Limit Transitions for BC Type Multivariable Orthogonal Polynomials[1]

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Abstract. Limit transitions will be derived between the five parameter family of Askey-Wilson polynomials, the four parameter family of big q-Jacobi polynomials and the three parameter family of little q-Jacobi polynomials in n variables associated with root system BC. These limit transitions generalize the known hierarchy structure between these families in the one variable case. Furthermore it will be proved that these three families are q-analogues of the three parameter family of BC type Jacobi polynomials in n variables. The limit transitions will be derived by taking limits of q-difference operators which have these polynomials as eigenfunctions.

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1 Introduction

Recently, a five parameter family of BC_n-type Askey-Wilson polynomials and a four resp. three parameter family of BC_n-type big and little q-Jacobi polynomials were introduced (cf. [K1] and [S1]), and full orthogonality was established in both cases with the help of specific second order q-difference operators.

In the one variable case (BC_1), limit transitions are known from the Askey-Wilson polynomials to the big resp. little q-Jacobi polynomials, as well as a limit transition from big q-Jacobi polynomials to little q-Jacobi polynomials. These limit transitions show how the three families fit into the hierarchy of the Askey-Wilson scheme. Furthermore, the one variable Askey-Wilson polynomials and the big resp. little q-Jacobi polynomials are q-analoques of the classical Jacobi polynomials in the sense that the polynomials tend to the Jacobi polynomials when q tends to 1 (up to a possible dilation and translation).

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The main purpose of this paper is to generalize these limit transitions to the BC_n case. Explicit expressions of the polynomials, which immediately yield the limit formulas in the one variable case, are no longer available for $n > 1$. Instead, we will derive limit formulas for the multivariable polynomials from limit formulas for q-difference operators having these polynomials as eigenfunctions and from limit formulas for the corresponding eigenvalues. Crucial for the proof of the limit formulas is (2.2), which expresses the polynomials in terms of the operators and the eigenvalues.

Macdonald introduced in [M1, §4] techniques to construct multivariable polynomials, and to prove full orthogonality of these polynomials. In section 2 we will describe these techniques in a slightly more general setting. In this setting, the techniques can immediately be applied in the case of BC_n type Askey-Wilson polynomials and the BC_n type big resp. little q-Jacobi polynomials. We will introduce these three families in section 3, as well as the three parameter family of generalized Jacobi polynomials (cf. [V]). Generalized Jacobi polynomials are related with BC_n type Heckman-Opdam polynomials by a suitable change of variables. In section 4 we give for each family a selfadjoint, triangular operator. In each case, we compare the eigenvalues which are related in the natural BC_n type partial order. In section 5 we will prove limit transitions from BC_n type Askey-Wilson polynomials to BC_n type big and little q-Jacobi polynomials and a limit transition from BC_n type big q-Jacobi polynomials to BC_n type little q-Jacobi polynomials for parameter values which satisfy certain specific conditions. The results of section 4 give then an explicit subset of the parameter domain for which these conditions are satisfied. We will prove that the BC_n type Askey-Wilson polynomials and the BC_n type big and little q-Jacobi polynomials are q-analogues of generalized Jacobi polynomials (with a possible dilation and translation in the variables). In section 6 we will discuss possible extensions to the whole parameter domain of the limit transitions from BC_n type Askey-Wilson polynomials to BC_n type big resp. little q-Jacobi polynomials and the limit transition from BC_n type big q-Jacobi polynomials to BC_n type little q-Jacobi polynomials. Furthermore, we make some additional remarks about the limits $q \uparrow 1$ of the big and little q-Jacobi polynomials.

Notations and conventions Throughout this paper $\mathbf{N} = \{1, 2, \ldots\}$ will be the natural numbers and $\mathbf{N}_0$ will denote the set of natural numbers together with 0. The convention will be used that $\prod_{i=l}^{k} a_i = 1$ if $k < l$, $k, l \in \mathbf{N}$. If there is no confusion possible, the dependence on the parameters a, b, c, d, q, t will be omitted in the formulas. The concept of selfadjoint operator will only be used in the formal sense: a hermitian linear operator with respect to an inner product on a vector space.

2 Techniques for proving full orthogonality

The next propositions summarize in essence the method introduced by Macdonald in [M1] to construct his polynomials for general root systems and to prove full orthogonality of these polynomials. For convenience, we will give proofs of the propositions.

We start with a proposition concerning triangular operators.

Proposition 2.1 *Suppose there is given a linear space V over $\mathbf{C}$ with a linear basis $\{e_\lambda \,/\, \lambda \in I\}$ for V, I some index set and there is given a partial order $<$ on I such that $I(\lambda) := \{\mu \in I \,/\, \mu \leq \lambda\}$ is finite for all $\lambda \in I$. Suppose that $D : V \to V$ is a triangular linear operator with respect to the given basis and partial order, i.e.*

$$De_\lambda = \sum_{\mu \leq \lambda} c_{\lambda,\mu} e_\mu \quad \forall \lambda \in I, \tag{2.1}$$

for certain $c_{\lambda,\mu} \in \mathbf{C}$. Define

$$Q_\lambda := \left(\prod_{\mu < \lambda} \frac{D - c_{\mu,\mu}}{c_{\lambda,\lambda} - c_{\mu,\mu}} \right) e_\lambda \tag{2.2}$$

for $\lambda \in I$ satisfying $c_{\lambda,\lambda} \neq c_{\mu,\mu}$ for all $\mu < \lambda$. Then $Q_\lambda \in V$ satisfies
(a) *$Q_\lambda = e_\lambda + \sum_{\mu<\lambda} k_{\lambda,\mu} e_\mu$ for certain $k_{\lambda,\mu} \in \mathbf{C}$;*
(b) *$DQ_\lambda = c_{\lambda,\lambda} Q_\lambda$.*
These two properties characterize Q_λ uniquely.

Proof: Fix $\lambda \in I$ such that $c_{\lambda,\lambda} \neq c_{\mu,\mu}$ for all $\mu < \lambda$. The triangularity property of D shows that Q_λ satisfies property **(a)**. Let $V_\lambda \subset V$ be the finite dimensional subspace spanned by $\{e_\mu \,/\, \mu \leq \lambda\}$. Then D maps V_λ into itself. Denote D_λ for the restriction of D to V_λ. There exists a total order $\prec$ on $I(\lambda)$ such that $\mu \prec \nu$ if $\mu < \nu$ (define $\mu \prec \nu$ if $\#I(\mu) < \#I(\nu)$ for $\mu, \nu \in I(\lambda)$ and extend $\prec$ to a total order on $I(\lambda)$). This implies, together with (2.1), that

$$\det\left(\xi \mathrm{Id} - D_\lambda\right) = \prod_{\mu \leq \lambda} \left(\xi - c_{\mu,\mu}\right).$$

Hence $\prod_{\mu \leq \lambda} (D - c_{\mu,\mu}) = 0$ on V_λ by the theorem of Cayley and Hamilton. In particular, $\prod_{\mu \leq \lambda} (D - c_{\mu,\mu})\, e_\lambda = 0$, so $DQ_\lambda = c_{\lambda,\lambda} Q_\lambda$. The root $c_{\lambda,\lambda}$ of the characteristic polynomial $\det(\xi \mathrm{Id} - D_\lambda)$ has multiplicity one, hence D_λ has a one dimensional eigenspace corresponding to eigenvalue $c_{\lambda,\lambda}$. So **(a)** and **(b)** characterize Q_λ uniquely. $\square$

Adding to the property that D is triangular the property that D is selfadjoint with respect to some inner product, gives

Proposition 2.2 *Keep the notations and assumptions of proposition 2.1. Suppose furthermore that there is given an inner product $\langle\, .\, ,\, .\, \rangle$ on V, such that D is selfadjoint with respect to $\langle\, .\, ,\, .\, \rangle$. Define a new basis $\{P_\lambda \,/\, \lambda \in I\}$ of V by the following two conditions:*
(1) $P_\lambda = e_\lambda + \sum_{\mu<\lambda} d_{\lambda,\mu} e_\mu$ *for some constants* $d_{\lambda,\mu}$,
(2) $\langle P_\lambda, e_\mu\rangle = 0$ *for* $\mu < \lambda$.
Then we have
(a) $DP_\lambda = c_{\lambda,\lambda} P_\lambda \quad \forall \lambda \in I$.
(b) $P_\lambda = Q_\lambda$ *for* $\lambda \in I$ *satisfying* $c_{\lambda,\lambda} \neq c_{\mu,\mu}$ *for all* $\mu < \lambda$.
(c) $\langle P_\lambda, P_\mu\rangle = 0$ *if* $\lambda < \mu$ *or* $\mu < \lambda$, *or if* $\lambda \neq \mu$ *and* $c_{\lambda,\lambda} \neq c_{\mu,\mu}$.

