POSITIVE CONVOLUTION STRUCTURES ASSOCIATED WITH QUANTUM GROUPS

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1. Introduction

Hypergroups originated as abstractions of convolution algebras of measures on locally compact groups, see for instance Jewett [9]. Gelfand pairs and orthogonal systems of special functions which (for certain parameter values) can be interpreted as spherical functions on Gelfand pairs, are good sources of commutative hypergroups.

Quantum groups are generalizations of groups. They were discovered during the last decade and provide a good setting for q-special functions. See Drinfel'd [5] and Woronowicz [24] for two quite different approaches to general quantum groups and Koornwinder [13] for a survey about the interpretation of orthogonal polynomials on quantum groups.

This paper is in particular meant for workers in hypergroups and in (q-special) orthogonal polynomials. I hope it will provide them with new examples and that it will give them new sources of inspiration.

The first sections contain a short introduction to Hopf algebras, quantum groups and the example of quantum $SU(2)$. Then, starting at §7, I show how quantum group analogues of Gelfand pairs give rise to positivity of linearization coefficients for spherical "functions" and to positivity of multiplication on the dual of the C*-algebra of biinvariant elements with respect to the quantum subgroup. I suspect that the resulting structures are hypergroups, although I leave the proof of this to my friends of the hypergroup community. In the example of the $SU(2)$ quantum group the spherical "functions" are expressible in terms of little q-Legendre polynomials. In §9 the case of Askey-Wilson polynomials is discussed, which has a quantum group interpretation, but not as a straightforward quantum Gelfand pair. In §10 my old technique [11] of deriving positivity of linearization coefficients from addition formulas is adapted to a class of addition formulas which one meets in the q-world. Finally, §11 deals with a class of orthogonal polynomials in two non-commuting variables yielding (in the case of quantum group interpretation) a non-commutative hypergroup with commutative dual hypergroup.

I conclude this introduction with some notation used in the theory of q-hypergeometric series. See Gasper & Rahman [8] for more information about this theory. Let q be some complex number, usually taken between 0 and 1. Put

$$(a;q)_k := (1-a)(1-aq)\dots(1-aq^{k-1}), \quad k=1,2,\dots; \qquad (a;q)_0 := 1.$$

For $|q|<1$ let

$$(a;q)_\infty := \lim_{k\to\infty}(a;q)_k.$$

Probability Measures on Groups X, Edited by H. Heyer
Plenum Press, New York, 1991

Put

$$(a_1, a_2, \ldots, a_r; q)_k := (a_1; q)_k (a_2; q)_k \ldots (a_r; q)_k.$$

Define the *q-hypergeometric series* by

$${}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z \right] := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k \left((-1)^k q^{k(k-1)/2} \right)^{s-r+1} z^k}{(b_1, \ldots, b_s; q)_k (q; q)_k}.$$

Note that the summand at the right hand side simplifies if $r = s+1$. Note also that the series at the right hand side terminates after the term $k = n$ if one of the parameters $a_1, \ldots, a_r$ equals q^{-n} for some $n \in \mathbb{Z}_+$.

2. Compact Gelfand pairs

We recall the definition of a compact Gelfand pair. A good introduction to harmonic analysis on Gelfand pairs is Faraut [6]. Let G be a compact group with closed subgroup K. Let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of G. Let $\mathcal{H}(\pi)$ be the (finite-dimensional) Hilbert space on which (a representative of) $\pi \in \hat{G}$ acts. Put

$$m_\pi := \dim\{v \in \mathcal{H}(\pi) \mid \pi(k)\, v = v \ \ \forall k \in K\}.$$

Then (G, K) is called a *compact Gelfand pair* if $m_\pi = 0$ or 1 for all $\pi \in \hat{G}$.

Assume that (G, K) is a compact Gelfand pair. Let $(G/K)^\wedge$ denote the set of all $\pi \in \hat{G}$ for which $m_\pi = 1$. Such representations are called *spherical*. Let π be spherical. Choose a K-fixed unit vector v_π in $\mathcal{H}(\pi)$. Then the *spherical function* associated with π is defined by

$$\phi_\pi(x) := (v_\pi, \pi(x)\, v_\pi), \quad x \in G.$$

Note that it is uniquely determined by π. Spherical functions satisfy the *product formula*

$$\phi_\pi(x)\, \phi_\pi(y) = \int_K \phi_\pi(xky)\, dk, \quad x, y \in G. \tag{2.1}$$

This can be rewritten as

$$\phi_\pi(x)\, \phi_\pi(y) = \int_G \phi_\pi(z)\, d\mu_{x,y}(z), \quad x, y \in G, \tag{2.2}$$

where, for each $x, y \in G$, $\mu_{x,y}$ is a positive Borel measure on G. This product formula is associated with a positive convolution structure for the K-biinvariant functions and measures on G. It yields one of the standard examples of a (compact commutative) hypergroup. See Jewett [9, Theorem 8.2A].

If ρ and σ are spherical representations for the compact Gelfand pair (G, K) then the corresponding spherical functions ϕ_ρ and ϕ_σ are positive definite, hence the product $\phi_\rho\, \phi_\sigma$ is also positive definite. Any positive definite K-biinvariant function on G has an expansion with nonnegative coefficients in terms of the spherical functions. Hence

$$\phi_\rho(x)\, \phi_\sigma(x) = \sum_{\tau \in (G/K)^\wedge} c_{\rho,\sigma}(\tau)\, \phi_\tau(x), \quad x \in G,$$

where the coefficients $c_{\rho,\sigma}(\tau)$ are non-negative. In fact, only finitely many of them are non-zero. This product formula is associated with a positive dual convolution structure for the measures on the discrete set $(G/K)^\wedge$. Again this yields a standard example of a hypergroup, dual to the hypergroup of the previous paragraph (cf. Gallardo & Gebuhrer [7, §2.2.1]).

As an example (see Vilenkin, [23, Ch.9]), let $G := O(d)$, the group of orthogonal $d \times d$ matrices, and let $K := O(d-1)$, the subgroup leaving the first standard basis vector fixed. Then (G, K) is a Gelfand pair and $(G/K)^\wedge$ can be identified with $\mathbb{Z}_+$, the set of nonnegative integers. For $n \in \mathbb{Z}_+$, a model for $\mathcal{H}(\pi_n)$ is the space of spherical harmonics of degree n on S^{d-1}, the unit sphere in $\mathbb{R}^d$. The corresponding spherical function ϕ_n can be expressed in terms of ultraspherical polynomials as follows. Put

$$a_\theta := \begin{pmatrix} \cos\theta & -\sin\theta & & & 0 \\ \sin\theta & \cos\theta & & & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Then

$$\phi_n(K a_\theta K) = \frac{P_n^{(\alpha,\alpha)}(\cos\theta)}{P_n^{(\alpha,\alpha)}(1)},$$

where $\alpha = \frac{1}{2}d - \frac{3}{2}$ and $P_n^{(\alpha,\beta)}(x)$ is defined as the orthogonal polynomial of degree n orthogonal with respect to the measure $(1-x)^\alpha(1+x)^\beta\,dx$ on the interval $[-1,1]$. For general α, β these polynomials are called *Jacobi polynomials* and for $\alpha = \beta$ *ultraspherical polynomials.* For $\alpha = 0, \frac{1}{2}, 1, \ldots$ the two dual hypergroup structures associated with the ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)$ come from the group interpretation. But the product formula and dual product formula for the ultraspherical polynomials persist for all real $\alpha \geq -\frac{1}{2}$, and so do the two corresponding hypergroup structures, dual to each other. See Lasser [16].

3. The function algebra on a group as a Hopf algebra

Let G be a group and $\mathrm{Fun}(G)$ be the space of all complex-valued functions on G. It will provide an example of a commutative Hopf algebra. It is good to keep this example in mind when going to non-commutative Hopf algebras. We can observe the following structures, operations and identities in $\mathrm{Fun}(G)$.

(i) $\mathrm{Fun}(G)$ is an associative algebra with identity 1
where $(fg)(x) := f(x)\,g(x)$ (pointwise multiplication)
and $1(x) := 1$.
The algebra is also commutative.

(ii) There is an algebra homomorphism $\Delta\colon \mathrm{Fun}(G) \to \mathrm{Fun}(G \times G)$ (*comultiplication*)
given by $(\Delta f)(x,y) := f(xy)$.

(iii) It satisfies $(\Delta \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \Delta) \circ \Delta$ (*coassociativity*).

(iv) There is an algebra homomorphism $\varepsilon\colon \mathrm{Fun}(G) \to \mathbb{C}$ (*counit*)
given by $\varepsilon(f) := f(e)$.

(v) It satisfies $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$.

(vi) There is a linear mapping $S\colon \mathrm{Fun}(G) \to \mathrm{Fun}(G)$ (*antipode*)
given by $(Sf)(x) := f(x^{-1})$.

(vii) Define the linear operator $m\colon \mathrm{Fun}(G \times G) \to \mathrm{Fun}(G)$ by $m(f \otimes g) := fg$.
Here $(f \otimes g)(x,y) := f(x)\,g(y)$.
Then $(m(f \otimes g))(x) = (f \otimes g)(x,x)$.
More generally put $(mF)(x) := F(x,x)$, $F \in \mathrm{Fun}(G \times G)$.

