3) \qquad g.(\zeta, \zeta_{n+1}) = (g.\zeta, \mu(\zeta,g)\zeta_{n+1}),$$

where the action of G on the ζ-space $\mathbb{C}^{n+1}$ is a group action and μ is a *multiplier* with respect to this action, i.e., a complex-valued function on $\mathbb{C}^{n+1} \times G$ satisfying

$$(2.4) \qquad \mu(\zeta, g_1 g_2) = \mu(g_2.\zeta, g_1)\mu(\zeta, g_2).$$

The G-action on $\mathbb{C}^{n+1}$ and μ are completely determined by the data in the following table:

g	$g.\zeta$	$\mu(\zeta,g)$
m_T	$(\zeta_0, (\det T)^{\frac{1}{2}} T.(\zeta_1, \ldots, \zeta_n))$	$(\det T)^{-\frac{1}{2}}$
a_s	$(e^{2s}\zeta_0, e^s\zeta_1, \ldots, e^s\zeta_n)$	e^{-s}
$n_{z,t}$	$(\zeta_0 + t + i\lvert z\rvert^2 + 2i \sum_{j=1}^n \zeta_j \overline{z_j}, \zeta_1 + z_1, \ldots, \zeta_n + z_n)$	1
m_w	$(-\frac{1}{\zeta_0}, \frac{\zeta_1}{\zeta_0}, \ldots, \frac{\zeta_n}{\zeta_0})$	ζ_0

Table 1

In terms of the coordinates $\zeta_0, \ldots, \zeta_{n+1}$ the differential operator Δ (cf.(2.1)) takes the form

$$(2.5) \qquad \Delta = |\zeta_{n+1}|^{-2}\Big[\sum_{j=1}^{n}\Big(-\frac{\partial^2}{\partial\zeta_j\partial\bar\zeta_j} + 2i\zeta_j\frac{\partial^2}{\partial\zeta_j\partial\bar\zeta_0} - 2i\,\bar\zeta_j\frac{\partial^2}{\partial\bar\zeta_j\partial\zeta_0}\Big) + $$
$$+ 2i(\zeta_0 - \bar\zeta_0)\frac{\partial^2}{\partial\zeta_0\partial\bar\zeta_0} + 2i\,\bar\zeta_{n+1}\frac{\partial^2}{\partial\bar\zeta_{n+1}\partial\zeta_0} - 2i\zeta_{n+1}\frac{\partial^2}{\partial\zeta_{n+1}\partial\bar\zeta_0}\Big].$$

Then, for α, β in $\mathbb{C}$ and f a smooth function on the ζ-space $\mathbb{C}^{n+1}$:

$$(2.6) \qquad \Delta(\zeta_{n+1}^{-\beta}\overline{\zeta_{n+1}}^{-\alpha}f(\zeta)) = \zeta_{n+1}^{-\beta-1}\overline{\zeta_{n+1}}^{-\alpha-1}\Delta_{\alpha,\beta}f(\zeta),$$

where

$$(2.7) \qquad \Delta_{\alpha,\beta} := \sum_{j=1}^{n}\left(-\frac{\partial^2}{\partial\zeta_j\partial\overline{\zeta}_j} + 2i\,\zeta_j\frac{\partial^2}{\partial\zeta_j\partial\overline{\zeta}_0} - 2i\,\overline{\zeta}_j\frac{\partial^2}{\partial\overline{\zeta}_j\partial\zeta_0}\right) + \\ + 2i(\zeta_0-\overline{\zeta}_0)\frac{\partial^2}{\partial\zeta_0\partial\overline{\zeta}_0} + 2i\,\beta\frac{\partial}{\partial\overline{\zeta}_0} - 2i\alpha\frac{\partial}{\partial\zeta_0}.$$

For fixed g in G and for (ζ,ζ_{n+1}) in $\mathbb{C}^{n+1}\times\mathbb{C}$ write

$$(\zeta',\zeta'_{n+1}) := g.(\zeta,\zeta_{n+1}).$$

Let Δ' denote the operator (2.4) with ζ_j replaced by ζ'_j, and similarly $\Delta'_{\alpha,\beta}$.

LEMMA 2.1. *For smooth functions* f *on* $\mathbb{C}^{n+1}$ *and with* $\zeta' := g.\zeta$, *where* g *in* G *is fixed, we have*

$$(2.8) \qquad \Delta_{\alpha,\beta}(\mu(\zeta,g)^{-\beta}\overline{\mu(\zeta,g)}^{-\alpha}f(\zeta')) = \mu(\zeta,g)^{-\beta-1}\overline{\mu(\zeta,g)}^{-\alpha-1}\Delta'_{\alpha,\beta}f(\zeta').$$

PROOF. The G-invariance of Δ can be expressed by the formula

$$\Delta F(\zeta',\zeta'_{n+1}) = \Delta' F(\zeta',\zeta'_{n+1}).$$

Hence

$$(*) \qquad \Delta(\zeta'^{-\beta}_{n+1}\overline{\zeta'_{n+1}}^{-\alpha}f(\zeta')) = \Delta'(\zeta'^{-\beta}_{n+1}\overline{\zeta'_{n+1}}^{-\alpha}f(\zeta')).$$

The right hand side of (*) equals

$$(**) \qquad \zeta'^{-\beta-1}_{n+1}\overline{\zeta'_{n+1}}^{-\alpha-1}\Delta'_{\alpha,\beta}f(\zeta'),$$

by the use of (2.6). The left hand side of (*) equals

$$\Delta(\zeta_{n+1}^{-\beta}\bar{\zeta}_{n+1}^{-\alpha}\mu(\zeta,g)^{-\beta}\overline{\mu(\zeta,g)}^{-\alpha}f(\zeta')),$$

by the use of (2.3), and, by (2.6), this can be written as

$$(***)\qquad \zeta_{n+1}^{-\beta-1}\bar{\zeta}_{n+1}^{-\alpha-1}\Delta_{\alpha,\beta}(\mu(\zeta,g)^{-\beta}\overline{\mu(\zeta,g)}^{-\alpha}f(\zeta')).$$

Now formula (2.8) follows by (2.3) and the equality of (**) and (***). □

Let $D_{n+1} := \{\zeta \in \mathbb{C}^{n+1} \mid |\zeta_1|^2+\ldots+|\zeta_n|^2 < \mathrm{Im}\zeta_0\}$.