Proof: **(a)** Fix $\lambda \in I$. Using the triangularity of D and the explicit form of P_λ, we have that

$$DP_\lambda = c_{\lambda,\lambda} e_\lambda + \sum_{\mu<\lambda} g_\mu e_\mu$$

for certain $g_\mu \in \mathbf{C}$. Furthermore we have for all $\mu < \lambda$ that

$$\langle DP_\lambda, e_\mu\rangle = \langle P_\lambda, De_\mu\rangle = \sum_{\nu\leq\mu} c_{\mu,\nu}\langle P_\lambda, e_\nu\rangle = 0.$$

If $c_{\lambda,\lambda} \neq 0$ then it follows immediately that $DP_\lambda / c_{\lambda,\lambda}$ satisfies the defining conditions of P_λ. If $c_{\lambda,\lambda} = 0$, then

$$\langle DP_\lambda, DP_\lambda\rangle = \sum_{\mu<\lambda} \overline{g_\mu}\langle DP_\lambda, e_\mu\rangle = 0,$$

so then $DP_\lambda = 0 = c_{\lambda,\lambda} P_\lambda$.
(b) It follows from **(a)** that $c_{\lambda,\lambda} \in \mathbf{R}$. Hence we have, again by **(a)**, that $\langle Q_\lambda, P_\mu\rangle = 0$ for all $\mu < \lambda$. Thus Q_λ satisfies condition (**2**). Q_λ satisfies also condition **(1)** according to proposition 2.1, hence $Q_\lambda = P_\lambda$.
(c) Case $\mu < \lambda$ resp. $\mu > \lambda$ is immediate from the definitions, while the case $\mu \neq \lambda$ and $c_{\lambda,\lambda} \neq c_{\mu,\mu}$ follows from

$$(c_{\lambda,\lambda} - c_{\mu,\mu})\langle P_\lambda, P_\mu\rangle = 0 \tag{2.3}$$

which is a consequence of the selfadjointness of D and **(a)**. □

In the applications of these propositions, the operators and inner products usually depend on real or complex parameters, and continuity resp. rationality arguments in these parameters are sometimes needed. The following two propositions deal with the dependence of P_λ on an arbitrary parameter set. We will use the same notations as in the first two propositions but we assume that the inner product, and the selfadjoint, triangular operator depend on a parameter $s \in J$, with J an arbitrary topological space. Hence, for fixed $s \in J$, we denote $\langle\, .\, ,\, .\, \rangle_s$ for the inner product on V, D_s for the selfadjoint (w.r.t. $\langle\, .\, ,\, .\, \rangle_s$) triangular (w.r.t. $\{e_\mu \,/\, \mu \in I\}$) operator, $c_{\lambda,\mu}(s)$ $(\mu \leq \lambda)$ for the coefficients in the expansion of $D_s e_\lambda$ w.r.t. the basis $\{e_\mu \,/\, \mu \in I\}$, $P_\lambda(s)$ $(\lambda \in I)$ for the new basis defined with respect to $\langle\, .\, ,\, .\, \rangle_s$, and $d_{\lambda,\mu}(s)$ $(\mu < \lambda)$ for the coefficients in the expansion of $P_\lambda(s)$ with respect to the basis $\{e_\mu \,/\, \mu \in I\}$.

Proposition 2.3 *Suppose that for all $\lambda, \mu \in I$, the functions $s \mapsto \langle e_\lambda, e_\mu \rangle_s : J \to \mathbf{C}$ are continuous. Then:*
(a) *The functions $s \mapsto d_{\lambda,\mu}(s) : J \to \mathbf{C}$ are continuous for all $\lambda, \mu \in I$.*
(b) *Suppose that the set $\{s \in J \,/\, c_{\lambda,\lambda}(s) \neq c_{\mu,\mu}(s)\}$ is dense in J for all $\lambda \neq \mu$. Then*

$$\langle P_\lambda(s), P_\mu(s) \rangle_s = 0 \quad \forall s \in J \text{ if } \mu \neq \lambda. \tag{2.4}$$

Proof: Let $\nu < \lambda$. We have

$$0 = \langle P_\lambda(s), e_\nu \rangle_s = \langle e_\lambda, e_\nu \rangle_s + \sum_{\mu < \lambda} d_{\lambda,\mu}(s) \langle e_\mu, e_\nu \rangle_s.$$

For fixed $\lambda \in I$ this gives for every $s \in J$ an inhomogeneous system of linear equations in $d_{\lambda,\mu}(s)$ $(\mu < \lambda)$. Since the e_μ's are linearly independent, we have that $\det(\langle e_\mu, e_\nu \rangle_s)_{\mu,\nu<\lambda} \neq 0$ for all $s \in J$. Hence the system has a unique solution for every $s \in J$, and Cramer's rule together with the continuity assumption on $\langle e_\rho, e_\sigma \rangle_s$ implies that the solution $d_{\lambda,\mu}(s)$ $(\mu < \lambda)$ depends continuously on s.
(b) Part **(a)** implies that $\langle P_\lambda(s), P_\mu(s) \rangle_s$ depends continuously on s, so **(b)** follows directly from proposition 2.2 **(c)**. □

Let us fix some notations and conventions about rational functions. Let $t_1, \ldots, t_m$ be independent (complex) variables. Let $\mathbf{C}[t]$ be the $\mathbf{C}$-algebra of polynomials in $t_1, \ldots, t_m$ and $\mathbf{C}\,(t)$ the field of rational functions in $t_1, \ldots, t_m$ over $\mathbf{C}$. For each $h \in \mathbf{C}\,(t)$, define the domain of h by

$$\mathrm{dom}(h) := \{t^0 \in \mathbf{C}^m \,/\, \exists\, p, q \in \mathbf{C}[t] \text{ such that } h = p/q \text{ and } q(t^0) \neq 0.\}$$

$\mathrm{dom}(h) \subset \mathbf{C}^m$ is open and dense, and h defines a continuous function from $\mathrm{dom}(h)$ to $\mathbf{C}$ by specialization.