(viii) The antipode satisfies $(m \circ (S \otimes \mathrm{id}) \circ \Delta)(f) = \varepsilon(f)\,1 = (m \circ (\mathrm{id} \otimes S) \circ \Delta)(f)$, $f \in \mathrm{Fun}(G)$.
The first identity in the above antipode formula is evident from:

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ & \big\downarrow \Delta & \\ (x,y) & \longmapsto & f(x,y) \\ & \big\downarrow S\otimes\mathrm{id} & \\ (x,y) & \longmapsto & f(x^{-1}y) \\ & \big\downarrow m & \\ x & \longmapsto & f(x^{-1}x) = f(e) = \varepsilon(f) = \varepsilon(f)\,1(x) \end{array}$$

In particular consider the example of a complex algebraic group. Consider the algebra $\mathrm{Pol}(G)$ of polynomial functions on G instead of $\mathrm{Fun}(G)$. Then the algebraic tensor product $\mathrm{Pol}(G) \otimes \mathrm{Pol}(G)$ can be identified with $\mathrm{Pol}(G \times G)$ such that $(f \otimes g)(x, y) = f(x)\, g(y)$.

4. Hopf algebras and quantum groups

We can now define a *Hopf algebra* as an associative (not necessarily commutative) algebra $\mathcal{A}$ with identity 1 such that the statements (i)–(viii) of §3 are satisfied with $\mathrm{Fun}(G)$ and $\mathrm{Fun}(G \times G)$ being replaced by $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{A}$, respectively. However, the indented lines of §3 do not apply to the case of a general Hopf algebra. See Sweedler [20] and Abe [1] for the general theory of Hopf algebras. It can be proved that the antipode is antimultiplicative: $S(ab) = S(b)\, S(a)$. Sometimes we will use a symbolic notation for the comultiplication:

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}.$$

The underlying idea is that $\Delta(a)$ is a finite sum of (not uniquely determined) elements $a_{(1)} \otimes a_{(2)}$. This notation is extended to iterated comultiplication:

$$((\Delta \otimes \mathrm{id}) \circ \Delta)(a) = ((\mathrm{id} \otimes \Delta) \circ \Delta)(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.$$

It is difficult to give a straightforward definition of a quantum group. It is a rather virtual object, but to some extent it can be implicitly defined as follows. We say that the *algebra of polynomial functions on a quantum group* is defined as a non-commutative Hopf algebra $\mathcal{A}_q$ which is a deformation with deformation parameter q of a commutative Hopf algebra $\mathcal{A}_1$, where $\mathcal{A}_1$ is the algebra of polynomial functions on a complex algebraic group. See Drinfel'd [5].

As an example let G be the complex algebraic group

$$SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}.$$

Let $\alpha, \beta, \gamma, \delta$ be the polynomial functions on $SL(2, \mathbb{C})$ which send $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to a, b, c, d, respectively. Then $\mathrm{Pol}(SL(2, \mathbb{C}))$ is the commutative algebra generated by $\alpha, \beta, \gamma, \delta$ with relation $\alpha\delta - \beta\gamma = 1$. The comultiplication acting on the generators is given by

$$\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{4.1}$$

which should be read in the sense of matrix multiplication, e.g.,

$$\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma.$$

For the counit we have

$$\varepsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.2}$$

and for the antipode

$$S \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

These formulas determine Δ, ε and S on the whole algebra, by continuation as algebra homomorphisms. (The anti-algebra homomorphism S is an algebra homomorphism on a commuatative algebra.)

We now define a q-deformation $\mathcal{A}_q = \mathrm{Pol}(SL_q(2,\mathbb{C}))$ of $\mathrm{Pol}(SL(2,\mathbb{C}))$. It will be the algebra generated by the non-commuting variables $\alpha, \beta, \gamma, \delta$ with relations

$$\begin{aligned} &\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \\ &\beta\gamma = \gamma\beta, \quad \alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma, \quad \alpha\delta - q\beta\gamma = 1. \end{aligned} \tag{4.3}$$

Note that, for $q = 1$, these relations say that everything commutes and that $\alpha\delta - \beta\gamma = 1$, so we are back then at $\mathrm{Pol}(SL(2,\mathbb{C}))$. The definitions of comultiplication Δ and counit ε are still as in (4.1) and (4.2), while

$$S\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}.$$

Then extend Δ and ε as algebra homomorphisms and S as anti-algebra homomorphism. It has to be verified as a non-trivial fact, that these extensions are compatible with the relations (4.3). However, it is easily verified that these extensions satisfy the Hopf algebra axioms, since they already verify these axioms on the generators. Thus we have implicitly defined the quantum group $SL_q(2,\mathbb{C})$ by defining the non-commutative algebra of "polynomial functions" on it.

5. Hopf $*$-algebras and compact matrix quantum groups

Let G be a complex algebraic group and let the subgroup G_0 be a real form of G. Note that a polynomial function on G is completely determined by its restriction to G_0. We can now make $\mathrm{Pol}(G)$ into a $*$-algebra by defining

$$f^*(x) := \overline{f(x)}, \quad x \in G_0.$$

Then Δ and ε become $*$-homomorphisms.

As an example let $G := SL(2,\mathbb{C})$ and $G_0 := SU(2)$. (Note that G_0 is now a *compact* real form.) Then

$$\begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}.$$

Then $*$ extends to $\mathrm{Pol}(SL(2,\mathbb{C}))$ as an antilinear antimultiplicative mapping, while antimultiplicative is here just multiplicative, because the algebra is commutative.

A *Hopf $*$-algebra* is defined as a Hopf algebra $\mathcal{A}$ with involution $*$ such that $\mathcal{A}$ is a $*$-algebra and Δ and ε are $*$-homomorphisms. It can be proved that, if in a Hopf $*$-algebra the antipode S is invertible, we have the identity

$$S \circ * \circ S \circ * = \mathrm{id}.$$

We can make the Hopf algebra $\mathcal{A}_q = \mathrm{Pol}(SL_q(2,\mathbb{C}))$ into a Hopf $*$-algebra for $0 < q < 1$. Put

$$\begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} := \begin{pmatrix} \delta & -q\gamma \\ -q^{-1}\beta & \alpha \end{pmatrix}.$$

Then $*$ can be extended to $\mathcal{A}_q$ as an antilinear antimultiplicative mapping. The underlying quantum group is called $SU_q(2)$ and the Hopf $*$-algebra is denoted by $\mathrm{Pol}(SU_q(2))$. See Woronowicz [24], [25].

We define a *matrix corepresentation* of a Hopf algebra $\mathcal{A}$ as a square matrix $t = (t_{i,j})_{i,j=1,\ldots,n}$ with entries $t_{i,j} \in \mathcal{A}$ such that

$$\Delta(t_{i,j}) = \sum_k t_{i,k} \otimes t_{k,j} \quad \text{and} \quad \varepsilon(t_{i,j}) = \delta_{i,j}.$$

If $\mathcal{A}$ is a Hopf $*$-algebra then a matrix corepresentation t is called *unitary* if

$$t^*_{i,j} = S(t_{j,i}).$$

Note that, for our example $\mathcal{A}_q$, the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a unitary matrix corepresentation. Moreover, the entries of this matrix corepresentation generate $\mathcal{A}_q$ as an algebra.

For the next topic we neeed some facts about tensor products of C*-algebras (cf. Takesaki [21, Ch.4, §4]). Let A_1 and A_2 be C*-algebras with unit element. The algebraic tensor product $A_1 \otimes_{\text{alg}} A_2$ is a $*$-algebra. A norm on it is called a *C*-norm* if $||xy|| \leq ||x||\,||y||$ and $||x^*x|| = ||x||^2$ for all $x, y \in A_1 \otimes_{\text{alg}} A_2$ and it is called a *cross norm* if

$$||a_1 \otimes a_2|| = ||a_1||\,||a_2||$$

for all $a_1 \in A_1$, $a_2 \in A_2$. It can be shown that each C*-norm on $A_1 \otimes_{\text{alg}} A_2$ is a cross norm. In particular, the norm

$$||a|| := \sup_{\pi_1, \pi_2} ||(\pi_1 \otimes \pi_2)(a)||, \quad a \in A_1 \otimes_{\text{alg}} A_2, \tag{5.1}$$

where π_1 and π_2 run over all representations of A_1 and A_2, defines a C*-cross norm on $A_1 \otimes_{\text{alg}} A_2$, which is called the *injective C*-cross norm.* Whenever we write $A_1 \otimes A_2$, we mean the completion of $A_1 \otimes_{\text{alg}} A_2$ with respect to this norm. This is a C*-algebra called the *injective C*-tensor product* of A_1 and A_2. The injective C*-cross norm is smallest among all possible C*-norms on $A_1 \otimes_{\text{alg}} A_2$.

A special class of quantum groups is given by the *compact matrix quantum groups.* We will characterize them by their Hopf algebras $\mathcal{A}$ of "polynomial functions". We require that $\mathcal{A}$ is a Hopf $*$-algebra such that:

(i) $\mathcal{A}$ is generated as a $*$-algebra by the matrix entries $t_{i,j}$ of a unitary matrix corepresentation $t = (t_{i,j})$.