Then G acts transitively on D_{n+1} and the stabilizer of $(i,0,\ldots,0)$ in G is K. Hence $D_{n+1} = G/K$. Also G acts transitively on $\partial D_{n+1} \cup \{\infty\}$ and the stabilizer of $(0,\ldots,0)$ in G is $MA\bar{N}$. Hence $\partial D_{n+1} \cup \{\infty\} = G/MA\bar{N}$.

Introduce new coordinates $(t,x,z_1,\ldots,z_n) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n$ on the ζ-space $\mathbb{C}^{n+1}$ by

$$(\zeta_0,\zeta_1,\ldots,\zeta_n) = (t+i(|z|^2+x),z_1,\ldots,z_n).$$

Write $z = (z_1,\ldots,z_n)$. Note that $\zeta \in D_{n+1} \Longleftrightarrow x > 0$; $\zeta \in \partial D_{n+1} \Longleftrightarrow x = 0$. Also:

$$(2.9)\qquad (t+i(|z|^2+x),z_1,\ldots,z_n) = n_{z,t}a_{\frac{1}{2}\log x}(i,0,\ldots,0),$$

$$(2.10)\qquad (t+i|z|^2,z_1,\ldots,z_n) = n_{z,t}(0,\ldots,0).$$

This identifies D_{n+1} with NA and ∂D_{n+1} with $N \simeq H_n$ and the local action of G on ∂D_{n+1} can be transplanted to H_n.

The operator $\Delta_{\alpha,\beta}$ expressed in terms of the coordinates $t,x,z_1,\ldots,z_n$ takes the form

$$(2.11)\qquad \Delta_{\alpha,\beta} = \sum_{j=1}^{n}\left(-\frac{\partial^2}{\partial z_j\partial\bar{z}_j} + i\frac{\partial}{\partial t}\left(z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial\bar{z}_j}\right)\right) + $$
$$ - |z|^2\frac{\partial^2}{\partial t^2} + i(\beta-\alpha)\frac{\partial}{\partial t} - x\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right) + (n-\alpha-\beta)\frac{\partial}{\partial x}.$$

If

(2.12) $\qquad \alpha = \frac{1}{2}(n-\gamma), \quad \beta = \frac{1}{2}(n+\gamma), \quad \gamma \in \mathbb{C}.$

then the restriction L_γ of $\Delta_{\alpha,\beta}$ to $x = 0$ will be a differential operator on ∂D_{n+1}:

$$L_\gamma = \sum_{j=1}^{n} \left(- \frac{\partial^2}{\partial z_j \partial \bar{z}_j} + i \frac{\partial}{\partial t} \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) \right) + - |z|^2 \frac{\partial^2}{\partial t^2} + i\gamma \frac{\partial}{\partial t} . \tag{2.13}$$

Now we obtain from Lemma 2.1 and Table 1:

<u>THEOREM 2.2</u>. *For smooth functions* f *on* H_n *and for* g *in* G *we have*

$$L_\gamma(\mu(z,t;g)^{-\beta}\, \overline{\mu(z,t;g)}^{-\alpha} f(g.(z,t))) = \tag{2.14}$$

$$= \mu(z,t;g)^{-\beta-1}\, \overline{\mu(z,t;g)}^{-\alpha-1} (L_\gamma f)(g.(z,t)),$$

where the local action of G *and* μ *are specified by:*

g	$g.(z,t)$	$\mu(z,t;g)$
m_T	$((\det T)^{\frac{1}{2}} Tz, t)$	$(\det T)^{-\frac{1}{2}}$
a_s	$(e^s z, e^{2s} t)$	e^{-2s}
$n_{z',t'}$	$(z',t')(z,t)$	1
m_W	$\left(\frac{z}{t+i\|z\|^2}, - \frac{t}{t^2+\|z\|^4} \right)$	$t+i\|z\|^2$

Table 2

In other words, L_γ is left H_n-invariant and invariant under the action $T.(z,t) = (Tz,t)$ of $U(n)$,

$$(2.15) \qquad L_\gamma(f(Rz,R^2t)) = R^2(L_\gamma f)(Rz,R^2t)$$

and

$$(2.16) \qquad L_\gamma\Big((|z|^2+it)^{-\alpha}(|z|^2-it)^{-\beta}f\Big(\frac{z}{t+i|z|^2}, -\frac{t}{t^2+|z|^4}\Big)\Big)$$

$$= (|z|^2+it)^{-\alpha-1}(|z|^2-it)^{-\beta-1}(L_\gamma f)\Big(\frac{z}{t+i|z|^2}, \frac{-t}{t^2+|z|^4}\Big) .$$

We call the function $K_\gamma f$ defined by

$$(2.17) \qquad (K_\gamma f)(z,t) := (|z|^2+it)^{-\alpha}(|z|^2-it)^{-\beta}f\Big(\frac{z}{t+i|z|^2}, \frac{-t}{t^2+|z|^4}\Big)$$

the *Kelvin transform* of f.

COROLLARY 2.3. *If* $L_\gamma f = 0$ *on* H_n *then* $L_\gamma(K_\gamma f) = 0$ on $H_n\backslash\{(0,0)\}$.

3. HARMONICS ON THE HEISENBERG GROUP

Throughout assume (2.12) and $\pm\gamma \neq n,n+2,n+4,\ldots$. Define

$$(3.1) \qquad \Phi_\gamma(z,t) := c_\gamma(|z|^2+it)^{-\alpha}(|z|^2-it)^{-\beta},$$

where

$$(3.2) \qquad c_\gamma := \Gamma(\alpha)\Gamma(\beta)2^{n-2}\pi^{-n-1}.$$

Then Φ_γ is a fundamental solution of L_γ at 0 (with respect to standard normalisation of Lebesgue measure):

$$(3.3) \qquad L_\gamma\Phi_\gamma = \delta,$$

cf. FOLLAND & STEIN [6,§6]. Actually, the fact that $L_\gamma\Phi_\gamma = 0$ outside 0 follows from (2.16). By the use of the analyticity of Φ_γ outside 0 and the left H_n-invariance of L_γ it now follows that L_γ is hypoelliptic and real analytic hypoelliptic, cf. FOLLAND & STEIN [6,§7]. In particular if f is a distribution on an open subset of H_n containing 0 and if f is L_γ-*harmonic*, i.e.

$$(3.4)\qquad L_\gamma f = 0,$$

then f is real analytic, so it can be expanded as a power series around zero. Because of (2.15) this power series can be rearranged such that

$$(3.5)\qquad f = \sum_{m=0}^{\infty} f_m,$$

with absolute and uniform convergence in some neighbourhood of 0 and where f_m is a (solid) Heisenberg harmonic of degree m:

DEFINITION 3.1. A function f on H_n is called H_n-*homogeneous of degree* m if

$$(3.6)\qquad f(Rz,R^2t) = R^m f(z,t), \quad R > 0.$$

DEFINITION 3.2. A *(solid) Heisenberg harmonic of degree* m on H_n is a polynomial in $z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n$, t which is H_n-homogeneous of degree m and L_γ-harmonic.