Definition 2.4 *Let $J \subset \mathbf{C}^m$ open, or $J \subset \mathbf{R}^m$ open. Consider $\mathbf{R}^m$ as subset of $\mathbf{C}^m$ in the usual manner. A function $f : J \to \mathbf{C}$ is said to have a rational extension if there exists a rational function $\tilde{f} \in \mathbf{C}(t)$ such that f and $\tilde{f}$ coincide as functions on $J \cap dom(\tilde{f})$. Clearly, if $\tilde{f}$ exists, then it is unique, and it will be called the rational extension of f.*

Proposition 2.5 *Let J be an open subset of $\mathbf{R}^m$ or an open subset of $\mathbf{C}^m$. Assume that the functions $s \mapsto c_{\lambda,\mu}(s) : J \to \mathbf{C}$ have rational extensions $\tilde{c}_{\lambda,\mu}$ for all $\mu \leq \lambda$. Suppose that $\tilde{c}_{\lambda,\lambda} \neq \tilde{c}_{\mu,\mu}$ as rational functions if $\mu < \lambda$. Define a dense open set $dom_\lambda \subset \mathbf{C}^m$ by*

$$dom_\lambda := \{s \in W_\lambda \,/\, \tilde{c}_{\lambda,\lambda}(s) \neq \tilde{c}_{\mu,\mu}(s) \quad \forall \mu < \lambda\},$$

with

$$W_\lambda := \bigcap_{\mu \le \nu \le \lambda} dom\,(\tilde{c}_{\nu,\mu})\,.$$

Then
(a) *The functions* $s \mapsto d_{\lambda,\mu}(s) : J \to \mathbf{C}$ *have rational extensions* $\tilde{d}_{\lambda,\mu}$ *for all* $\mu < \lambda$. *The domain of* $\tilde{d}_{\lambda,\mu}$ *contains the set* dom_λ.
(b) *The functions* $P_\lambda(s)$ *and the equation* $D_s P_\lambda(s) = c_{\lambda,\lambda}(s) P_\lambda(s)$ *remain meaningful and valid, by continuation of rational functions, for* $s \in dom_\lambda$.
(c) *Suppose that* $\tilde{c}_{\lambda,\lambda} \neq \tilde{c}_{\mu,\mu}$ *as rational functions for* $\lambda, \mu \in I$, $\mu \neq \lambda$, *and that* $s \mapsto \langle e_\lambda, e_\mu \rangle_s : J \to \mathbf{C}$ *is continuous for all* $\lambda, \mu \in I$. *Then* $\langle P_\lambda(s), P_\mu(s) \rangle_s = 0$ *for all* $s \in J$ *if* $\lambda \neq \mu$.

Proof: (a) Let $x_{\nu,\rho}$ $(\rho \le \nu \le \lambda)$ be independent variables. Proposition 2.2**(b)** gives that there are polynomials $p_{\lambda,\mu} \in \mathbf{C}[\{x_{\nu,\rho}\}_{\rho \le \nu \le \lambda}]$ such that

$$P_\lambda(s) = e_\lambda + \sum_{\mu < \lambda} \left(\frac{p_{\lambda,\mu}\left(\{c_{\nu,\rho}(s)\}_{\rho \le \nu \le \lambda}\right)}{\prod_{\nu < \lambda} \left(c_{\lambda,\lambda}(s) - c_{\nu,\nu}(s)\right)} \right) e_\mu \tag{2.5}$$

for $s \in J$ such that $c_{\lambda,\lambda}(s) \neq c_{\nu,\nu}(s)$ for all $\nu < \lambda$. Hence for all $\mu < \lambda$, the rational function $\tilde{d}_{\lambda,\mu}$ given by

$$\tilde{d}_{\lambda,\mu} = \frac{p_{\lambda,\mu}\left(\{\tilde{c}_{\nu,\rho}\}_{\rho \le \nu \le \lambda}\right)}{\prod_{\nu < \lambda} \left(\tilde{c}_{\lambda,\lambda} - \tilde{c}_{\nu,\nu}\right)} \tag{2.6}$$

is a rational extension of $d_{\lambda,\mu} : J \to \mathbf{C}$, and the domain of $\tilde{d}_{\lambda,\mu}$ contains dom_λ.
(b) is clear.
(c) follows from proposition 2.3 **(b)**.

□

Remark 2.6 *Note that the polynomials* $p_{\lambda,\mu}$ *in (2.5) and (2.6) are completely determined by the partially ordered set* $(I, <)$. *They do not depend on the choice of the inner product* $\langle\,.\,,\,.\,\rangle_s$ *or on the choice of the basis vectors* e_λ. *Furthermore, the polynomial* $p_{\lambda,\mu}$ *can be chosen homogeneous of total degree* $\#\{\nu \in I \,/\, \nu < \lambda\}$.

Remark 2.7 *For the limit transitions from* BC_n *type Askey-Wilson polynomials to* BC_n *type big resp. little q-Jacobi polynomials we will apply these propositions for* J *being an open subset of* $\mathbf{R}$. *In this case, note that if a continuous function* $f : J \to \mathbf{C}$ *has a rational extension* $\tilde{f}$, *then* $J \subset dom(\tilde{f})$. *This implies that if we have all the assumptions of proposition 2.5 with* $m = 1$, *then the rational expression* $\tilde{d}_{\lambda,\mu}(s)$ *(see (2.6)) coincides with the function* $d_{\lambda,\mu}(s)$ *for all* $s \in J$ *and all* $\mu < \lambda$.

3 Families of BC_n type orthogonal polynomials

In this section, we fix a $q \in (0,1)$. Let P^+ be the partitions of length $\leq n$, so

$$P^+ := \{\lambda = (\lambda_1, \ldots, \lambda_n) \,/\, \lambda_1 \geq \ldots \geq \lambda_n \geq 0\}. \tag{3.1}$$

Define a partial order on P^+ in the following way: $\mu, \lambda \in P^+$ then

$$\mu \leq \lambda \Leftrightarrow \sum_{j=1}^{i} \mu_j \leq \sum_{j=1}^{i} \lambda_j \quad i = 1, \ldots, n. \tag{3.2}$$

Remark 3.1 *Choose for the root system $R = R^+ \cup (-R^+)$ of type BC_n, the positive roots R^+ by*

$$R^+ = \{e_i\}_{i=1}^n \cup \{e_i \pm e_j\}_{1 \leq i < j \leq n} \cup \{2e_i\}_{i=1}^n, \tag{3.3}$$

with $\{e_i\}_{i=1}^n$ the standard orthonormal basis for $\mathbf{R}^n$, then P^+ coincides with the set of dominant weights, and $\lambda > \mu$ for $\lambda, \mu \in P^+$ iff $\lambda - \mu$ is a sum of positive roots (cf. [K1]).

Let $\mathcal{A} = \mathbf{C}[x_1, \ldots, x_n]$ be the $\mathbf{C}$-algebra of polynomials in the independent indeterminates $x_1, \ldots, x_n$ and let $A = \mathbf{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the algebra of Laurent polynomials in $x_1, \ldots, x_n$. The Weyl group $\mathcal{S}$ corresponding to the root system of type A_{n-1} is isomorphic to the permutation group of $\{1, \ldots, n\}$, so it acts in an obvious way on $\mathbf{N}_0^n$. This induces an action of $\mathcal{S}$ on $\mathcal{A}$. The algebra of symmetric polynomials, denoted $\mathcal{A}^{\mathcal{S}}$, is the subalgebra of $\mathcal{A}$ consisting of $\mathcal{S}$-invariant polynomials in the variables $x_1, \ldots, x_n$.

The Weyl group W corresponding to the root system of type BC_n is isomorphic to the semidirect product of $(\mathbf{Z}/2\mathbf{Z})^n$ and $\mathcal{S}$. It acts in an obvious way on $\mathbf{Z}^n$. This induces an action of W on A. Denote A^W for the subalgebra of A consisting of W-invariant Laurent polynomials in the variables $x_1, \ldots, x_n$.