(ii) For $a \in \mathcal{A}$ define $||a||$ as the supremum of all operator norms $||\pi(a)||$, where π runs through all $*$-representations of the $*$-algebra $\mathcal{A}$ on a Hilbert space. (Then $||a||$ is finite because the generating matrix $t_{i,j}$ is unitary.) Suppose that $||\,.\,||$ is *nondegenerate*, i.e., $a = 0$ if $||a|| = 0$. (This condition is satisfied if $\mathcal{A}$ has a faithful $*$-representation. This is the case for $\mathcal{A}_q = \text{Pol}(SU_q(2))$, cf. Woronowicz [24].)

If $\mathcal{A}$ satisfies the above conditions then it has, in the above norm, a completion to a C*-algebra A and Δ extends to a homomorphism $\Delta: A \to A \otimes A$ of C*-algebras. To see this, recall that $A \otimes A$ is the injective C* tensor product of A and A. For $a \in \mathcal{A}$ we have

$$||\Delta(a)|| = \sup_{\pi_1, \pi_2} ||(\pi_1 \otimes \pi_2)(\Delta(a))|| \leq \sup_{\pi} ||\pi(a)|| = ||a||,$$

where π_1, π_2 and π run over all $*$-representations of A and where we used (5.1) and the fact that $a \mapsto (\pi_1 \otimes \pi_2)(\Delta(a))$ is a $*$-representation of $\mathcal{A}$.

The pair (A, t) is now a *compact matrix pseudogroup* in the sense of Woronowicz [24].

For compact linear groups G (equivalently, compact Lie groups), the algebra $\mathcal{A}$ will be the linear span of the matrix elements of the irreducible unitary representations of G, while A will be the commutative C*-algebra $C(G)$ of continuous functions on G.

We conclude this section with a description of the Hopf algebra operations induced by a Hopf algebra $\mathcal{A}$ on its algebraic linear dual $\mathcal{A}^*$. For $f, g \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$ we put

$$(fg)(a) := (f \otimes g)(\Delta(a)), \quad (\Delta(f))(a \otimes b) := f(ab), \quad (S(f))(a) := f(S(a)).$$

Then fg and $S(f)$ belong to $\mathcal{A}^*$, but, in general, $\Delta(f)$ belongs to $(\mathcal{A} \otimes \mathcal{A})^*$, not to $\mathcal{A}^* \otimes \mathcal{A}^*$. For the unit element of $\mathcal{A}^*$ we can take the counit of $\mathcal{A}$. The counit of $\mathcal{A}^*$ is given by $f \mapsto f(1)$. If $\mathcal{A}$ is a Hopf $*$-algebra then we define an involution on $\mathcal{A}^*$ by

$$f^*(a) := \overline{f(S(a)^*)}.$$

It can now be verified that the Hopf $*$-algebra axioms are valid on $\mathcal{A}^*$ (except that Δ sends $\mathcal{A}^*$ to $\mathcal{A}^* \otimes \mathcal{A}^*$). If $\mathcal{A}$ and A are associated with a compact matrix quantum group and if A^* is the continuous linear dual of A then, for $f, g \in A^*$, fg as defined above is a well-defined element of A^*.

6. Haar and Schur

Let $\mathcal{A}$ be the Hopf $*$-algebra associated with a compact matrix quantum group and let A be its C*-algebra completion. Then Woronowicz [24] shows that there exists a *Haar functional*, a generalization of the Haar measure on a compact group:

Theorem 6.1: There is a unique continuous linear functional $h: A \to \mathbb{C}$ such that
(i) $h(1) = 1$;
(ii) $h(aa^*) \geq 0$ for all $a \in A$ and $a = 0$ if $h(aa^*) = 0$.
(iii) $(h \otimes \mathrm{id})(\Delta(a)) = h(a)\,1 = (\mathrm{id} \otimes h)(\Delta(a))$.

We say that two matrix corepresentations s and t of a Hopf algebra $\mathcal{A}$ are *equivalent* if they have the same size and if there is a complex scalar invertible matrix b of the same size such that $bs = tb$. Call a matrix corepresentation t *irreducible* if it is not equivalent to a matrix corepresentation of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

Let $\{t^\alpha\} = \{(t^\alpha_{m,n})\}$ be a complete set of representatives of equivalence classes of irreducible unitary matrix corepresentations of a Hopf $*$-algebra $\mathcal{A}$. Then it can be shown that the $t^\alpha_{m,n}$ form a basis of $\mathcal{A}$. Woronowicz [24] also proves a generalization of Schur's orthogonality relations:

Theorem 6.2: There is an algebra homomorphism $\phi: \mathcal{A} \to \mathbb{C}$ such that

$$h(t^\alpha_{m,k}\,(t^\beta_{n,l})^*) = \frac{\delta_{\alpha,\beta}\,\delta_{m,n}\,\phi(t^\alpha_{l,k})}{\phi(\sum_j t^\alpha_{j,j})},$$

$$h((t^\alpha_{k,m})^*\,t^\beta_{l,n}) = \frac{\delta_{\alpha,\beta}\,\delta_{m,n}\,\overline{\phi((t^\alpha_{l,k})^*)}}{\phi(\sum_j t^\alpha_{j,j})}.$$

7. Positive dual convolution structures from quantum groups

Let the Hopf $*$-algebra $\mathcal{A}$ and the Hopf C*-algebra A be associated with a compact matrix quantum group as in §5. Let $\mathcal{A}^*$ be the algebraic linear dual of $\mathcal{A}$. Define the involution $f \mapsto f^*: \mathcal{A}^* \to \mathcal{A}^*$ as in §5, by

$$f^*(a) := \overline{f(S(a)^*)}, \quad a \in \mathcal{A}.$$

Definition 7.1: $a \in \mathcal{A}$ is *positive definite* if

$$(f^* \otimes f)(\Delta(a)) \geq 0 \quad \forall f \in \mathcal{A}^*.$$

Proposition 7.2: Let $(t_{i,j})_{i,j=1,\ldots,n}$ be a unitary matrix corepresentation of $\mathcal{A}$. Let $(a_{i,j})_{i,j=1,\ldots,n}$ be a positive definite complex hermitian matrix. Then

$$a := \sum_{i,j=1}^{n} a_{i,j}\,t_{i,j}$$

is positive definite.

Proof:

$$\Delta(a) = \sum_{i,j,k} a_{i,j}\,t_{i,k} \otimes t_{k,j}.$$

Hence

$$\begin{aligned}(f^* \otimes f)(\Delta(a)) &= \sum_{i,j,k} a_{i,j}\,f^*(t_{i,k})\,f(t_{k,j}) \\ &= \sum_{i,j,k} a_{i,j}\,\overline{f(S(t_{i,k})^*)}\,f(t_{k,j}) \\ &= \sum_k \Big(\sum_{i,j} a_{i,j}\,\overline{f(t_{k,i})}\,f(t_{k,j})\Big) \geq 0,\end{aligned}$$

where we used $S(t_{i,k})^* = (t^*_{k,i})^* = t_{k,i}$. □

Let $\{(t^\alpha_{i,j})_{i,j=1,\ldots,d_\alpha}\}$ be a complete set of representatives of irreducible unitary corepresentations of $\mathcal{A}$.

Proposition 7.3: Let $a \in \mathcal{A}$ be expressed in terms of the $t^\alpha_{i,j}$ by

$$a = \sum_{\alpha,i,j} a^\alpha_{i,j}\, t^\alpha_{i,j}.$$

Then a is positive definite iff, for each α, $(a^\alpha_{i,j})$ is a positive definite hermitian matrix.

Proof: One direction follows from Proposition 7.2. Now suppose a is positive definite. Fix β and let $c_1, \ldots, c_{d_\beta}$ be complex numbers. Let f be the linear functional on $\mathcal{A}$ which sends $t^\alpha_{i,j}$ to $\delta_{\alpha,\beta}\,\delta_{i,1}\,c_j$. Then

$$0 \leq (f^* \otimes f)(\Delta(a)) = \sum_{\alpha,i,j,k} a^\alpha_{i,j}\, \overline{f(t^\alpha_{k,i})}\, f(t^\alpha_{k,j}) = \sum_{i,j} a^\beta_{i,j}\, \overline{c_i}\, c_j.$$

Since the c_i are arbitrary, the matrix $(a^\beta_{i,j})$ is positive definite hermitian. $\square$

Corollary 7.4: Let $a, b \in \mathcal{A}$ be positive definite. Then $\varepsilon(a) \geq 0$, $S(a) = a^*$ and ab is positive definite.