Because of the U(n)-invariance of L_γ and property (3.6), the class of Heisenberg harmonics of degree m can be decomposed as a direct sum of subspaces on which U(n) acts irreducibly. These subspaces were obtained explicitly by GREINER [9] in the case n = 1 and by DUNKL [3] in the general case, later also by KORÁNYI [17] with a different proof. Here we will obtain these subspaces in yet another way, somewhat related to Korányi's argument.

DEFINITION 3.3. The space $H_{k,\ell}$ of *complex (solid) spherical harmonics of bidegree* (k,ℓ) on $\mathbb{C}^n$ consists of all polynomials P in $z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n$, homogeneous of degree k in the z_j's and homogeneous of degree ℓ in the $\bar{z}_j$'s and satisfying

(3.7) $$\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j} P = 0.$$

PROPOSITION 3.4 (cf. KOORNWINDER [15], RUDIN [19, §12.2]).

(a) *The group* U(n) *acts irreducibly on each space* $\mathcal{H}_{k,\ell}$ $(k,\ell = 0,1,2,\ldots;$ *if* $n = 1$ *then, moreover,* k *or* $\ell = 0)$.

(b) *Representations of* U(n) *on different spaces* $\mathcal{H}_{k,\ell}$ *are inequivalent.*

(c) $$L^2(S^{2n-1}) = \bigoplus_{k,\ell} \mathcal{H}_{k,\ell} \mid S^{2n-1}.$$

(d) $$N_{k,\ell} := \dim \mathcal{H}_{k,\ell} = \frac{(n+k+\ell-1)(n+k-2)!(n+\ell-2)!}{k!\ell!(n-1)!(n-2)!}.$$

(e) *If* $\{Y_1,\ldots,Y_{N_{k,\ell}}\}$ *is an orthonormal basis of* $\mathcal{H}_{k,\ell} \mid S^{2n-1}$ *then*

$$\sum_{j=1}^{N_{k,\ell}} Y_j(\xi)\overline{Y_j(\eta)} = |S_{2n-1}|^{-1} N_{k,\ell} R^{n-2}_{k,\ell}(\xi\cdot\eta), \quad \xi,\eta \in S^{2n-1},$$

where the disk polynomial $R^{\alpha}_{k,\ell}$ *is defined in terms of Jacobi polynomials* $P_n^{(\alpha,\beta)}$ *by*

$$R^{\alpha}_{k,\ell}(re^{i\phi}) := \frac{P^{(\alpha,|k-\ell|)}_{k\wedge\ell}(2r^2-1)}{P^{(\alpha,|k-\ell|)}_{k\wedge\ell}(1)} r^{|k-\ell|} e^{i(k-\ell)\phi}.$$

(f) *If* F *is a bihomogeneous polynomial of bidegree* (k,ℓ) *on* $\mathbb{C}^n$ *then*

$$F(z) = \sum_{j=0}^{k\wedge\ell} |z|^{2j} Y_j(z) \quad \textit{with} \quad Y_j \in \mathcal{H}_{k-j,\ell-j}.$$

THEOREM 3.5. *The space of Heisenberg harmonics of degree* m *on* H_n *is spanned by the functions*

(3.8) $$(z,t) \mapsto C^{(\alpha+\ell,\beta+k)}_{\frac{1}{2}(m-k-\ell)}(t+i|z|^2)\, Y(z),$$

where $m-k-\ell \geq 0$ *and even and* $Y \in \mathcal{H}_{k,\ell}$.

PROOF. First we show that the function (3.8) is a Heisenberg harmonic of degree m. Clearly, it is a H_n-homogeneous polynomial of degree m, so it is left to prove that the function is L_γ-harmonic. By (2.13) and (3.7) $L_\gamma Y = 0$, where $Y(z,t) := Y(z)$. It follows from Corollary 2.3 and the bihomogeneity of Y that

$$0 = L_\gamma\left((t-i|z|^2)^{-\alpha}(t+i|z|^2)^{-\beta}Y\left(\frac{z}{t+i|z|^2}\right)\right) =$$

$$= L_\gamma((t-i|z|^2)^{-\alpha-\ell}(t+i|z|^2)^{-\beta-k}Y(z)).$$

By the left N-invariance of L_γ and by (1.1) we obtain

$$0 = L_\gamma((1+t-i|z|^2)^{-\alpha-\ell}(1+t+i|z|^2)^{-\beta-k}Y_{k,\ell}(z)) =$$

$$= L_\gamma\left(\sum_{r=0}^{\infty}(-1)^r C_r^{(\alpha+\ell,\beta+k)}(t+i|z|^2)Y_{k,\ell}(z)\right),\quad t^2+|z|^4<1.$$

The result follows by use of (2.15).