Since $\mathrm{Card}\,(\mathcal{S}\mathbf{n} \cap P^+) = 1$ for all $\mathbf{n} \in \mathbf{N}_0^n$, we have that the symmetric monomial functions $\{m_\lambda \,/\, \lambda \in P^+\}$ defined by

$$m_\lambda(x) := \sum_{\mu \in \mathcal{S}\lambda} x^\mu,$$

with $x^\mu := x_1^{\mu_1} \ldots x_n^{\mu_n}$, form a $\mathbf{C}$-basis for $\mathcal{A}^{\mathcal{S}}$. Similarly, the monomials $\{\tilde{m}_\lambda \,/\, \lambda \in P^+\}$ defined by

$$\tilde{m}_\lambda(x) := \sum_{\mu \in W\lambda} x^\mu$$

form a $\mathbf{C}$-basis for A^W, since $\mathrm{Card}\,(W\mathbf{z} \cap P^+) = 1$ for all $\mathbf{z} \in \mathbf{Z}^n$.

Let $u, v \in \mathbf{R}, u < v$. Define the Jackson ($q$-)integral of f over $[u, v]$ by

$$\int_u^v f(x)\, d_q x := \int_0^v f(x)\, d_q x - \int_0^u f(x)\, d_q x,$$
$$\int_0^v f(x)\, d_q x := (1-q) \sum_{k=0}^{\infty} f(vq^k) v q^k,$$

provided that the infinite sums converge absolutely. If f is continuous on $[u, v]$, then

$$\lim_{q \uparrow 1} \int_u^v f(x)\, d_q x = \int_u^v f(x) dx, \tag{3.4}$$

with dx the Lebesgue measure. The q-shifted factorial is defined by

$$(u; q)_v := \frac{(u; q)_\infty}{(q^v u; q)_\infty},$$
$$(u; q)_\infty := \prod_{k=0}^{\infty} (1 - uq^k),$$

for $u, v \in \mathbf{C}$ such that $q^v u \neq q^{-k}$ for all $k \in \mathbf{N}_0$. If $v \in \mathbf{N}_0$, then this yields

$$(u; q)_v = \prod_{k=0}^{v-1} \left(1 - uq^k\right),$$

which we will use as a definition of $(u; q)_v$ for arbitrary $v \in \mathbf{N}_0$, $u \in \mathbf{C}$. Denote

$$(u_1, \ldots, u_r; q)_v := \prod_{j=1}^{r} (u_j; q)_v,$$

and denote

$${}_{r+1}\phi_r \left[\begin{matrix} u_1, \ldots, u_{r+1} \\ v_1, \ldots, v_r \end{matrix}; q, z \right] := \sum_{k=0}^{\infty} \frac{(u_1, \ldots, u_{r+1}; q)_k\, z^k}{(v_1, \ldots, v_r, q; q)_k}$$

for the q-hypergeometric series.

We now first define the BC_n-type Askey-Wilson polynomials (cf. [K1]). Define the weight function $\delta(x; a, b, c, d; q, t)$ by

$$\delta(x_1, \ldots, x_n) := \delta^+(x_1, \ldots, x_n) \delta^+(x_1^{-1}, \ldots, x_n^{-1}), \tag{3.5}$$

$$\delta^+(x) := \prod_{i=1}^{n} \frac{\left(x_i^2; q\right)_\infty}{(ax_i, bx_i, cx_i, dx_i; q)_\infty} \prod_{1 \leq k < l \leq n} \frac{\left(x_k x_l^{-1}, x_k x_l; q\right)_\infty}{\left(t x_k x_l^{-1}, t x_k x_l; q\right)_\infty}. \tag{3.6}$$

Assume that $|a|, |b|, |c|, |d| \leq 1$, and that if a, b, c, d are complex, then they appear in conjugate pairs. Assume also that the pairwise products of a, b, c, d

are not equal to 1. Denote V_{AW} for the set of parameters (a,b,c,d) which satisfy these conditions. Denote $du := du_1 \dots du_n$ and $e^{iu} := (e^{iu_1}, \dots, e^{iu_n})$. Suppose $t \in (0,1)$, then

$$\langle f, g\rangle_{AW,t} := \int .. \int_{[-\pi,\pi]^n} f(e^{iu})\overline{g(e^{iu})}\delta(e^{iu};t)du \quad f,g \in A^W \tag{3.7}$$

is a hermitian inner product on A^W.

Definition 3.2 *Let $(a,b,c,d) \in V_{AW}$ and $t \in (0,1)$. The Askey-Wilson polynomials $\{P_\lambda^{AW}(x;a,b,c,d;q,t) \,/\, \lambda \in P^+\}$ are defined by the following two conditions:*
(1) $P_\lambda^{AW}(t) = \tilde{m}_\lambda + \sum_{\mu<\lambda;\mu\in P^+} d_{\lambda,\mu}^{AW}(t)\tilde{m}_\mu$, *certain* $d_{\lambda,\mu}^{AW}(t) \in \mathbf{C}$
(2) *If $\mu < \lambda$ and $\mu \in P^+$, then $\langle P_\lambda^{AW}(t), \tilde{m}_\mu\rangle_{AW,t} = 0$.*

For the one variable case, there is no t-dependence, and explicit expressions of the Askey-Wilson polynomials $\{P_m^{AW}(x;a,b,c,d;q) \,/\, m \in \mathbf{N}_0\}$ are given by

$$P_m^{AW}(x;a,b,c,d;q) = \frac{(ab,ac,ad;q)_m}{a^m\,(q^{m-1}abcd;q)_m}\,{}_4\phi_3\left[\begin{matrix} q^{-m}, q^{m-1}abcd, ax, ax^{-1} \\ ab, ac, ad\end{matrix};q,q\right]. \tag{3.8}$$

Usually the Askey-Wilson polynomials are written as function of $\frac{x+x^{-1}}{2}$ and normalized differently (cf. [AW]).

For the BC_n-type big q-Jacobi polynomials, we define an inner product on $\mathcal{A}^{\mathcal{S}}$ as follows (cf. [S1]). Let $c,d > 0$, and

$$a \in (\frac{-c}{dq}, \frac{1}{q}), b \in (\frac{-d}{cq}, \frac{1}{q}),$$

or $a = cz, b = -d\bar{z}$ with $z \in \mathbf{C}\backslash\mathbf{R}$. Denote V_B^q for the set of parameters (a,b,c,d) which satisfy these conditions. Fix some $(a,b,c,d) \in V_B^q$. Define $\langle\,.\,,\,.\,\rangle_{B,n,q,t}^{a,b,c,d}$ for $t \in (0,1)$ on $\mathcal{A}^{\mathcal{S}}$ by:

$$\langle f, g\rangle_{B,t} := \sum_{j=0}^{n}\langle f,g\rangle_{j,B,t} \quad f,g \in \mathcal{A}^{\mathcal{S}}, \tag{3.9}$$

with $\langle f,g\rangle_{j,B,t}$ given by the following multidimensional Jackson integral:

$$\int_{x_1=0}^{c}\int_{x_2=0}^{tx_1}\cdots\int_{x_j=0}^{tx_{j-1}}\int_{x_{j+1}=-dt^{n-j-1}}^{0}\int_{x_{j+2}=-dt^{n-j-2}}^{qt^{-1}x_{j+1}}\cdots$$
$$\cdots\int_{x_n=-d}^{qt^{-1}x_{n-1}} f(x)\overline{g(x)}w_j(x;t)d_qx,$$

with $d_qx := d_qx_n \dots d_qx_1$ and weight function $w_j(x;a,b,c,d;q,t)$ given by

$$w_j(x;t) := d_j^\tau\left(\prod_{i=1}^{n}\frac{(qx_i/c, -qx_i/d;q)_\infty}{(qax_i/c, -qbx_i/d;q)_\infty}\right)\Delta_\tau^j(x), \tag{3.10}$$

with $t = q^\tau$ and

$$\Delta_\tau^j(x) := \Delta(x) \left(\prod_{\substack{1\le k<m\le n \\ k\le j}} |x_k|^{2\tau-1} \left(q^{1-\tau} \frac{x_m}{x_k}; q \right)_{2\tau-1} \right) \times \prod_{j<k<m\le n} |x_m|^{2\tau-1} \left(q^{1-\tau} \frac{x_k}{x_m}; q \right)_{2\tau-1}, \tag{3.11}$$