Proof: Write

$$a = \sum_{\alpha,i,j} a^\alpha_{i,j}\, t^\alpha_{i,j}, \quad b = \sum_{\alpha,i,j} b^\alpha_{i,j}\, t^\alpha_{i,j}.$$

Then

$$\begin{aligned}
\varepsilon(a) &= \sum_{\alpha,i,j} a^\alpha_{i,j}\, \varepsilon(t^\alpha_{i,j}) = \sum_{\alpha,i} a^\alpha_{i,i} \geq 0,\\
a^* &= \sum_{\alpha,i,j} \overline{a^\alpha_{i,j}}\, (t^\alpha_{i,j})^* = \sum_{\alpha,i,j} a^\alpha_{j,i}\, S(t^\alpha_{j,i}) = S(a),\\
ab &= \sum_{\alpha,\beta,i,j,k,l} a^\alpha_{i,j}\, b^\beta_{k,l}\, t^\alpha_{i,j}\, t^\beta_{k,l} = \sum_{\alpha,\beta,i,j,k,l} c^{\alpha,\beta}_{i,k;j,l}\, t^{\alpha,\beta}_{i,k;j,l} \quad \text{is positive definite,}
\end{aligned}$$

where $c^{\alpha,\beta}_{i,k;j,l} := a^\alpha_{i,j}\, b^\beta_{k,l}$ and $t^{\alpha,\beta}_{i,k;j,l} := t^\alpha_{i,j}\, t^\beta_{k,l}$. We used that $(c_{i,k;j,l})$ is again a positive definite hermitian matrix and the tensor product corepresentation $(t^{\alpha,\beta}_{i,k;j,l})$ is again unitary. $\square$

Let J be a subset of the set of α's labeling the equivalence classes of irreducible unitary corepresentations of $\mathcal{A}$. For each $\alpha \in J$ let z_α be a nonzero positive definite element of $\mathrm{Span}\{t^\alpha_{m,n} \mid m,n = 1, \ldots, d_\alpha\}$. Suppose that

$$\mathcal{Z} := \bigoplus_{\alpha \in J} \mathbb{C}\, z_\alpha$$

is a subalgebra with 1 of $\mathcal{A}$. Now the following proposition is evident from Corollary 7.4 and Proposition 7.3.

Proposition 7.5: Let $\alpha, \beta \in J$. Then

$$z_\alpha\, z_\beta = \sum_{\gamma \in J} c_{\alpha,\beta}(\gamma)\, z_\gamma$$

with $c_{\alpha,\beta}(\gamma) \geq 0$.

A class of examples of such subalgebras $\mathcal{Z}$ is provided as follows. Let the Hopf $*$-algebra $\mathcal{B}$ and the Hopf C*-algebra B be associated with another compact matrix quantum group and let $\Psi: \mathcal{A} \to \mathcal{B}$ be a surjective Hopf $*$-algebra homomorphism. By the definition of the norms on $\mathcal{A}$ and $\mathcal{B}$ (cf. §5), Ψ has a continuos extension to a C*-homomorphism $\Psi: A \to B$. By Dixmier [4, Corollaire 1.8.3] $\Psi(A)$ will be closed in B. Hence $\Psi(A) = B$. Then we say that the quantum group associated with $\mathcal{B}$ and B is a *quantum subgroup* of the quantum group associated with $\mathcal{A}$ and A.

We say that $a \in \mathcal{A}$ is *left (right) invariant with respect to* $\mathcal{B}$ if

$$(\Psi \otimes \mathrm{id})(\Delta(a)) = 1_{\mathcal{B}} \otimes a \quad \text{resp.} \quad (\mathrm{id} \otimes \Psi)(\Delta(a)) = a \otimes 1_{\mathcal{B}}. \tag{7.1}$$

The left (right) $\mathcal{B}$-invariant elements form a $*$-subalgebra with unit of $\mathcal{A}$.

Definition 7.6: The pair $(\mathcal{A}, \mathcal{B})$ is called a *quantum Gelfand pair* if for each irreducible unitary matrix corepresentation $(t_{i,j})$ of $\mathcal{A}$ the dimension of vectors $(c_1, \ldots, c_n)$ in $\mathbb{C}^n$ such that

$$\sum_j c_j \Psi(t_{i,j}) = c_i 1_{\mathcal{B}}, \qquad i = 1, \ldots, n,$$

is 0 or 1.
Equivalently: for each $(t_{i,j})$ the dimension of biinvariant elements with respect to $\mathcal{B}$ in Span $\{t_{i,j}\}$ is 0 or 1.

If the above dimension is 1 then we can make a unitary basis transformation such that

$$(\Psi(t_{i,j})) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}.$$

Then $t_{1,1}$ is the unique, up to a constant factor, $\mathcal{B}$-biinvariant element in Span$\{t_{i,j}\}$.

Let $(\mathcal{A}, \mathcal{B})$ be a quantum Gelfand pair. Let $\mathcal{Z}$ be the unital subalgebra of $\mathcal{B}$-biinvariant elements of $\mathcal{A}$. Then

$$\mathcal{Z} = \sum_{\alpha \in J} \mathbb{C}\, t^{\alpha}_{1,1},$$

where the $(t^{\alpha}_{i,j})$, $\alpha \in J$, form a maximal set of inequivalent irreducible unitary corepresentations with dimension of $\mathcal{B}$-biinvariant elements in Span$\{t^{\alpha}_{i,j}\}$ equal to 1. Note that the $t^{\alpha}_{1,1}$ are positive definite.

Example 7.7: Consider $\mathcal{A}_q = \mathrm{Pol}(SU_q(2))$. Let $\mathcal{B}$ be the algebra generated by z and z^{-1} with relations $zz^{-1} = 1 = z^{-1}z$. This becomes a Hopf algebra with $\Delta(z) := z \otimes z$ and it becomes a Hopf $*$-algebra with $z^* := z^{-1}$. Evidently, it can be identified with the algebra of polynomial functions on the circle group and it has as C*-algebra completion B the algebra of continuous functions on the circle group. Now we define a surjective homomorphism $\Psi: \mathcal{A}_q \to \mathcal{B}$ of Hopf $*$-algebras by

$$\begin{pmatrix} \Psi(\alpha) & \Psi(\beta) \\ \Psi(\gamma) & \Psi(\delta) \end{pmatrix} := \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

Up to equivalence, there is for each positive dimension $2l+1$ ($l = 0, \frac{1}{2}, 1, \ldots$) a unique irreducible unitary matrix corepresentation t^l of $\mathcal{A}_q$. The representative $(t^l_{m,n})_{m,n=-l,-l+1,\ldots,l}$ can be chosen such that $\Psi(t^l_{m,n}) = \delta_{m,n}\, z^{-2n}$. Then the pair $(\mathcal{A}_q, \mathcal{B})$ is a quantum Gelfand pair and $J = \{0, 1, 2, \ldots\} = \mathbb{Z}_+$. Now $t^l_{0,0}$ is $\mathcal{B}$-biinvariant for $l \in \mathbb{Z}_+$ and

$$t^l_{0,0} = p_l(\gamma\gamma^*; q^2),$$

where $p_l(\,.\,;q^2)$ is the orthogonal polynomial of degree l with respect to the weights q^{2k} on the points q^{2k} $(k=0,1,2,\ldots)$, with normalization $p_l(0;q^2)=1$. The orthogonality can be derived from the Schur orthogonality relations of Theorem 6.2 together with an explicit expression for the Haar functional acting on polynomials of $\gamma\gamma^*$. See Vaksman & Soĭbel'man [22], Masuda e.a. [17] and Koornwinder [12] for these results. These orthogonal polynomials are called *little q-Legendre polynomials*, a special case of *little q-Jacobi polynomials* (cf. Andrews & Askey [2]). They can be written in terms of q-hypergeometric series as

$$p_l(x;q) = {}_2\phi_1\left[\begin{matrix} q^{-n}, q^{n+1} \\ q \end{matrix}; q, qx\right].$$

It follows from Proposition 7.5 that the little q-Legendre polynomials have a linearization formula

$$p_l\, p_m = \sum_k c_{l,m}(k)\, p_k \quad \text{with} \quad c_{l,m}(k) \geq 0.$$

This is a first example of a positivity result for a (dual) convolution structure obtained from quantum groups. I could not find this positivity result (maybe proved by analytic methods) in the literature.

8. Positive convolution structures from quantum groups

Let A be a C*-algebra. Call $a \in A$ *positive* if $a = b^*b$ for some $b \in A$. Let A^* be the continuous linear dual of A. Call a linear mapping from A to another C*-algebra *positive* if it sends positive elements to positive elements. In particular, call $f \in A^*$ *positive* if $f(a) \geq 0$ for all positive $a \in A$. Let now $\mathcal{A}$ and A be associated with a compact matrix quantum group as in §5.

Proposition 8.1: If $f, g \in A^*$ are positive then so is fg.

Proof: There are $*$-representations σ, τ of A on Hilbert spaces and vectors v, w in the corresponding representation spaces such that

$$f(a) = (\sigma(a)v, v), \quad g(a) = (\tau(a)w, w) \quad \text{for all } a \in A,$$

cf. Takesaki [21, Ch.I, Theorem 9.14]. Then the tensor product π of σ and τ defined by

$$\pi(a) := (\sigma \otimes \tau)(\Delta(a))$$

is again a $*$-representation of A and

$$(fg)(a) = (f \otimes g)(\Delta(a)) = (\pi(a)\, v \otimes w, v \otimes w).$$

Hence fg is positive. □

Let $\mathcal{B}$ and B be associated with a quantum subgroup of the quantum group associated with $\mathcal{A}$ and A, as in §7, where Ψ maps $\mathcal{A}$ onto $\mathcal{B}$ and A onto B. We can extend the definition of left or right $\mathcal{B}$-invariant elements of $\mathcal{A}$, given by (7.1), to a definition of left or right B-invariant elements a of A:

$$(\Psi \otimes \mathrm{id})(\Delta(a)) = 1_B \otimes a \quad \text{resp.} \quad (\mathrm{id} \otimes \Psi)(\Delta(a)) = a \otimes 1_B.$$

Proposition 8.2: Let h_B be the Haar functional on B. Let the continuous mappings L and R from A to A be defined by

$$L(a) := (h_B\Psi \otimes \mathrm{id})\,\Delta(a), \quad R(a) := (\mathrm{id} \otimes h_B\Psi)\,\Delta(a). \tag{8.1}$$

(We wrote $h_B\Psi$ for $h_B \circ \Psi$.) Then L and R are commuting projections on the closed subspaces of left respectively right B-invariant elements. Moreover, L and R are positive mappings.