Conversely, let F be a Heisenberg harmonic of degree m. Then F must be a linear combination of functions

$$(z,t)\mapsto t^r F(z),$$

where F is a bihomogeneous polynomial of degree (p,q) and 2r+p+q = m. Hence, by Prop. 3.4 F must be a linear combination of functions

$$(z,t)\mapsto t^r|z|^{2s}Y(z),$$

where $Y\in H_{k,\ell}$ and $2r+2s+k+\ell = m$, i.e.,

$$(*)\qquad F(z,t) = \sum_{\substack{k,\ell \\ m-k-\ell\geq 0 \\ \text{and even}}}\sum_j f_{k,\ell;j}(t,|z|^2)Y_{k,\ell;j}(z),$$

where, for each k,ℓ, the $Y_{k,\ell;j}$-s form a basis for $H_{k,\ell}$ and $f_{k,\ell;j}$ is a homogeneous polynomial of degree $\frac{1}{2}(m-k-\ell)$. Now, again by Prop. 3.4 and by the U(n)-invariance of L_γ,

$$L_\gamma(f_{k,\ell;j}(t,|z|^2)Y_{k,\ell;j}(z)) = g_{k,\ell;j}(t,|z|^2)Y_{k,\ell;j}(z)$$

for some homogeneous polynomial $g_{k,\ell;j}$

of degree $\frac{1}{2}(m-k-\ell)-1$. Since $L_\gamma F = 0$, it follows that each of the terms in the right hand side of (*) is L_γ-harmonic, so we are left to prove that, if the function $(z,t) \mapsto f(t,|z|^2)Y(z)$ is L_γ-harmonic with $Y \in H_{k,\ell}$, f a homogeneous polynomial of degree r, then f is unique up to a constant factor. We prove this by complete induction with respect to r. It is clearly true for r = 0. Suppose it is proved for degree (f) = r-1. Suppose $f_i(t,|z|^2)Y(z)$ is L_γ-harmonic for i = 1,2, degree $(f_i) = r$. Then $\frac{\partial}{\partial t}(f_i(t,|z|^2)Y(z))$ is L_γ-harmonic of degree $2r-2+k+\ell$ (cf. (2.13)), so, by the induction hypothesis, there are λ,μ, not both zero, such that $\frac{\partial}{\partial t}(\lambda f_1+\mu f_2) = 0$. Hence

$$\lambda f_1(t,|z|^2)+\mu f_2(t,|z|^2) = c|z|^{2\ell},$$

so $c|z|^{2\ell}Y_{r,s}(z)$ satisfies (3.7). Thus, by Prop. 3.4, c = 0. Hence f_1 and f_2 are proportional. □

4. THE HEISENBERG BALL

4.1. The Dirichlet problem

The region

$$B_{H_n} := \{(z,t) \in H_n \mid |z|^4+t^2 < 1\} \tag{4.1}$$

is called the *Heisenberg ball*. We are interested in the *Dirichlet problem* for L_γ $(\pm\gamma\neq n,n+2,\ldots)$ on the Heisenberg ball:

For given f in $C(\partial B_{H_n})$ does there exist a unique function u in $C^\infty(B_{H_n}) \cap C(\overline{B_{H_n}})$ such that

(i) $L_\gamma u = 0$ on B_{H_n},

(ii) $u = f$ on B_{H_n}?

For $\gamma = 0$ the problem was solved by GAVEAU [8], who used probabilistic methods, and by JERISON [12], who used analytic methods. For certain $\gamma \neq 0$ the problem was solved by JERISON [13], to some extent.

In particular, we are interested in solving the Dirichlet problem by finding an explicit *Poisson kernel* P_γ on $B_{H_n} \times \partial B_{H_n}$ such that the desired solution u is expressed in terms of f by

$$(4.2) \qquad u(z,t) = \int_{\partial B_{H_n}} f(z',t')P_\gamma(z,t;z',t')ds(z',t').$$

This problem is still open for all γ.

Let us introduce "spherical" coordinates ρ,ϕ,ξ adapted to the Heisenberg ball by

$$(4.3) \qquad (z,t) = (\rho\ \sin^{\frac{1}{2}}\phi\xi,\rho^2\cos\phi),\ \rho \geq 0,\ 0 \leq \phi \leq \pi,\ \xi \in S^{2n-1}.$$

In terms of the coordinates ρ,ϕ,ξ the special Heisenberg harmonics (3.8) take the form

$$(4.4) \qquad (\rho,\phi,\xi) \to \rho^m(\sin\phi)^{\frac{1}{2}(k+\ell)}C^{(\alpha+\ell,\beta+k)}_{\frac{1}{2}(m-k-\ell)}(e^{i\phi})Y(\xi).$$

4.2. Green's formula for L_γ

The differential operators $Z_j,\bar{Z}_j$ ($j = 1,\ldots,n$) and T on H_n, defined by

$$(4.5) \qquad \begin{cases} Z_j := \dfrac{\partial}{\partial z_j} + i\bar{z}_j\dfrac{\partial}{\partial t}, \\ \bar{Z}_j := \dfrac{\partial}{\partial \bar{z}_j} - iz_j\dfrac{\partial}{\partial t}, \\ T := \dfrac{\partial}{\partial t}, \end{cases}$$

form a basis for the left invariant vector fields on H_n. L_γ can be expressed in terms of these operators by

$$(4.6) \qquad L_\gamma = -\tfrac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j+\bar{Z}_jZ_j) + i\gamma T.$$

If we introduce real coordinates x_j, y_j ($j = 1,\ldots,n$), t by $z_j = x_j+iy_j$ then

$$(4.7)\qquad L_\gamma = -\tfrac{1}{4}\sum_{j=1}^{n}\Big[\Big(\frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}\Big)\Big(\frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}\Big) + \Big(\frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}\Big)\Big(\frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}\Big)\Big] + i\gamma\frac{\partial}{\partial t}.$$

Hence L_γ has principal symbol

$$(4.8)\qquad P_{L_\gamma}((x+iy,t),(\xi,\eta,\tau)) = -\tfrac{1}{4}\sum_{j=1}^{n}[(\xi_j+2y_j\tau)^2+(\eta_j-2x_j\tau)^2],$$

$$(x+iy,t)\in H_n,\ (\xi,\eta,\tau)\in \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R},$$

which shows that L_γ is not elliptic. An associated bilinear form on vector fields $(\xi,\eta,\tau),(\xi',\eta',\tau')$ on H_n is defined by

$$(4.9)\qquad ((\xi,\eta,\tau)\mid(\xi',\eta',\tau'))_{H_n}\Big|_{(x+iy,t)} := -\tfrac{1}{4}\sum_{j=1}^{n}[(\xi_j+2y_j\tau)(\xi'_j+2y_j\tau') + (\eta_j-2x_j\tau)(\eta'_j-2x_j\tau')].$$

Now let Ω be a nonempty open connected bounded subset of $\mathbb{R}^n$ with smooth boundary and let $\nu = (\nu_x,\nu_y,\nu_t)$ denote the outward normal at a point of $\partial\Omega$ in terms of the (x,y,t) coordinates. Write $dx\,dy$ instead of $dx_1,\dots,dx_n dy_1,\dots,dy_n$. Let ds be the surface element on $\partial\Omega$. Let $u,v\in C^2(\Omega)\cap C^1(\bar\Omega)$. Let

$$\nabla u := \Big(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n},\frac{\partial u}{\partial y_1},\dots,\frac{\partial u}{\partial y_n},\frac{\partial u}{\partial t}\Big).$$

Then *Green's formula* for L_γ reads:

$$(4.10)\qquad \int_\Omega (uL_\gamma v - vL_{-\gamma}u)\,dx\,dy\,dt = \int_{\partial\Omega}[u(\nabla v|\nu)_{H_n} - v(\nabla u\,|\nu)_{H_n} + i\gamma uv\nu_t]\,ds$$

(cf. GAVEAU [8, Corollaire après Lemme 4] if $\gamma = 0$).