$\Delta(x) := \prod_{i<j}(x_i - x_j)$ the Vandermonde determinant, and with $d_j^\tau = d_j^\tau(c,d)$ a positive constant given by

$$d_j^\tau := \prod_{\substack{1\le k<m\le n \\ k\le j}} |y_{mk}|^{2\tau-1} \frac{(q^{1-\tau} y_{mk}^{-1}; q)_{2\tau-1}}{(q^{1-\tau} y_{mk}; q)_{2\tau-1}}, \quad y_{mk} := \frac{-d}{c} q^{(n-m-k+1)\tau}. \tag{3.12}$$

Definition 3.3 *Let $(a,b,c,d) \in V_B^q$ and $t \in (0,1)$.*
The big q-Jacobi polynomials $\{P_\lambda^B(\,.\,; a,b,c,d;q,t) \,/\, \lambda \in P^+\}$ are defined by the following two conditions: $\lambda \in P^+$, then:
(1) *$P_\lambda^B(t) = m_\lambda + \sum_{\mu<\lambda;\mu\in P^+} d_{\lambda,\mu}^B(t) m_\mu$ for some $d_{\lambda,\mu}^B(t) \in \mathbf{C}$,*
(2) *$\langle P_\lambda^B(t)\,,\, m_\mu \rangle_{B,t} = 0$ if $\mu < \lambda$, $\mu \in P^+$.*

For the one variable case, there is no t-dependence, and explicit expressions for the big q-Jacobi polynomials $\{P_m^B(x;a,b,c,d;q) \,/\, m \in \mathbf{N}_0\}$ are given by (cf. [AA2])

$$P_m^B(x;a,b,c,d;q) = \frac{(qa;q)_m\,(-qad/c;q)_m}{(q^{m+1}ab;q)_m\,(qa/c)^m}\, {}_3\phi_2 \left[\begin{matrix} q^{-m}, q^{m+1}ab, qxa/c \\ qa, -qad/c \end{matrix} ; q, q \right]. \tag{3.13}$$

Usually the big q-Jacobi polynomials are normalized such that the explicit expression is given by only the ${}_3\phi_2$ part of (3.13).

The little q-Jacobi polynomials are defined as follows (cf. [S1]): Let $0 < a < \frac{1}{q}$ and $b < \frac{1}{q}$, and denote V_L^q for the set of parameters (a,b) which satisfy these conditions. Fix some $(a,b) \in V_L^q$. Define for $t \in (0,1)$ a hermitian inner product $\langle\,.\,,\,.\,\rangle_{L,n,q,t}^{a,b}$ on $\mathcal{A}^{\mathcal{S}}$ by

$$\langle f,g \rangle_{L,t} := \int_{x_1=0}^{1} \int_{x_2=0}^{tx_1} \cdots \int_{x_n=0}^{tx_{n-1}} f(x)\overline{g(x)} v(x;t) d_q x \quad f,g \in \mathcal{A}^{\mathcal{S}} \tag{3.14}$$

with weight function $v(x;a,b;q,t)$ given by

$$v(x;t) := \left(\prod_{i=1}^n \frac{(qx_i;q)_\infty}{(qbx_i;q)_\infty} x_i^\alpha \right) \Delta_\tau(x), \quad (a = q^\alpha, t = q^\tau) \tag{3.15}$$

$$\Delta_\tau(x) := \Delta(x) \prod_{1\le i<j\le n} |x_i|^{2\tau-1} \left(q^{1-\tau} \frac{x_j}{x_i}; q \right)_{2\tau-1}. \tag{3.16}$$

Definition 3.4 *Let* $(a, b) \in V_L^q$ *and* $t \in (0, 1)$*. The little q-Jacobi polynomials* $\{P_\lambda^L(\,.\,; a, b; q, t) \,/\, \lambda \in P^+\}$ *are defined by the following two conditions:* $\lambda \in P^+$*, then:*
(1) $P_\lambda^L(t) = m_\lambda + \sum_{\mu<\lambda;\mu\in P^+} d_{\lambda,\mu}^L(t) m_\mu$ *for some* $d_{\lambda,\mu}^L(t) \in \mathbf{C}$*,*
(2) $\langle\, P_\lambda^L(t), m_\mu \,\rangle_{L,t} = 0$ *if* $\mu < \lambda$*,* $\mu \in P^+$*.*

For the one variable case, there is no t-dependence, and explicit expressions for the little q-Jacobi polynomials $\{P_m^L(x; a, b, c, d; q) \,/\, m \in \mathbf{N}_0\}$ are given by (cf. [AA1]):

$$P_m^L(x; a, b; q) := \frac{(-1)^m q^{\binom{m}{2}} \, (qa; q)_m}{(q^{m+1}ab; q)_m} \, {}_2\phi_1 \left[\begin{matrix} q^{-m}, q^{m+1}ab \\ qa \end{matrix} ; q, qx \right]. \tag{3.17}$$

Usually the little q-Jacobi polynomials are normalized such that the explicit expression is given by only the ${}_2\phi_1$ part of (3.17).

Finally, we define two families of 'classical' BC_n type orthogonal polynomials. Let $\alpha, \beta > -1$ and $\tau > 0$. Denote V_J for the set of parameters (α, β, τ) which satisfies these conditions. Define an hermitian inner product $\langle\,.\,,\,.\,\rangle_{J,\tau}^{\alpha,\beta}$ on $\mathcal{A}^{\mathcal{S}}$ by

$$\langle f, g \rangle_{J,\tau}^{\alpha,\beta} := \frac{1}{n!} \int_{x_1=0}^1 \cdots \int_{x_n=0}^1 f(x) \overline{g(x)} v_J(x; \alpha, \beta; \tau) dx \quad f, g \in \mathcal{A}^{\mathcal{S}}$$

with $v_J(x; \alpha, \beta; \tau) := \left(\prod_{i=1}^n (1 - x_i)^\beta x_i^\alpha \right) |\Delta(x)|^{2\tau}$.

Definition 3.5 *Let* $(\alpha, \beta, \tau) \in V_J$*. The generalized Jacobi polynomials* $\{P_\lambda^J(x; \alpha, \beta; \tau) \,/\, \lambda \in P^+\}$ *are defined by the following two conditions:*
(1) $P_\lambda^J(\alpha, \beta; \tau) = m_\lambda + \sum_{\mu<\lambda;\mu\in P^+} d_{\lambda,\mu}^J(\alpha, \beta; \tau) m_\mu$ *for some* $d_{\lambda,\mu}^J(\alpha, \beta; \tau) \in \mathbf{C}$*.*
(2) $\langle P_\lambda^J(\alpha, \beta; \tau), m_\mu \rangle_{J,\tau}^{\alpha,\beta} = 0$ *if* $\mu < \lambda$*.*

In the one variable case, the Jacobi polynomials $\{P_m^J(x; \alpha, \beta) \,/\, m \in \mathbf{N}_0\}$ are independent of τ, and are explicitly given by

$$P_m^J(x; \alpha, \beta) := \frac{(-1)^m \, (\alpha + 1)_m}{(m + \alpha + \beta + 1)_m} \, {}_2F_1 \left[\begin{matrix} -m, m + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix} ; x \right] \tag{3.18}$$

with

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; z \right] := \sum_{n=0}^\infty \frac{(a)_n \, (b)_n}{(c)_n \, n!} z^n$$

the hypergeometric function and

$$(a)_n := a(a + 1) \ldots (a + n - 1) \qquad (n \in \mathbf{N})$$

the Pochhammer symbol, $(a)_0 := 1$. Usually the Jacobi polynomials are written as functions of $1 - 2x$ and normalized differently (cf. [EM], §10.8).