Proof: We prove the projection property for L. (The proof for R is similar.) If $a \in A$ is left B-invariant then clearly $L(a) = a$. Now we will show that, for arbitrary $a \in A$, $L(a)$ is left B-invariant:

$$\begin{aligned}(\Psi \otimes \mathrm{id})\,\Delta(L(a)) &= (\Psi \otimes \mathrm{id})\,\Delta\,(h_B\Psi \otimes \mathrm{id})\,\Delta(a)\\ &= (\Psi \otimes \mathrm{id})\,(h_B\Psi \otimes \mathrm{id} \otimes \mathrm{id})\,(\mathrm{id} \otimes \Delta)\,\Delta(a)\\ &= (h_B \otimes \mathrm{id} \otimes \mathrm{id})\,(\Psi \otimes \Psi \otimes \mathrm{id})\,(\Delta \otimes \mathrm{id})\,\Delta(a)\\ &= (h_B \otimes \mathrm{id} \otimes \mathrm{id})\,(\Delta_B \otimes \mathrm{id})\,(\Psi \otimes \mathrm{id})\,\Delta(a)\\ &= 1_B \otimes \big((h_B \otimes \mathrm{id})\,(\Psi \otimes \mathrm{id})\,\Delta(a)\big) = 1_B \otimes L(a).\end{aligned}$$

For the commuting property of L and R observe that both LR and RL are equal to

$$(h_B\Psi \otimes \mathrm{id} \otimes h_B\Psi) \circ (\Delta \otimes \mathrm{id}) \circ \Delta.$$

Finally, to see that, for instance, L is positive, observe that Δ and Ψ, being C*-homomorphisms, are positive mappings, and that h_B is a positive linear functional on B, so $h_B\Psi$ is a positive linear functional on A. By Takesaki [21, Ch.IV, Cor. 4.25] it follows that the mapping $h_B\Psi \otimes \mathrm{id}$ is positive from $A \otimes A$ to A. □

Note that the B-biinvariant elements in A form a C*-subalgebra of A. We will denote it by A_{BB}.

We call $f \in A^*$ *left (right) B-invariant* if

$$(\Psi \otimes f)(\Delta(a)) = f(a)\,1_B \quad \text{resp.} \quad (f \otimes \Psi)(\Delta(a)) = f(a)\,1_B \quad \text{for all } a \in A. \tag{8.2}$$

Proposition 8.3: The left (right) B-invariant elements in A^* form a right (left) ideal in A^*. In particular, the B-biinvariant elements form an algebra. Furthermore, the mapping sending elements of A^* to their restriction to A_{BB} establishes a bijection between the space of B-biinvariant elements of A^* and the continuous linear dual A^*_{BB} of A_{BB}. This is also a bijection between the positive B-biinvariant elements of A^* and the positive elements of A^*_{BB}.

Proof: If $f, g \in A^*$, $a \in A$ and f is left B-invariant then

$$(\Psi \otimes fg)(\Delta(a)) = \sum_{(a)} \Psi(a_{(1)})\,f(a_{(2)})\,g(a_{(3)}) = \sum_{(a)} f(a_{(1)})\,g(a_{(2)})\,1_B = (fg)(a)\,1_B.$$

Hence the left B-invariant elements in A^* form a right ideal. Similarly for the other case. Next observe from (8.1) and (8.2) that a left B-invariant element f in A^* will satisfy

$$f(a) = f(L(a)), \quad a \in A.$$

So such an element f is determined by its restriction to the left B-invariant elements of A. Conversely, if f is a continuous linear functional on the the space of left B-invariant elements of A and if

$$g(a) := f(L(a)), \quad a \in A,$$

then g is a left B-invariant element of A^*. This follows from (8.2) and the fact that

$$(\Psi \otimes L)(\Delta(a)) = 1_B \otimes L(a), \quad a \in A.$$

The proof of this last identity is by a similar string of identities as at the end of the proof of Proposition 8.2. Similar statements also hold for right B-invariant elements. Combination of these statements together with the commuting of L and R proves the bijection statement about the restriction mapping in the proposition. Regarding the action on positive elements of this bijection observe that clearly the restriction of a positive $f \in A^*$ to A_{BB} is positive. Conversely, if $f \in A^*_{BB}$ is positive and $g := f \circ LR$ then g is positive since L and R are positive. □

Suppose now that $(\mathcal{A},\mathcal{B})$ is a quantum Gelfand pair as defined in §7. Let

$$\mathcal{Z} = \sum_{\alpha\in J} \mathbb{C}\, t_{1,1}^{\alpha}$$

as in §7.

Proposition 8.4: Let $f,g \in A^*$ be B-biinvariant. Then $f(t_{i,j}^{\alpha}) \neq 0$ implies $\alpha \in J$ and $i=j=1$ and

$$(fg)(t_{1,1}^{\alpha}) = f(t_{1,1}^{\alpha})\, g(t_{1,1}^{\alpha}), \quad \alpha \in J. \tag{8.3}$$

In particular, the B-biinvariant $f \in A^*$ form a commutative algebra. Finally, $f^*(a) = \overline{f(a)}$ for all $a \in A$.

Proof:

$$f(t_{i,j}^{\alpha})\, 1_B = (f \otimes \Psi)(\Delta(t_{i,j}^{\alpha})) = \sum_k f(t_{i,k}^{\alpha})\, \Psi(t_{k,j}^{\alpha}).$$

Hence

$$f(t_{i,j}^{\alpha}) = f(t_{i,j}^{\alpha})\, h_B(1_B) = \sum_k f(t_{i,k}^{\alpha})\, h_B(\Psi(t_{k,j}^{\alpha})) = 0 \quad \text{if } j \neq 1 \text{ or } \alpha \notin J.$$

Similarly,

$$f(t_{i,j}^{\alpha}) = \sum_k h_B(\Psi(t_{i,k}^{\alpha}))\, f(t_{k,j}^{\alpha}) = 0 \quad \text{if } i \neq 1 \text{ or } \alpha \notin J.$$

This yields the first statement of the proposition. Next, if $\alpha \in J$:

$$(fg)(t_{1,1}^{\alpha}) = \sum_k f(t_{1,k}^{\alpha})\, g(t_{k,1}^{\alpha}) = f(t_{1,1}^{\alpha})\, g(t_{1,1}^{\alpha}) = (gf)(t_{1,1}^{\alpha}).$$

So $fg = gf$ when restricted to $\mathcal{A}$, and by continuity on A.
Finally, we can establish $f^*(a) = \overline{f(a)}$ on $\mathcal{A}$ (and therefore on A) by observing that

$$f^*(t_{i,j}^{\alpha}) = \overline{f(S(t_{i,j}^{\alpha})^*)} = \overline{f(t_{j,i}^{\alpha})}. \qquad \square$$

The commutative algebra of B-biinvariant elements in A^* generalizes the commutative algebra of K-biinvariant measures for a Gelfand pair (G,K). In particular, if the C*-algebra of B-biinvariant elements in A is commutative then it can be identified with $C(X)$ for some compact Hausdorff space X and its continuous linear dual with the space of Borel measures on X. Then we obtain a positive convolution product on this measure space.

Let $e_{i,j}^{\alpha} \in \mathcal{A}^*$ be defined by

$$e_{i,j}^{\alpha}(t_{k,l}^{\beta}) := \delta_{\alpha,\beta}\, \delta_{i,k}\, \delta_{j,l}.$$

An arbitrary $f \in \mathcal{A}^*$ has the form

$$f = \sum_{\alpha,i,j} f_{i,j}^{\alpha}\, e_{i,j}^{\alpha} \tag{8.4}$$

for certain complex $f_{i,j}^{\alpha}$ (infinitely many of them may be nonzero). Any $f \in A^*$ is determined by its restriction to $\mathcal{A}$, so can be written in the form (8.4), but not each $f \in \mathcal{A}^*$ can be lifted to a continuous linear functional on A. By Proposition 8.4, the expression (8.4) reduces for a B-biinvariant $f \in A^*$ to

$$f = \sum_{\alpha\in J} f_{1,1}^{\alpha}\, e_{1,1}^{\alpha}.$$

Proposition 8.5: Let $0 \neq a \in \mathcal{A}$ be $\mathcal{B}$-invariant and have the property that

$$(fg)(a) = f(a)\, g(a)$$

for all B-biinvariant $f, g \in A^*$. Then $a = t^\alpha_{1,1}$ for some $\alpha \in J$.