Let us rewrite the right hand side of (4.10) in terms of spherical coordinates in the case $\Omega = \rho\, B_{H_n}$. Then:

$$(4.11)\qquad \nu = |\nabla(\rho^4)|^{-1}\nabla(\rho^4),$$

$$(4.12)\qquad |\nabla(\rho^4)|^{-1}ds = \tfrac{1}{4}\rho^{2n-2}(\sin\phi)^{n-1}d\phi d\xi$$

($d\xi$ surface element on S^{2n-1}),

$$(\nabla u|\nabla(\rho^4))|_{H_n} = -|z|^2 \sum_k \left(x_k \frac{\partial u}{\partial x_k} + y_k \frac{\partial u}{\partial y_k} + 2t \frac{\partial u}{\partial t}\right) + - t\sum_k \left(y_k \frac{\partial u}{\partial x_k} - x_k \frac{\partial u}{\partial y_k}\right).$$

Define

$$(4.13)\qquad \frac{\partial u}{\partial \theta}(z,t) := \frac{\partial}{\partial \theta} u(e^{i\theta}z,t)|_{\theta=0}.$$

Then:

$$(4.14)\qquad (\nabla u|\nabla(\rho^4))|_{H_n} = -\rho^3 \sin\phi \frac{\partial u}{\partial \rho} + \rho^2 \cos\phi \frac{\partial u}{\partial \theta}.$$

Hence, (4.10) takes the form

$$(4.15)\qquad \int_{\rho B_{H_n}} (uL_\gamma v - vL_{-\gamma}u)\,dxdydt = \tfrac{1}{4}\rho^{2n} \int_0^\pi \int_{S^{2n-1}} [\rho\sin\phi(-u\frac{\partial v}{\partial \rho} + v\frac{\partial u}{\partial \rho}) + + \cos\phi(u\frac{\partial v}{\partial \theta} - v\frac{\partial u}{\partial \theta}) + 2i\gamma\cos\phi\, uv](\sin\phi)^{n-1}d\phi d\xi.$$

Now apply (4.15) to the case of two Heisenberg harmonics of type (4.4):

$$u(\rho,\phi,\xi) = \rho^{2m+k+\ell}(\sin\phi)^{\frac{1}{2}(k+\ell)}C_m^{(\beta+\ell,\alpha+k)}(e^{i\phi})Y(\xi),$$

$$v(\rho,\phi,\xi) = \rho^{2m'+k+\ell}(\sin\phi)^{\frac{1}{2}(k+\ell)}C_{m'}^{(\alpha+k,\beta+\ell)}(e^{i\phi})\overline{Y(\xi)},$$

where $0 \neq Y \in \mathcal{H}_{k,\ell}$. Then we obtain

$$\int_0^\pi ((-m'+m)\sin\phi + i(\gamma+\ell-k)\cos\phi).$$

$$\cdot\ (\sin\phi)^{n+k+\ell-1} C_m^{(\frac{1}{2}(n+\gamma)+\ell,\frac{1}{2}(n-\gamma)+k)}(e^{i\phi}) \cdot$$

$$\cdot\ C_{m'}^{(\frac{1}{2}(n-\gamma)+k,\frac{1}{2}(n+\gamma)+\ell)}(e^{i\phi})d\phi = 0.$$

By application of *Carlson's theorem* (cf. TITCHMARSH [20,§5.81]) we conclude that:

$$(4.16) \qquad \int_0^\pi ((-m'+m)\sin\phi + i(\alpha-\beta)\cos\phi)(\sin\phi)^{\alpha+\beta-1} \cdot$$
$$\cdot\ C_m^{(\alpha,\beta)}(e^{i\phi})C_{m'}^{(\beta,\alpha)}(e^{i\phi})d\phi = 0, \qquad \mathrm{Re}(\alpha+\beta) > 0.$$

Unfortunately, this does not provide a biorthogonality for the functions $C_k^{(\alpha,\beta)}$ since the weight function depends on m,m'. Only in the case $\alpha = \beta$, (4.16) reduces to the orthogonality for Gegenbauer polynomials (cf. (1.9)). Formula (4.16) was also obtained by Dunkl (personal communication, unpublished).

4.3. Remarks on the Poisson Kernel

In [9] the spherical harmonics on H_1 and the functions $C_k^{(\alpha,\beta)}$ were derived in an attempt to construct the Poisson kernel for L_γ on B_{H_1}. This is analogous to the construction of the classical Poisson kernel in the unit ball in $\mathbb{R}^n$. The next step is to obtain orthogonality relations among the $C_k^{(\alpha,\beta)}$-s. This we have not been able to do yet. For instance, $L^2(S^{n-1})$ splits uniquely into a direct sum of O(n)-irreducible subspaces (spaces of spherical surface harmonics of a fixed degree), while $L^2(B_{H_n})$ contains each irreducible representation of U(n) occurring on some $H_{k,\ell}$, countably many times (cf.(4.4) and Theorem 3.5). Furthermore, an application of Green's formula shows that the classical spherical surface harmonics of different degree are orthogonal, while in the Heisenberg case we obtain (4.16) only. These difficulties probably have connection with the fact that there is no natural group acting transitively on the Heisenberg unit sphere. Another related fact may be that the equation $L_\gamma u = \lambda u$ probably does not separate in any coordinates adapted to the Heisenberg ball. (However, observe that $L_\gamma u = 0$ does separate in the sense of [16, Definition 2.1].)