The generalized Jacobi polynomials are closely related to the Heckman-Opdam polynomials of type BC_n. The BC_n type Heckman-Opdam polynomials are defined as follows (cf. [HO],[H1]). We will use the notation introduced in remark 3.1. Denote $\langle\, . \, , \, . \,\rangle$ for the standard hermitian inner product on $\mathbf{C}^n$, so $\langle e_i, e_j\rangle = \delta_{i,j}$. A multiplicity function k is a function $k : R \to \mathbf{C}$ such that $k_\alpha = k_{w\alpha}$ for all $\alpha \in R, w \in W$. $k = (k_\alpha)_{\alpha\in R}$ is completely determined by $k_1 := k_{e_1}$, $k_2 := k_{e_1+e_2}$ and $k_3 := k_{2e_1}$, so we will sometimes denote $k = (k_1, k_2, k_3)$. Let V_{HO} be the set of parameters (k_1, k_2, k_3) such that $k_1 + k_3 > -\frac{1}{2}, k_3 > -\frac{1}{2}$ and $k_2 > 0$. Define a hermitian inner product on A^W for $k \in V_{HO}$ by

$$\langle f, g\rangle_k := \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\delta_J(\theta; k)d\theta \quad f, g \in A^W,$$

with $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_n})$ and weight function

$$\begin{aligned} \delta_J(\theta; k) &:= \prod_{\alpha\in R} \left(e^{\frac{1}{2}i\langle\alpha,\theta\rangle} - e^{-\frac{1}{2}i\langle\alpha,\theta\rangle}\right)^{k_\alpha} \\ &= c(k)\prod_{j=1}^{n}(\sin^2(\theta_j/2))^{k_1+k_3}(\cos^2(\theta_j/2))^{k_3}\prod_{l<m}|\sin^2(\theta_l/2) - \sin^2(\theta_m/2)|^{2k_2} \end{aligned}$$

with $c(k) = 4^{n(k_1+2k_3)+n(n-1)k_2}$.

Definition 3.6 *Let $k \in V_{HO}$. The BC_n type Heckman-Opdam polynomials $\{P_\lambda^{HO}(x; k) \,/\, \lambda \in P^+\}$ are defined by the following two conditions:*
(1) $P_\lambda^{HO}(k) = \tilde{m}_\lambda + \sum_{\mu<\lambda;\mu\in P^+} d_{\lambda,\mu}^{HO}(k)\tilde{m}_\mu$ *for some* $d_{\lambda,\mu}^{HO}(k) \in \mathbf{C}$,
(2) $\langle P_\lambda^{HO}(k), \tilde{m}_\mu\rangle_k = 0$ *if* $\mu < \lambda$.

Note that

$$m_\lambda\left(\sin^2(\theta/2)\right) = (-4)^{-|\lambda|}\,\tilde{m}_\lambda\left(e^{i\theta}\right) + \sum_{\mu<\lambda} b_{\lambda,\mu}\tilde{m}_\mu\left(e^{i\theta}\right)$$

for certain constants $b_{\lambda,\mu}$. A calculation shows then that the defining conditions for P_λ^J (definition 3.5) with $\alpha = k_1 + k_3 - \frac{1}{2}, \beta = k_3 - \frac{1}{2}$ and $\tau = k_2$ become the defining conditions for $P_\lambda^{HO}(k)$ (definition 3.6) under the change of variables $x_i := \sin^2(\theta_i/2)$ $(i = 1, \ldots, n)$, up to the constant $(-4)^{|\lambda|}$. So the relation between Heckman-Opdam polynomials of type BC_n and the generalized Jacobi polynomials is given by

$$P_\lambda^{HO}(e^{i\theta}; k) = (-4)^{|\lambda|}P_\lambda^J\left(\sin^2(\theta/2); k_1 + k_3 - \frac{1}{2}, k_3 - \frac{1}{2}, k_2\right) \tag{3.19}$$

for $\lambda \in P^+$, with $\sin^2(\theta/2) := (\sin^2(\theta_1/2), \ldots, \sin^2(\theta_n/2))$.

In the one variable case, we have the following limit transitions: $m \in \mathbf{N}_0$, then

$$\lim_{\epsilon \to 0} \left(\frac{\epsilon (cd)^{\frac{1}{2}}}{q^{\frac{1}{2}}} \right)^m P_m^{AW} \left(\frac{q^{\frac{1}{2}} x}{\epsilon (cd)^{\frac{1}{2}}}; \epsilon q^{\frac{1}{2}} a \, (d/c)^{\frac{1}{2}}, \epsilon^{-1} q^{\frac{1}{2}} (c/d)^{\frac{1}{2}}, -\epsilon^{-1} q^{\frac{1}{2}} (d/c)^{\frac{1}{2}}, \right.$$
$$\left. -\epsilon q^{\frac{1}{2}} b (c/d)^{\frac{1}{2}}; q \right) = P_m^B(x; a, b, c, d; q) \quad (3.20)$$

for $(a, b, c, d) \in V_B^q$,

$$\lim_{\epsilon \to 0} \left(\frac{\epsilon}{q^{\frac{1}{2}}} \right)^m P_m^{AW} \left(\frac{q^{\frac{1}{2}} x}{\epsilon}; \epsilon q^{\frac{1}{2}} b, \epsilon^{-1} q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}} a; q \right) = P_m^L(x; a, b; q) \quad (3.21)$$

for $(a, b) \in V_L^q$ and

$$\lim_{d \downarrow 0} P_m^B(x; b, a, 1, d; q) = P_m^L(x; a, b; q) \quad (3.22)$$

for $(a, b) \in V_L^q$ (cf. [K2],[K3] and [S1]). These three limit transitions induce the hierarchy structure between these three families of orthogonal polynomials within the Askey-Wilson scheme. For the limit q tends to 1, we have the following limits in the one variable case:

$$\lim_{q \uparrow 1} P_m^{AW} \left(x; c, \frac{q^{\alpha+1}}{c}, \frac{q^{\beta+1} d}{c}, \frac{c}{d}; q \right) = k_m^{c,d} P_m^J \left(\frac{1 + c^2 - c(x + x^{-1})}{(1-d)(1 - c^2/d)}; \alpha, \beta \right) \quad (3.23)$$

for $\alpha, \beta > -1$ and $c, d \neq 0$, $d \neq 1$, $c^2 \neq d$ with $k_m^{c,d} := \left(\frac{(d-1)(1-c^2/d)}{c} \right)^m$,

$$\lim_{q \uparrow 1} P_m^B(x; q^\alpha, q^\beta, c, d; q) = (-(c+d))^m \, P_m^J \left(\frac{c - x}{c + d}; \alpha, \beta \right) \quad m \in \mathbf{N}_0 \quad (3.24)$$

for $\alpha, \beta > -1$ and $c, d > 0$, and

$$\lim_{q \uparrow 1} P_m^L(x; q^\alpha, q^\beta; q) = P_m^J(x; \alpha, \beta) \quad m \in \mathbf{N}_0 \quad (3.25)$$

for $\alpha, \beta > -1$. These limit transitions follow immediately from the explicit expressions for the one variable orthogonal polynomials ((3.8), (3.13), (3.17) and (3.18)). We will generalize these limit transitions to the BC_n case.