Proof: Put

$$f = \sum_{\alpha \in J} f^\alpha_{1,1}\, e^\alpha_{1,1}, \quad g = \sum_{\alpha \in J} g^\alpha_{1,1}\, e^\alpha_{1,1}, \quad a = \sum_{\alpha \in J} a^\alpha_{1,1}\, t^\alpha_{1,1}.$$

Then

$$\sum_{\alpha,\beta,\gamma} f^\alpha_{1,1}\, g^\beta_{1,1}\, a^\gamma_{1,1}\, (e^\alpha_{1,1}\, e^\beta_{1,1})(t^\gamma_{1,1}) = \sum_{\alpha,\beta,\gamma,\delta} f^\alpha_{1,1}\, g^\beta_{1,1}\, a^\gamma_{1,1}\, a^\delta_{1,1}\, e^\alpha_{1,1}(t^\gamma_{1,1})\, e^\beta_{1,1}(t^\delta_{1,1}).$$

Hence, by (8.3),

$$\sum_\gamma f^\gamma_{1,1}\, g^\gamma_{1,1}\, a^\gamma_{1,1} = \sum_{\gamma,\delta} f^\gamma_{1,1}\, g^\delta_{1,1}\, a^\gamma_{1,1}\, a^\delta_{1,1}.$$

Take

$$f^\gamma_{1,1} := \delta_{\gamma,\alpha}, \quad g^\gamma_{1,1} := \delta_{\gamma,\beta}.$$

Then

$$a^\alpha_{1,1}\, \delta_{\alpha,\beta} = a^\alpha_{1,1}\, a^\beta_{1,1} \quad \text{for all } \alpha, \beta.$$

This forces $a^\beta_{1,1} = \delta_{\alpha,\beta}$ for some α. □

Next we will derive product formulas for the $t^\alpha_{1,1}$ which generalize (2.1) and (2.2). We have

$$(\Delta \otimes \mathrm{id})\, \Delta(t^\alpha_{1,1}) = \sum_{k,l} t^\alpha_{1,k} \otimes t^\alpha_{k,l} \otimes t^\alpha_{l,1}.$$

Hence

$$(\mathrm{id} \otimes h_B \Psi \otimes \mathrm{id})\, (\Delta \otimes \mathrm{id})\, \Delta(t^\alpha_{1,1}) = t^\alpha_{1,1} \otimes t^\alpha_{1,1}, \quad \alpha \in J. \tag{8.5}$$

This generalizes (2.1). Now let $f, g \in A^*$. Define $\mu_{f,g} \in A^*$ by

$$\mu_{f,g}(a) := (f \otimes h_B \Psi \otimes g)\, (\Delta \otimes \mathrm{id})\, \Delta(a) = \big(f\, (h_B \circ \Psi)\, g\big)(a). \tag{8.6}$$

Then

$$\mu_{f,g}(t^\alpha_{1,1}) = f(t^\alpha_{1,1})\, g(t^\alpha_{1,1}), \quad \alpha \in J. \tag{8.7}$$

This generalizes (2.2). If f, g are positive (as elements of A^*) then $\mu_{f,g}$ is positive, since $h_B \circ \Psi$ is positive. Observe that, if f, g are B-biinvariant, then $\mu_{f,g}$ is B-biinvariant and

$$\mu_{f,g} = fg.$$

Example 8.6: Now consider our standard example $\mathcal{A}_q = \mathrm{Pol}(SU_q(2))$. Denote its C*-completion by A_q. The $\mathcal{B}$-biinvariant elements of $\mathcal{A}_q$ are precisely the polynomials in $\gamma\gamma^*$, so they form a commutative algebra and $(A_q)_{BB}$ will be a commutative C*-algebra $C(X)$ for some compact Hausdorff space X. We will determine X.

The irreducible $*$-representations of $\mathcal{A}_q$ on a Hilbert space can be classified (cf. Vaksman & Soĭbel'man [22]):

(a) one-dimensonal representations χ_θ $(0 \le \theta < 2\pi)$ such that $\chi_\theta(\alpha) := e^{i\theta}$, $\chi_\theta(\gamma) := 0$.

(b) ∞-dimensional representations π_θ $(0 \le \theta < 2\pi)$ on a Hilbert space with orthonormal basis $e_0, e_1, \ldots$ such that

$$\pi_\theta(\alpha)\, e_n := \begin{cases} \sqrt{1-q^{2n}}\, e_{n-1} & , n > 0, \\ 0 & , n = 0, \end{cases}$$

$$\pi_\theta(\gamma)\, e_n := e^{i\theta}\, q^n\, e_n.$$

Hence, for a polynomial f,

$$||\pi_\theta(f(\gamma\gamma^*))|| = \sup_{n\in\mathbb{Z}_+} |f(q^{2n})|,$$

so

$$||f(\gamma^*\gamma)|| = \sup_{n\in\mathbb{Z}_+} |f(q^{2n})|.$$

Thus $A_q = C(X)$ with

$$X := \{0\} \cup \{1, q^2, q^4, \ldots\}.$$

We conclude from the earlier results in this section that the little q-Legendre polynomials (cf. Example 7.7) satisfy a product formula

$$p_l(x; q^2)\, p_l(y; q^2) = \int_X p_l(z; q^2)\, d\mu_{x,y}(z), \quad x, y \in X,$$

where $\mu_{x,y}$ is a positive measure on X. In fact, the measure has been explicitly computed in Koornwinder [15]. We have

$$p_l(q^x; q)\, p_l(q^y; q) = (1-q) \sum_{z=0}^{\infty} p_l(q^z; q)\, K(q^x, q^y, q^z; q)\, q^z$$

with

$$K(q^x, q^y, q^z; q) := \frac{(q^{x+1}; q)_\infty\, (q^{y+1}; q)_\infty\, (q^{z+1}; q)_\infty\, q^{xy+xz+yz}}{(q;q)_\infty^2\, (1-q)} \times \left\{ {}_3\phi_2 \left[\begin{matrix} q^{-x}, q^{-y}, q^{-z} \\ 0, 0 \end{matrix} ; q, q \right] \right\}^2 .$$

9. Askey-Wilson polynomials and $SU_q(2)$

Askey-Wilson polynomials (cf. Askey & Wilson [3]) are defined by

$$p_n(\cos\theta; a, b, c, d \mid q) := a^{-n}\, (ab, ac, ad; q)_n\, {}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} ; q, q \right]. \tag{9.1}$$

They are polynomials of degree n in $\cos\theta$ and they are symmetric in a, b, c, d. Assume a, b, c, d are real, or if complex, appear in conjugate pairs, and that $|a|, |b|, |c|, |d| \leq 1$, but the pairwise products of a, b, c, d have absolute value less than one. Then they satisfy orthogonality relations

$$\frac{1}{2\pi} \int_0^\pi p_n(\cos\theta)\, p_m(\cos\theta)\, w(\cos\theta)\, d\theta = \delta_{n,m}\, h_n, \tag{9.2}$$

where

$$w(\cos\theta) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty}, \tag{9.3}$$

$$\frac{h_n}{h_0} = \frac{(1 - q^{n-1}abcd)\, (q, ab, ac, ad, bc, bd, cd; q)_n}{(1 - q^{2n-1}abcd)\, (abcd; q)_n}, \tag{9.4}$$

and

$$h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}. \tag{9.5}$$

If a, b, c, d are real, or if complex, appear in conjugate pairs, and the pairwise products of a, b, c, d are not ≥ 1, but one or more of a, b, c, d have absolute value larger than one, then the Askey-Wilson polynomials satisfy orthogonality relations

$$\frac{1}{2\pi} \int_0^\pi p_n(\cos\theta)\, p_m(\cos\theta)\, w(\cos\theta)\, d\theta + \sum_k p_n(x_k)\, p_m(x_k)\, w_k = 0, \quad n \neq m,$$

where $w(\cos\theta)$ is as above, the w_k are certain positive weights and the x_k are the points $(eq^k + e^{-1}q^{-k})/2$ with e any of the parameteters a, b, c or d whose absolute value is larger than one. The sum is over the $k \in \mathbb{Z}_+$ with $|eq^k| > 1$.

As another preliminary to this section we describe the Hopf algebra $\mathcal{U}_q$, which is a q-deformation of the universal enveloping algebra of the Lie algebra $sl(2, \mathbb{C})$. As an algebra, $\mathcal{U}_q$ is generated by the non-commuting variables A, A^{-1}, B, C with relations

$$AA^{-1} = 1 = A^{-1}A, \quad AB = qBA, \quad AC = q^{-1}CA, \quad BC - CB = \frac{A^2 - A^{-2}}{q - q^{-1}}.$$

Comultiplication, counit, antipode and involution are defined on the generators by

$$\begin{aligned}
&\Delta(A^{\pm1}) = A^{\pm1} \otimes A^{\pm1}, \quad \Delta(B) = A \otimes B + B \otimes A^{-1}, \quad \Delta(C) = A \otimes C + C \otimes A^{-1}, \\
&\varepsilon(A) = 1 = \varepsilon(A^{-1}), \quad \varepsilon(B) = 0 = \varepsilon(C), \\
&S(A) = A^{-1}, \quad S(A^{-1}) = A, \quad S(B) = -q^{-1}B, \quad S(C) = -qC, \\
&A^* = A, \quad (A^{-1})^* = A^{-1}, \quad B^* = C, \quad C^* = B.
\end{aligned}$$

Two Hopf $*$-algebras $\mathcal{A}$ and $\mathcal{U}$ are said to be in *nondegenerate duality* if there is a bilinear form $\langle\, .\, , .\, \rangle$ on $\mathcal{U} \times \mathcal{A}$ such that

$$\begin{aligned}
&\langle u, a\rangle = 0 \quad \forall a \in \mathcal{A} \Longrightarrow u = 0, \qquad \langle u, a\rangle = 0 \quad \forall u \in \mathcal{U} \Longrightarrow a = 0, \\
&\langle \Delta(u), a \otimes b\rangle = \langle u, ab\rangle, \quad \langle u \otimes v, \Delta(a)\rangle = \langle uv, a\rangle, \\
&\langle 1, a\rangle = \varepsilon(a), \quad \langle u, 1\rangle = \varepsilon(u), \\
&\langle S(u), a\rangle = \langle a, S(u)\rangle, \quad \langle u^*, a\rangle = \overline{\langle u, S(a)^*\rangle}.
\end{aligned}$$

Thus $\mathcal{U}$ is embedded in the algebraic linear dual $\mathcal{A}^*$ of $\mathcal{A}$ and we may write $u(a)$ instead of $\langle u, a\rangle$. Note that the above formulas are compatible with the formulas for the Hopf algebra operations on $\mathcal{A}^*$ as given at the end of §5. If $\mathcal{U}$ and $\mathcal{A}$ are both generated as an algebra by certain generators then $\langle\, .\, , .\, \rangle$ is already determined by the values $\langle u, a\rangle$ for u, a belonging to the generators.