A well-known method of obtaining the Poisson kernel for Δ on the unit ball uses the Kelvin transform, Green's formula and the fundamental solution. However, in terms of the coordinates ρ,ϕ,ξ, formula (2.17) reads:

$$(4.17)\qquad (K_\gamma f)(\rho,\phi,\xi) = \rho^{-2n} e^{i\gamma(\pi/2-\phi)} f(\rho^{-1},\pi-\phi,e^{-i\phi}\xi).$$

Hence, in general we have

$$(K_\gamma f)(1,\phi,\xi) \neq f(1,\phi,\xi)$$

and the method used for the unit ball fails here.

Another way of deriving the Poisson kernel for Δ on the unit ball is to derive first a Poisson kernel for each $O(n)$-irreducible subspace of $L^2(S^{n-1})$ separately and next to sum up all these kernels. The summands are easily found because f in an $O(n)$-irreducible subspace of $L^2(S^{n-1})$ is a spherical surface harmonic of degree n, which has an harmonic extension f to the ball given by $u(x) := |x|^n f(|x|^{-1}x)$. Let us try to do the same for the Heisenberg ball. Suppose that the Dirichlet problem is solvable and allow some formal reasoning. Under the action of $U(n)$ the space $C(\partial B_{H_n})$ splits into subspaces $C_{k,\ell}(\partial B_{H_n})$ on which $U(n)$ acts as on $H_{k,\ell}$: $C_{k,\ell}(\partial B_{H_n})$ will be spanned by functions of the form $(\phi,\xi) \mapsto f(\phi)(\sin\phi)^{\frac{1}{2}(k+\ell)}Y(\xi)$, where $Y \in H_{k,\ell}$. By the $U(n)$-invariance of L_γ, the L_γ-harmonic continuation to the interior of such a function will have the form $(\rho,\phi,\xi) \mapsto u_{f,k,\ell}(\rho,\phi)(\sin\phi)^{\frac{1}{2}(k+\ell)}Y(\xi)$. Hence, in terms of the coordinates ρ,ϕ,ξ and for functions f in $C_{k,\ell}(\partial B_{H_n})$, formula (4.2) will take the form

$$(4.18)\qquad u(\rho,\phi,\xi) = \frac{1}{N_{k,\ell}|S_{2n-1}|}\int_0^\pi \int_{S^{2n-1}} f(\phi',\xi')\left(\frac{\sin\phi}{\sin\phi'}\right)^{\frac{1}{2}(k+\ell)} \cdot$$

$$\cdot\, P_{\gamma;k,\ell}(\rho,\phi;\phi')R^{n-2}_{k,\ell}(\xi\cdot\xi')\,d\phi'\,d\xi'.$$

Here we used Prop.3.4(e). The kernel $P_{\gamma;k,\ell}$ will have the property

$$(4.19)\qquad \int_0^\pi C_m^{(\frac{1}{2}(n-\gamma)+\ell,\frac{1}{2}(n+\gamma)+k)}(e^{i\phi'})P_{\gamma;k,\ell}(\rho,\phi;\phi')\,d\phi' =$$

$$= \rho^{2m+k+\ell} C_m^{(\frac{1}{2}(n-\gamma)+\ell,\frac{1}{2}(n+\gamma)+k)}(e^{i\phi}).$$

Formula (4.19) defines $P_{\gamma;k,\ell}$ if the functions $\phi \mapsto C_k^{(\alpha,\beta)}(e^{i\phi})$ are, in some sense, complete on $[0,\pi]$. This is, of course, true in the Gegenbauer case $C_k^{(\alpha,\alpha)}$ ($\gamma \in \mathbb{Z}$ and $\ell - k = \gamma$ in the case (4.19)). In view of (1.10) and (1.11) $\{C_k^{(\alpha,0)}\}$ and $\{C_k^{(0,\beta)}\}$ are also complete: *Mergelyan's Theorem* (cf. RUDIN [18, Theorem 20.5]) states that every continuous function on $\{e^{i\phi} \mid 0 \le \phi \le \pi\}$ can be uniformly approximated by polynomials in one complex variable.

In section 5 we show that if P_γ exists then the family

$$\{\phi \mapsto C_m^{(\alpha+k,\beta+\ell)}(e^{i\phi})\}_{k=0,1,\ldots}$$

is dense for $k,\ell \in \mathbb{Z}$ and $\alpha = \frac{1}{2}(n-\gamma)$, $\beta = \frac{1}{2}(n+\gamma)$.

5. THE EXPANSION OF THE TRANSLATE OF THE FUNDAMENTAL SOLUTION

Let Φ_γ be the fundamental solution of L_γ at 0 as defined by (3.1). By using the left H_n-invariance of L_γ and the obvious identity

$$\Phi_\gamma(z,t) = \Phi_{-\gamma}((z,t)^{-1})$$

we obtain

$$L_\gamma(\Phi_\gamma((z',t')^{-1}(z,t)) = 0,$$

$$L'_{-\gamma}(\Phi_\gamma((z',t')^{-1}(z,t)) = 0,$$

where $(z,t) \neq (z',t')$ in both cases. Here $L'_{-\gamma}$ means the differential operator $L_{-\gamma}$ expressed in terms of the primed variables. The function $\Phi_\gamma((\cdot)^{-1}(z,t))$ is analytic in a neighbourhood of 0 ($(z,t)\neq 0$) and can thus be expanded in terms of $L_{-\gamma}$-Heisenberg harmonics. The expansion coefficients will be L_γ-harmonic functions of (z,t) ($|z|^4+t^2$ large). In fact, Dunkl [3, Theorem 1.6] explicitly obtained these coefficients. He proved it by using an addition theorem for Heisenberg harmonics, which he first derived.

However, the coefficients depending on (z,t) can be recognized as Kelvin transforms of L_γ-Heisenberg harmonics. This suggests a new and shorter proof of Dunkl's formula, which we will present now.