4 Selfadjoint, triangular operators and their eigenvalues

In this section, fix $q \in (0, 1)$. For each family of BC_n type polynomials defined in section 3, we will introduce a selfadjoint, triangular operator. By application of the propositions of section 2, we can conclude that the polynomials are joint eigenfunctions of the operator. Proposition 2.3**(b)** gives an alternative description of the polynomials for a subset of the parameter domain. This turns out to be crucial for the proofs of the limit transitions. In each case, we will investigate this subset of the parameter domain more carefully at the end of the section by comparing eigenvalues of the operators which are related by the partial order.

We start with defining the selfadjoint, triangular operators in each case.

Define a second order q-difference operator $D^{a,b,c,d}_{AW,q,t}$ by

$$\left(D_{AW}f\right)(x) := \sum_{i=1}^{n} \left(\psi_i^{AW}(x)\left(T_{q,i}f - f\right)(x) + \phi_i^{AW}(x)\left(T_{q^{-1},i}f - f\right)(x)\right) \tag{4.1}$$

for $f \in A^W$, with

$$(T_{q,i}f)(x) := f(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n), \tag{4.2}$$

the q-shift in the i^{th} component, and functions $\psi_i^{AW}(x; a, b, c, d; q, t)$ and $\phi_i^{AW}(x; a, b, c, d; q, t)$ given by

$$\psi_i^{AW}(x) := \frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{(1-x_i^2)(1-qx_i^2)} \prod_{l \neq i} \frac{(1-tx_ix_l)(1-tx_ix_l^{-1})}{(1-x_ix_l)(1-x_ix_l^{-1})}, \tag{4.3}$$

$$\phi_i^{AW}(x) := \psi_i^{AW}(x_1^{-1}, \ldots, x_n^{-1}). \tag{4.4}$$

We have (cf. [K1], lemma 5.2):

Proposition 4.1 *Let $\lambda \in P^+$. For arbitrary $a, b, c, d, t \in \mathbf{C}$, there exist constants $c^{AW}_{\lambda,\mu}(a, b, c, d; q, t) \in \mathbf{C}$ $(\mu \leq \lambda)$ depending polynomially on a, b, c, d, t and rationally on q, such that*

$$D_{AW,t}\tilde{m}_\lambda = \sum_{\mu \leq \lambda} c^{AW}_{\lambda,\mu}(t)\tilde{m}_\mu.$$

The leading term $c^{AW}_{\lambda,\lambda}(a, b, c, d; q, t)$ will be denoted by $a^{AW}_\lambda(a, b, c, d; q, t)$ and is given by

$$a^{AW}_\lambda := \sum_{j=1}^{n} \left(q^{-1}abcdt^{2n-j-1}(q^{\lambda_j} - 1) + t^{j-1}(q^{-\lambda_j} - 1)\right). \tag{4.5}$$

The nature of the dependance of $c^{AW}_{\lambda,\mu}$ on a, b, c, d, t, q follows by inspection of the proof of lemma 5.2 in [K1]. In [K1] it is also proved that $D^{a,b,c,d}_{AW,q,t}$ is selfadjoint with respect to $\langle\, .\, ,\, .\, \rangle^{a,b,c,d}_{AW,q,t}$ if $(a, b, c, d) \in V_{AW}$ and $t \in (0, 1)$, and that $\langle \tilde{m}_\lambda, \tilde{m}_\mu \rangle_{AW,t}$ is continuous in t for $t \in (0, 1)$, for all $\lambda, \mu \in P^+$. If $(a, b, c, d) \in \mathbf{C}^4$ with $abcd \notin \{1, q^{-1}, q^{-2}, \ldots\}$ then we have $a^{AW}_\lambda(t) = a^{AW}_\mu(t)$ as polynomials in

t if and only if $\lambda = \mu$. So application of the propositions in section 2 shows that $P_\lambda^{AW}(t)$ is an eigenfunction of $D_{AW,t}$ with eigenvalue $a_\lambda^{AW}(t)$ for all $t \in (0,1)$, and it gives full orthogonality of the polynomials.

For the big q-Jacobi case, we define a second order q-difference operator $D_{B,q,t}^{a,b,c,d}$ by replacing in the expression of D_{AW} (formula (4.1)), ψ_i^{AW} by ψ_i^B and ϕ_i^{AW} by ϕ_i^B with $\psi_i^B(x;a,b,c,d;q,t)$ given by

$$\psi_i^B(x) := qt^{n-1}\left(a - \frac{c}{qx_i}\right)\left(b + \frac{d}{qx_i}\right)\prod_{l\neq i}\frac{x_l - tx_i}{x_l - x_i}$$

and $\phi_i^B(x;a,b,c,d;q,t)$ given by

$$\phi_i^B(x) := \left(1 - \frac{c}{x_i}\right)\left(1 + \frac{d}{x_i}\right)\prod_{l\neq i}\frac{x_i - tx_l}{x_i - x_l}.$$

For the little q-Jacobi case, define $D_{L,q,t}^{a,b} := D_{B,q,t}^{b,a,1,0}$, so denote

$$\psi_i^L(x;a,b;q,t) := \psi_i^B(x;b,a,1,0;q,t),$$

$$\phi_i^L(x;a,b;q,t) := \phi_i^B(x;b,a,1,0;q,t).$$

We have (cf. [S1]):

Proposition 4.2 *Let $\lambda \in P^+$.*
(1) *For arbitrary $a,b,c,d,t \in \mathbf{C}$, there exist constants $c_{\lambda,\mu}^B(a,b,c,d;q,t) \in \mathbf{C}$ ($\mu \leq \lambda$) depending polynomially on a,b,c,d,t and Laurent polynomially on q, such that*

$$D_{B,t}m_\lambda = \sum_{\mu\leq\lambda} c_{\lambda,\mu}^B(t)m_\mu.$$

The leading term $c_{\lambda,\lambda}^B(a,b,c,d;q,t)$ is independent of c and d and will be denoted by $a_\lambda^{B,L}(a,b;q,t)$. $a_\lambda^{B,L}$ is given by

$$a_\lambda^{B,L} := \sum_{j=1}^n \left(qabt^{2n-j-1}(q^{\lambda_j} - 1) + t^{j-1}(q^{-\lambda_j} - 1)\right). \tag{4.6}$$

(2) *For arbitrary $a,b,t \in \mathbf{C}$, there exist constants $c_{\lambda,\mu}^L(a,b;q,t) \in \mathbf{C}$ ($\mu \leq \lambda$) depending polynomially on a,b,t and Laurent polynomially on q, such that*

$$D_{L,t}m_\lambda = \sum_{\mu\leq\lambda} c_{\lambda,\mu}^L(t)m_\mu.$$

We have $c_{\lambda,\lambda}^L(a,b;q,t) = a_\lambda^{B,L}(a,b;q,t)$.

Clearly part (2) of the proposition is a direct consequence of part (1), since it is clear that

$$c^L_{\lambda,\mu}(a,b;q,t) = c^B_{\lambda,\mu}(b,a,1,0;q,t), \quad \mu \le \lambda. \tag{4.7}$$

The Laurent polynomial dependance of $c^B_{\lambda,\mu}$ and $c^L_{\lambda,\mu}$ on q was not explicitly stated in [S1] but follows by inspection of the proof of proposition 4.2 in [S1].