The Hopf $*$-algebras $\mathcal{A}_q$ and $\mathcal{U}_q$ are in nondegenerate duality in the following way. The only pairs of generators (u, a) yielding nonzero $\langle u, a\rangle$ are:

$$\langle A^{\pm1}, \alpha\rangle = q^{\pm\frac{1}{2}}, \quad \langle A^{\pm1}, \delta\rangle = q^{\mp\frac{1}{2}}, \quad \langle B, \beta\rangle = 1, \quad \langle C, \gamma\rangle = 1.$$

If the Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ are in nondegenerate duality then define left and right actions of $\mathcal{U}$ on $\mathcal{A}$ by

$$u.a := (\mathrm{id} \otimes u)(\Delta(a)), \quad a.u := (u \otimes \mathrm{id})(\Delta(a)), \qquad u \in \mathcal{U},\ a \in \mathcal{A}.$$

In particular, consider this for $\mathcal{A}_q$ cand $\mathcal{U}_q$ and let $X \in \mathrm{Span}\{A - A^{-1}, B, C\}$. Then

$$\Delta(X) = A \otimes X + X \otimes A^{-1}$$

and

$$\begin{aligned}
X.a = 0 \quad &\& \quad X.b = 0 \Longrightarrow X.(ab) = 0, \\
a.X = 0 \quad &\& \quad b.X = 0 \Longrightarrow (ab).X = 0.
\end{aligned}$$

We say that $a \in \mathcal{A}_q$ is *left X-invariant* or *right X-invariant* if $X.a = 0$ or $a.X = 0$ and we call a *X-biinvariant* if $X.a = 0 = a.X$. Thus the left X-invariant, right X-invariant and X-biinvariant elements form subalgebras of $\mathcal{A}_q$. If $X = A - A^{-1}$ this coincides with left, right or biinvariance under the Hopf algebra $\mathcal{B}$ of §7. For other choices of X this is a quantum analogue of being invariant with respect to a subgroup of $SU(2)$ conjugate to $S(U(1) \times U(1))$.

Let $\sigma \in \mathbf{R}$. Let

$$X_\sigma := iB - iC - \frac{q^{-\sigma} - q^\sigma}{q^{-1} - q}(A - A^{-1}).$$

Then $X_\sigma^* = X_\sigma$. Let

$$\rho_\sigma := \tfrac{1}{2}\big(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + iq^{\frac{1}{2}}(q^{-\sigma} - q^\sigma)(\delta\gamma + \beta\alpha - \delta\beta - \gamma\alpha) + (q^{-\sigma} - q^\sigma)^2\beta\gamma\big).$$

Proposition 9.1: The algebra of X_σ-biinvariant elements in $\mathcal{A}_q$ coincides with the algebra of elements $p(\rho_\sigma)$, where p is a polynomial.

Let

$$c_m^{l,\sigma} := \frac{i^m\, q^{-(l+\sigma)m}\, q^{m^2/2}}{(q^2;q^2)_{l+m}^{1/2}\,(q^2;q^2)_{l-m}^{1/2}}\; {}_3\phi_2\left[\begin{matrix} q^{-2l+2m}, q^{-2l}, -q^{-2l-2\sigma} \\ q^{-4l}, 0 \end{matrix}; q^2, q^2\right].$$

Proposition 9.2: Let $l \in \frac{1}{2}\mathbb{Z}_+$. The dimension of the X_σ-biinvariant elements in the space $\mathrm{Span}\{t_{m,n}^l \mid m,n = -l, -l+1, \ldots, l\}$ is equal to 0 if $l \in \mathbb{Z}_+ + \frac{1}{2}$ and equal to 1 if $l \in \mathbb{Z}_+$. For $l \in \mathbb{Z}_+$ we have

$$\sum_{n,m=-l}^{l} c_m^{l,\sigma}\, \overline{c_n^{l,\sigma}}\, t_{n,m}^l = \frac{|c_l^{l,\sigma}|^2}{(q^{2l+2};q^2)_l}\, p_l(\rho_\sigma; -q^{2\sigma+1}, -q^{-2\sigma+1}, q, q \mid q^2), \tag{9.6}$$

where p_l is an Askey-Wilson polynomial. Both sides of the above formula are X_σ-biinvariant elements in $\mathrm{Span}\{t_{n,m}^l\}$ and a general X_σ-biinvariant element can be written as a linear combination of them (with l running through $\mathbb{Z}_+$).

The $c_m^{l,\sigma}$ can be extended to a unitary matrix, the elements of which consist of certain dual q-Krawtchouk polynomials. This amounts to an orthonormal basis transformation of the corepresentation space of t^l from a basis adapted to $\mathcal{B}$ to a basis of eigenvectors of $t^l(X_\sigma)$. See Koornwinder [13], [14] for the above results. (There slightly different expressions for X_σ and ρ_σ were used, and also (9.6) occurred there in a slightly different form, but it is easy to make the transition between the two versions.)

Now it follows from (9.6) and Proposition 7.2 that the two sides of (9.6) are positive definite. Moreover, the $p_l(\rho_\sigma; -q^{2\sigma+1}, -q^{-2\sigma+1}, q, q \mid q^2)$ span an algebra (of X_σ-biinvariant elements). Thus, by Proposition 7.5 we have a linearization formula

$$p_l\, p_m = \sum_k c_{l,m}(k)\, p_k \quad \text{with} \quad c_{l,m}(k) \geq 0,$$

where

$$p_l(x) := p_l(x; -q^{2\sigma+1}, -q^{-2\sigma+1}, q, q \mid q^2).$$

The case $\sigma = 0$ of this positivity result is well-known and goes back to Rogers, cf. Gasper & Rahman [8, §8.5]. The case of general σ may be new. The positivity of linearization for the little q-Legendre polynomials, obtained at the end of §7, is also the limit case $\sigma \to \infty$ of the present result, cf. Koornwinder [14, §6].

I did not yet succeed to extend the results of §8 to the case of X_σ-biinvariant elements. However, it can be expected that the positivity of the kernel in the continuous q-Legendre case $q = \frac{1}{4}$ of Rahman & Verma's product formula [19, (1.20)] has a quantum group interpretation.

10. Positivity of linearization coefficients from addition formulas

In Koornwinder [11] a method was described how the positivity of linearization coefficients for certain orthogonal systems can be obtained from the addition formula which they satisfy. The idea was that much group theoretic information about spherical functions is encoded in addition formulas and that addition formulas may persist for parameter values for which the group theoretic interpretation is lost. The addition formulas for q-hypergeometric orthogonal polynomials (sometimes derivable from a quantum group interpretation) usually have a different structure than the classical addition formulas. Fortunately we can adapt [11, Theorem 4.1] to the q-situation. This will give us a tool to derive positivity of linearization coefficients in cases where the quantum group approach of §7 is not applicable.

I use this occasion to point out an error in [11]: The second factors at the left hand sides of (3.4) and (4.3) should get a complex conjugate bar. Some consequent modifications should be made in the proofs of Theorem 3.1 and Theorem 4.1.

Theorem 10.1: Let X be a compact Hausdorff space. Let $\{p_n\}$ be a family of continuous functions on X such that

$$\int_X p_m(z)\,\overline{p_n(z)}\,d\alpha(z) = \pi_n^{-1}\,\delta_{m,n} \tag{10.1}$$

for some positive Borel measure α on X, where $0 < \pi_n < \infty$. Let $\{r_n\}$ be a family of continuous functions on $X \times X \times X$ such that. for each $x, y \in X$, $r_n(z; x, y) = 1$ and

$$\int_X r_m(z;x,y)\,\overline{r_n(z;x,y)}\,d\beta_{x,y}(z) = \rho_n(x)\,\rho_n(y)\,\delta_{m,n} \tag{10.2}$$

for some positive Borel measure $\beta_{x,y}$ on X, where $(x, y) \mapsto \beta_{x,y}(E)$ is continuous on $X \times X$ for all Borel sets E of X and ρ_n is continuous and strictly positive on X. Suppose that there is an addition formula of the form

$$p_n(z) = \sum_k c_{n,k}\,p_n^k(x)\,\overline{p_n^k(y)}\,r_k(z;x,y), \tag{10.3}$$

where p_n^k is continuous on X, $p_n^0 = p_n$, $c_{n,k} \geq 0$, $c_{n,0} > 0$, and for each n only finitely many terms in the sum are nonzero. Suppose that

$$p_m\,p_n = \sum_l a(m,n,l)\,\pi_l\,p_l, \tag{10.4}$$

Suppose that, if two of the three arguments of $a(m, n, l)$ are fixed, it is nonzero for only finitely many values of the remaining argument. Then $a(m, n, l) \geq 0$.