Let K_γ denote the Kelvin transform with respect to the (z,t) variables. Then, by (2.17):

$$(5.1)\qquad \Psi_\gamma(z,t;z',t') := K_\gamma(\Phi_\gamma((z',t')^{-1}(z,t))) = (|z|^2+it)^{-\alpha}(|z|^2-it)^{-\beta}\cdot$$

$$\cdot\Phi_\gamma((z',t')^{-1}(\frac{z}{t+i|z|^2}, -\frac{t}{t^2+|z|^4})) =$$

$$= c_\gamma(1+it|z'|^2-it'|z|^2+tt'+|z|^2|z'|^2-2iz'\cdot z)^{-\alpha}\cdot$$

$$\cdot\ (1-it|z'|^2+it'|z|^2+tt'+|z|^2|z'|^2+2iz\cdot z')^{-\beta},$$

$$(|z|^4+t^2)(|z'|^4+t'^2) < 1.$$

In this region Ψ_γ is real analytic in z,t,z',t' and L_γ-harmonic in (z,t), $L_{-\gamma}$-harmonic in (z',t'). Also:

$$(5.2)\qquad \Psi_\gamma(RTz,R^2t;R^{-1}Tz',R^{-2}t'^2) = \Psi_\gamma(z,t;z',t'),\quad R>0,\ T\in U(n).$$

For each k,ℓ choose a basis $\{Y_{k,\ell;j}\}$ for $H_{k,\ell}$ such that its restriction to S^{2n-1} is an orthonormal basis. Then it follows by Prop.3.4, Theor.3.5 and formulas (3.5), (5.2) that

$$\Psi_\gamma(z,t;z',t') = \sum_{m=0}^{\infty}\sum_{k,\ell=0}^{\infty}\sum_{j=1}^{N_{k,\ell}} a_{m;k,\ell}\cdot$$

$$\cdot\ C_m^{(\alpha+\ell,\beta+k)}(t+i|z|^2)Y_{k,\ell;j}(z)\cdot$$

$$\cdot\ C_m^{(\beta+k,\alpha+\ell)}(t'+i|z'|^2)\overline{Y_{k,\ell;j}(z')},$$

for certain coefficients $a_{m;k,\ell}$ (not depending on j). This expansion absolutely and uniformly converges for sufficiently small $(|z|^4+t^2)(|z'|^4+(t')^2)$.

It follows from (2.17) that

$$f(w,s) = (|w|^2+is)^{-\alpha}(|w|^2-is)^{-\beta}\ (K_\gamma f)(\frac{-w}{s+i|w|^2}, \frac{-s}{s^2+|w|^4}).$$

Hence

$$\Phi_\gamma((z',t')^{-1}(z,t)) = (|z|^2+it)^{-\alpha}(|z|^2-it)^{-\beta} \cdot$$
$$\cdot \sum_{m=0}^{\infty} \sum_{k,\ell=0}^{\infty} \sum_{j=1}^{N_{k,\ell}} a_{m;k,\ell}\, i^{k-\ell}(|z|^2+it)^{-m-\ell}(|z|^2-it)^{-m-k} \cdot$$
$$\cdot\ C_m^{(\alpha+\ell,\beta+k)}(-t+i|z|^2)Y_{k,\ell;j}(z)C_m^{(\beta+k,\alpha+\ell)}(t'+i|z'|^2)\overline{Y_{k,\ell;j}(z')},$$

so

$$\Phi_\gamma((z',t')^{-1}(z,t)) = \rho^{-2n} \sum_{m=0}^{\infty} \sum_{k,\ell=0}^{\infty} \sum_{j=1}^{N_{k,\ell}} b_{m;k,\ell} \cdot \tag{5.3}$$
$$\cdot\ \rho^{-2m-k-\ell} e^{i(-\gamma+\ell-k)\phi}(\sin\phi)^{\frac{1}{2}(k+\ell)}C_m^{(\beta+k,\alpha+\ell)}(e^{i\phi})Y_{k,\ell;j}(\xi) \cdot$$
$$\cdot\ (\rho')^{2m+k+\ell}(\sin\phi')^{\frac{1}{2}(k+\ell)}C_m^{(\beta+k,\alpha+\ell)}(e^{i\phi'})\overline{Y_{k,\ell;j}(\xi')},$$

where

$$b_{m;k,\ell} = (-1)^{m+k-\ell}e^{\frac{1}{2}i\gamma\pi}a_{m;k,\ell}.$$

Now we have absolute and uniform convergence for sufficiently small ρ'/ρ.

Let u be a $L_{-\gamma}$-harmonic function on $\rho\bar{B}_{H_n}$ of the form

$$u(z,t) := f(|z|^2,t)\overline{Y(z)},$$

where Y is in $H_{k,\ell}$ with L^2-norm 1. Then, by Prop.3.4, f is a C^∞-function on $\{(x,y)\,|\,x^2+y^2 \le \rho,\ x \ge 0\}$. Let v(z,t) be given by the left hand side of (5.3) with $|z'|^4+(t')^2 < \rho^4$. Apply Green's formula (4.15). We obtain

$$u(z',t') = \sum_{m=0}^{\infty} c_m(\rho'/\rho)^{2m}C_m^{(\beta+k,\alpha+\ell)}(e^{i\phi'})\overline{Y(z')}, \quad \rho' < \rho, \tag{5.4}$$

where

$$(5.5)\qquad c_m = \tfrac{1}{4} b_{m;k,\ell} \int_0^{\pi} e^{i(-\gamma+\ell-k)\phi} (\sin\phi)^{k+\ell+n-1} C_m^{(\beta+k,\alpha+\ell)}(e^{i\phi}) \cdot$$
$$\cdot \left[\sin\phi\left(\rho \frac{\partial}{\partial\rho} + 2(n+m+k+\ell)\right) + 2i(\gamma+k-\ell)\cos\phi\right] f(\rho^2\sin\phi, \rho^2\cos\phi)\, d\phi.$$

Convergence in (5.4) is still absolute and uniform for ρ' sufficiently small. If we make the particular choice

$$f(|z|^2, t) := C_{m'}^{(\beta+k,\alpha+\ell)}(t+i|z|^2)$$

then, obviously,

$$c_m = \delta_{m,m'}\rho^{2m}.$$

Hence (5.5) yields

$$\delta_{m,m'} = \tfrac{1}{2} b_{m;k,\ell} \int_0^{\pi} C_m^{(\beta+k,\alpha+\ell)}(e^{i\phi}) C_{m'}^{(\beta+k,\alpha+\ell)}(e^{i\phi}) \cdot$$
$$\cdot\, e^{i\phi(-\gamma+\ell-k)} (\sin\phi)^{k+\ell+n-1} [(m+m'+n+k+\ell)\sin\phi + i(\gamma+k-\ell)\cos\phi]\, d\phi.$$