Proof: Formula (10.4) is equivalent to

$$a(m,n,l) = \int_X p_m(z)\,p_n(z)\,\overline{p_l(z)}\,d\alpha(z),$$

hence to

$$p_m(z)\,\overline{p_n(z)} = \sum_l a(l,m,n)\,\pi_l\,\overline{p_l(z)}. \tag{10.5}$$

Substitute the addition formula (10.3) for $p_l(z)$ in (10.5) and integrate both sides of (10.5) over X with respect to $d\beta_{x,y}(z)$. Then multiply with $p_l(x)\,\overline{p_l(y)}/(\rho_0(x)\,\rho_0(y))$ and integrate over $X \times X$ with respect to $d\alpha(x)\,d\alpha(y)$. We obtain

$$\pi_l^{-1}\,c_{l,0}\,a(l,m,n) = \int_X \int_X \left(\int_X p_m(z)\,\overline{p_n(z)}\,d\beta_{x,y}(z)\right) \frac{p_l(x)\,\overline{p_l(y)}}{\rho_0(x)\,\rho_0(y)}\,d\alpha(x)\,d\alpha(y). \tag{10.6}$$

Substitute the addition formula (10.3) for $p_m(z)$ and $p_n(z)$ in (10.6) and apply (10.2). Then we obtain

$$a(l,m,n) = \frac{\pi_l}{c_{l,0}} \sum_i c_{m,i}\,c_{n,i} \left| \int_X p_m^i(x)\,\overline{p_n^i(x)}\,p_l(x)\,\frac{\rho_i(x)}{\rho_0(x)}\,d\alpha(x) \right|^2 \geq 0. \qquad \square$$

We now apply this theorem to the case of q-ultraspherical polynomials, for which Rahman & Verma [19] derived an addition formula of the form (10.3). Let $0 < a < 1$. Let X be the interval $[-1, 1]$. Let

$$p_n(z) := p_n(z; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}} \mid q) \tag{10.7}$$

(we used the notation (9.1) for Askey-Wilson polynomials) and take for $d\alpha(z)$ the corresponding orthogonality measure for these polynomials as given by (9.2), (9.3). Let

$$r_n(z; \cos\theta, \cos\phi) := p_n(z; ae^{i\theta+i\phi}, ae^{-i\theta-i\phi}, ae^{i\theta-i\phi}, ae^{i\phi-i\theta} \mid q)$$

and let $d\beta_{x,y}$ be the corresponding orthogonality measure given by (9.2), (9.3). Then it follows from (9.2), (9.4), (9.5) that (10.2) holds with

$$\rho_n(\cos\theta) = \left(\frac{2\pi\,(1-q^{n-1}a^4)\,(q^n a^4;q)_\infty}{(1-q^{2n-1}a^4)\,(q^{n+1};q)_\infty}\right)^{\frac{1}{2}} \frac{1}{(q^n a^2;q)_\infty\,(q^n a^2 e^{2i\theta};q)_\infty\,(q^n a^2 e^{-2i\theta};q)_\infty}.$$

Let

$$p_n^k(x) := p_{n-k}(x; aq^{k/2}, aq^{(k+1)/2}, -aq^{k/2}, -aq^{(k+1)/2} \mid q).$$

Then the Rahman-Verma addition formula has the form (10.3) with

$$c_{n,k} := \frac{(q;q)_n\,(a^4q^n, a^4q^{-1}, a^2q^{1/2}, -a^2q^{1/2}, -a^2;q)_k\,a^{n-k}}{(q;q)_k\,(q;q)_{n-k}\,(a^4q^{-1};q)_{2k}\,(a^2q^{1/2}, -a^2q^{1/2}, -a^2;q)_n}.$$

We conclude the positivity of the linearization coefficients for the q-ultraspherical polynomials (10.7) ($0 < a < 1$). This result goes back to Rogers, cf. Gasper & Rahman [8, §8.5].

11. Quantum disk polynomials and a non-commutative hypergroup

The spherical functions on the compact Gelfand pair $(U(n), U(n-1))$ can be expressed in terms of the *disk polynomials* $R_{k,l}^{n-2}(z)$. These are polynomials in z and $\bar{z}$ ($z \in \mathbb{C}$) defined in terms of normalized Jacobi polynomials $R_n^{(\alpha,\beta)} := P_n^{(\alpha,\beta)}/P_n^{(\alpha,\beta)}(1)$ by

$$R_{k,l}^{\alpha}(re^{i\theta}) := R_{\min\{k,l\}}^{(\alpha,|k-l|)}(2r^2-1)\,e^{i(k-l)\theta}.$$

See Koornwinder [10].

The quantum group analogue $(U_q(n), U_q(n-1))$ of this Gelfand pair was studied by Noumi, Yamada & Mimachi [18]. Associated with the quantum group $U_q(n)$ and the quantum subgroup $U_q(n-1)$ they have certain Hopf $*$-algebras $\mathcal{A}$ and $\mathcal{B}$. In [18, §4.3] they show that the $*$-subalgebra of $\mathcal{B}$-biinvariant elements in $\mathcal{A}$ can be identified with a certain $*$-algebra $\mathcal{A}_{\mathcal{BB}}$ generated by z and z^* with relation

$$z^*z - q^2 zz^* = 1 - q^2 \tag{11.1}$$

(here $0 < q < 1$) and that the Haar functional restricted to $\mathcal{A}_{\mathcal{BB}}$ is given by

$$h((z^*)^k z^l) = \delta_{k,l}\,\frac{(q^2;q^2)_k}{(q^{2\alpha+4};q^2)_k}, \quad \alpha = n-2. \tag{11.2}$$

Put

$$\zeta := 1 - zz^*.$$

Then they express the spherical elements in terms of *little q-Jacobi polynomials*

$$p_n^{(\alpha,\beta)}(x;q) := {}_2\phi_1\left[\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1} \\ q^{\alpha+1} \end{matrix}; q, qx\right],$$

which satisfy orthogonality relations

$$\int_0^1 p_n^{(\alpha,\beta)}(x;q)\,p_m^{(\alpha,\beta)}(x;q)\,x^\alpha\,\frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty}\,d_qx = 0, \quad n \neq m,$$

cf. Andrews & Askey [2]. Here the *q-integral* is defined by

$$\int_0^1 f(x)\,d_qx := (1-q)\sum_{k=0}^{\infty} f(q^k)\,q^k.$$

The expression of Noumi, Yamada & Mimachi for the spherical elements is

$$R^\alpha_{k,l}(z,z*;q^2) := \begin{cases} z^{k-l}\, p_l^{(\alpha,k-l)}(\zeta;q^2) & ,\ k \ge l, \\ p_k^{(\alpha,l-k)}(\zeta;q^2)\,(z^*)^{l-k} & ,\ k < l. \end{cases} \tag{11.3}$$

The orthogonality relations for the spherical elements now follow from the Schur orthogonality and (11.2). However, suppose that, outside the context of quantum groups, the relations (11.1), the elements (11.3) of the $*$-algebra $\mathcal{A}_{BB}$ generated by z and z^* and the functional (11.2) on $\mathcal{A}_{BB}$ (for arbitrary $\alpha > -1$) are given. Then it is still possible to derive the orthogonality relations

$$h((R^\alpha_{r,s}(z,z^*;q^2))^*\, R^\alpha_{k,l}(z,z^*;q^2)) = 0 = h(R^\alpha_{k,l}(z,z^*;q^2)\,(R^\alpha_{r,s}(z,z^*;q^2))^*), \quad (k,l) \ne (r,s). \tag{11.4}$$

One uses the identity

$$(z^*)^m\, z^m = (q^2\zeta;q^2)_m$$

(easily proved by complete induction with respect to m) and the *q-beta integral*

$$\int_0^1 t^{b-1}\, \frac{(qt;q)_\infty}{(q^a t;q)_\infty}\, d_q t = \frac{(1-q)\,(q;q)_\infty\,(q^{a+b};q)_\infty}{(q^a;q)_\infty\,(q^b;q)_\infty},$$

cf. Gasper & Rahman [8, (1.11.7), (1.10.14)]. Then it follows that

$$h((q^2\zeta;q^2)_m) = \frac{1-q^{2\alpha+2}}{1-q^2} \int_0^1 (q^2\zeta;q^2)_m\, \zeta^\alpha\, d_{q^2}\zeta$$

and this readily yields (11.4). Thus we have obtained a class of orthogonal polynomials in two non-commuting variables. We call them *quantum disk polynomials.*

By the quantum group interpretation for $\alpha = n-2$, the results of §7 and §8 can now be applied. By §7 we will obtain positivity of linearization coefficients for the (non-commuting) quantum disk polynomials, while §8 will establish a positive commutative multiplication on the continuous linear dual of the C*-closure of $\mathcal{A}_{BB}$.

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