By applying again Carlson's Theorem (cf. TITCHMARSH [20, 5.81]) in the case $m \neq m'$ and by applying (1.14) in the case $m = m'$ we obtain for all α,β in $\mathbb{C}$ with $\mathrm{Re}(\alpha+\beta) > 0$:

$$(5.6)\qquad \int_0^{\pi} C_m^{(\alpha,\beta)}(e^{i\phi}) C_{m'}^{(\alpha,\beta)}(e^{i\phi}) e^{i\phi(\beta-\alpha)} (\sin\phi)^{\alpha+\beta-1} \cdot$$
$$\cdot\, [(m+m'+\alpha+\beta)\sin\phi + i(\alpha-\beta)\cos\phi]\, d\phi =$$
$$= \frac{e^{\frac{1}{2}i(\beta-\alpha)\pi}\,\pi\,\Gamma(\alpha+\beta)\,(\alpha+\beta)_m}{2^{\alpha+\beta-2}\,\Gamma(\alpha)\,\Gamma(\beta)\,m!}\,\delta_{m,m'}.$$

This formula was also obtained by Dunkl (personal communication, unpublished). Like in (4.16), the weight function depends on m,m'. Only if $\alpha = \beta$ this dependence on m,m' can be divided out. Formula (5.6) with $m-m'$ even is a special case of (1.14).

Formula (5.6) with $m = m'$ yields the value of $b_{m;k,\ell}$ in (5.3):

$$(5.7) \qquad b_{m;k,\ell} = \frac{2^{n+k+\ell-1} e^{\frac{1}{2}i(\gamma+k-\ell)\pi} \Gamma(\beta+k)\Gamma(\alpha+\ell)m!}{\pi(m+k+\ell+n-1)!} .$$

Formula (5.7) together with (1.11) implies that, for each k,ℓ,j and for each $\varepsilon > 0$ the m-sum in (5.3) converges absolutely and uniformly if $\rho'/\rho \le 1-\varepsilon$. By combination of (5.3) with Prop.3.4(d) we get

$$(5.8) \qquad \Phi_\gamma((z',t')^{-1}(z,t)) = \rho^{-2n} \sum_{k,\ell=0}^{\infty} (\rho'/\rho)^{k+\ell} e^{i(-\gamma+\ell-k)\phi} \cdot$$

$$\cdot (\sin\phi \, \sin\phi')^{\frac{1}{2}(k+\ell)} |S_{2n-1}|^{-1} N_{k,\ell} R^{n-2}_{k,\ell}(\xi\cdot\xi') \cdot$$

$$\cdot \sum_{m=0}^{\infty} b_{m;k,\ell} (\rho'/\rho)^{2m} C_m^{(\beta+k,\alpha+\ell)}(e^{i\phi}) C_m^{(\beta+k,\alpha+\ell)}(e^{i\phi'})$$

with convergence of the m-sums as above. This formula coincides with DUNKL [3, Theorem 1.6].

Now we turn to the completeness question. First we have the interesting result:

THEOREM 5.1. *Let* u *be a* L_γ*-harmonic function on* B_{H_n} *which behaves under* U(n) *as the irreducible representation of* U(n) *on* $H_{k,\ell}$. *Then the expansion of* u *in terms of Heisenberg harmonics absolutely and uniformly converges on each compact subset of* B_{H_n}.

PROOF. Apply (5.4), (5.5), (5.7). □

THEOREM 5.2. *Suppose that the Dirichlet problem for* L_γ *on* B_{H_n} *is solvable for some* γ *and* n. *Then, for each* k *and for each continuous function* g *on* $[0,\pi]$ *there is a sequence* $g_1, g_2, \ldots,$ *of finite linear combinations of functions* $\phi \to \mathbb{C}_m^{(\alpha+\ell,\beta+k)}(e^{i\phi})$ *such that*

$$\lim_{j\to\infty} |g(\phi)-g_j(\phi)|(\sin\phi)^{\frac{1}{2}(k+\ell)} = 0.$$

PROOF. The function f defined by

$$f(\phi,\xi) := g(\phi)(\sin\phi)^{\frac{1}{2}(k+\ell)} Y(\xi)$$

($0 \neq Y \in H_{k,\ell}$) is continuous on ∂B_{H_n}. Suppose that $Y(\xi_0) = 1$ for some ξ_0 in S^{2n-1}. Let u be its L_γ-harmonic continuation to B_{H_n}. Then, by Theorem 5.2,

$$u(\rho,\phi,\xi) = \sum_{m=0}^{\infty} c_m \rho^{2m} C_m^{(\alpha+\ell,\beta+k)}(e^{i\phi})(\rho^2 \sin\phi)^{\frac{1}{2}(k+\ell)} Y(\xi)$$

with absolute and uniform convergence for ρ in compact subsets of $[0,1)$. Let $\varepsilon > 0$. For some $\rho < 1$ we have

$$|f(\phi,\xi)-u(\rho,\phi,\xi)| < \tfrac{1}{2}\varepsilon \quad \text{for all} \quad \phi,\xi$$

and for some M we have

$$\Big|\sum_{m=M+1}^{\infty} c_m \rho^{2m} C_m^{(\alpha+\ell,\beta+k)}(e^{i\phi})\Big| < \tfrac{1}{2}\varepsilon \quad \text{for all } \phi.$$

Hence

$$\Big|g(\phi) - \sum_{m=0}^{M} c_m \rho^{2m+k+\ell} C_m^{(\alpha+\ell,\beta+k)}(e^{i\phi})\Big| (\sin\phi)^{\frac{1}{2}(k+\ell)} < \varepsilon. \qquad \square$$

Since GAVEAU [8] and JERISON [12] showed the Dirichlet problem to be solvable for $\gamma = 0$ this shows:

<u>COROLLARY 5.3</u>.

$$\mathrm{Span}\{C_m^{(\alpha,\beta)}(e^{i\cdot})(\sin\cdot)^{|\alpha-\beta|}\}$$

is dense in $(\sin\cdot)^{|\alpha-\beta|}\, C([0,\pi])$ *with respect to the uniform norm if* $\alpha-\beta \in \mathbb{Z}$ *and* $\alpha \wedge \beta \in \{\frac{1}{2},1,3/2,\ldots\}$.

This was earlier conjectured by Dunkl (personal communication). In a recent preprint JERISON [13, Cor.10.2] solves some version of the Dirichlet problem for L_γ for certain nonzero values of γ. Theorem 5.2 applied to these cases will yield the completeness on $[0,\pi]$ of the $C_k^{(\alpha,\beta)}$-s for a larger set of parameter values α,β.

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