In [S1] it is also proved that $D^{a,b,c,d}_{B,q,t}$ is selfadjoint with respect to $\langle\, .\,,\,.\,\rangle^{a,b,c,d}_{B,q,t}$ if $(a,b,c,d) \in V^q_B$ and $t \in (0,1)$, and it is proved that $D^{a,b}_{L,q,t}$ is selfadjoint with respect to $\langle\, .\,,\,.\,\rangle^{a,b}_{L,q,t}$ if $(a,b) \in V^q_L$ and $t \in (0,1)$. Furthermore, $\langle m_\lambda, m_\mu\rangle_{B,t}$ and $\langle m_\lambda, m_\mu\rangle_{L,t}$ depend continuously on t for $t \in (0,1)$ for all $\lambda,\mu \in P^+$. If $(a,b) \in \mathbf{C}^2$ such that $ab \notin \{q^{-2}, q^{-3}, \dots\}$ then we have that $a^{B,L}_\lambda(t) = a^{B,L}_\mu(t)$ as polynomials in t if and only if $\mu = \lambda$. So we can apply the propositions of section 2. This gives that $P^B_\lambda(t)$ resp. $P^L_\lambda(t)$ is an eigenfunction of $D_{B,t}$ resp. $D_{L,t}$ with eigenvalue $a^{B,L}_\lambda(t)$ for all $t \in (0,1)$, and it proves full orthogonality in the big resp. little q-Jacobi case.

For the generalized Jacobi polynomials, denote $\partial_j := \frac{\partial}{\partial x_j}$. Define a second order differential operator $D^{\alpha,\beta}_{J,\tau}$ by

$$\begin{aligned} D^{\alpha,\beta}_{J,\tau} := \sum_{j=1}^n \big(&(x_j-1)x_j\partial_j^2 + ((2+\alpha+\beta)x_j - (\alpha+1))\partial_j \\ &+ 2\tau(x_j-1)x_j\Delta(x)^{-1}\left(\partial_j\Delta\right)(x)\partial_j \big). \end{aligned} \tag{4.8}$$

We will use the notations and definitions of remark 3.1, and we denote $\langle\, .\,,\,.\,\rangle$ for the standard inner prduct on $\mathbf{C}^n$, so $\langle e_i, e_j\rangle = \delta_{i,j}$. Define $\rho(\alpha,\beta,\tau) \in \mathbf{C}^n$ by

$$\rho(\alpha,\beta,\tau) := \frac{1}{2}\sum_{i=1}^n \left(\alpha+\beta+1+2(n-i)\tau\right) e_i. \tag{4.9}$$

We have the following proposition

Proposition 4.3 *Fix $\lambda \in P^+$. For arbitrary $\alpha,\beta,\tau \in \mathbf{C}$ there exist constants $c^J_{\lambda,\mu}(\alpha,\beta;\tau) \in \mathbf{C}$ ($\mu \le \lambda$) depending polynomially on α,β and τ, such that*

$$D_J m_\lambda = \sum_{\mu\le\lambda} c^J_{\lambda,\mu} m_\mu.$$

The leading term $c^J_{\lambda,\lambda}(\alpha,\beta;\tau)$ will be denoted by $a^J_\lambda(\alpha,\beta;\tau)$. a^J_λ is given by

$$a^J_\lambda(\alpha,\beta;\tau) := \langle \lambda, \lambda + 2\rho(\alpha,\beta,\tau)\rangle. \tag{4.10}$$

Proof: This can be proved by a straightforward calculation (compare with [V], p. 817). □

Furthermore, we have that $D^{\alpha,\beta}_{J,\tau}$ is selfadjoint with respect to $\langle\, .\, ,\, .\,\rangle^{\alpha,\beta}_{J,\tau}$ for all $(\alpha,\beta,\tau)\in V_J$ (compare with [V], p. 816, theorem 4.3. Be aware of the fact that the change of variables $x_i=\cos(\theta_i)$ in the proof of theorem 4.3 in [V] should be replaced by $x_i=\cos(2\theta_i)$). The first part of proposition 2.2 gives that

$$D^{\alpha,\beta}_{J,\tau}P^J_\lambda(\alpha,\beta;\tau)=a^J_\lambda(\alpha,\beta;\tau)P^J_\lambda(\alpha,\beta;\tau) \tag{4.11}$$

for all $\lambda\in P^+$ and $(\alpha,\beta,\tau)\in V_J$. Full orthogonality of the generalized Jacobi polynomials can not be proved with the help of the single selfadjoint, triangular operator D_J. Full orthogonality can be established by proving the existence of a commutative algebra consisting of selfadjoint, triangular differential operators generated by n independent differential operators. Then for fixed $(\alpha,\beta,\tau)\in V_J$ and fixed $\lambda,\mu\in P^+$ with $\lambda\neq\mu$, one can always find a differential operator in this commutative algebra such that its eigenvalue for $P^J_\lambda(\alpha,\beta;\tau)$ is different from its eigenvalue for $P^J_\mu(\alpha,\beta;\tau)$. With the help of this operator, it follows that $\langle P^J_\lambda(\alpha,\beta;\tau),P^J_\mu(\alpha,\beta;\tau)\rangle^{\alpha,\beta}_{J,\tau}=0$ (see [H1], [HO] and [H], in these papers it is done for Jacobi polynomials associated with arbitrary root systems).

We finish this section with comparing the eigenvalues related by the partial order $<$ on P^+. We use the notation introduced in remark 3.1. Denote $Q^+:=\mathbf{N}_0$- span$\{R^+\}$ for the positive cone of the root lattice. The set S of simple roots for R^+ is given by

$$S:=\{e_i-e_{i+1}\}_{i=1}^{n-1}\cup\{e_n\}. \tag{4.12}$$

For $r\in Q^+$, there exist unique $k_s(r)\in\mathbf{N}_0 (s\in S)$ such that $r=\sum_{s\in S}k_s(r)s$. Define the height of $r\in Q^+$ by $\mathrm{ht}(r):=\sum_{s\in S}k_s(r)$. Denote $\tilde{R}^+$ for the set of positive roots of the form e_i and e_i-e_j $(i<j)$.

Proposition 4.4 *Let $\mu,\lambda\in P^+$, with $\mu\leq\lambda$. Then we can walk from μ to λ while staying within P^+ by successively adding an element of $\tilde{R}^+$.*

Proof: It is sufficient to prove that for arbitrary $\mu<\lambda$, there exists $\alpha\in\tilde{R}^+$ such that $\mu+\alpha\in P^+$ and $\mu+\alpha\leq\lambda$, because induction with respect to $\mathrm{ht}(\lambda-\mu)$ will then give the desired result. So fix $\mu,\lambda\in P^+$, such that $\mu<\lambda$. Write

$$\lambda-\mu=\sum_{i=1}^{n-1}c_i(e_i-e_{i+1})+c_ne_n,$$

with $c_j\in\mathbf{N}_0$. So we have that $\lambda_k-\mu_k=c_k-c_{k-1}$ for $k=2,\ldots,n$ and $\lambda_1-\mu_1=c_1$. Furthermore we have that

$$\sum_{j=1}^{i}(\lambda_j-\mu_j)=c_i\quad i=1,\ldots,n.$$

Let $\{c_p,\ldots,c_{q-1}\}$ $(p<q)$ be a string such that $c_j>0$ for $j=p,\ldots,q-1$ and such that $c_{p-1}=0$ (or $p=1$) and $c_q=0$ (or $q=n+1$). Then $\mu_{p-1}\geq\lambda_{p-1}\geq$

$\lambda_p > \mu_p$ (or $p = 1$ and $\lambda_p > \mu_p$) and $\mu_q > \lambda_q \geq \lambda_{q+1} \geq \mu_{q+1}$ (or $q = n+1$, or $q = n$ and $\mu_q > \lambda_q$). So $\alpha = e_p - e_q \in \tilde{R}^+$ does the job for $q < n+1$, and $\alpha = e_p \in \tilde{R}^+$ for $q = n+1$. $\square$

Remark 4.5 *It is not always possible, if $\lambda \geq \mu$, to go within P^+ from μ to λ by successively adding a simple root. For example, take $\mu = (0,0)$ ($n = 2$) or $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_1 = \ldots = \mu_n$ ($n \geq 3$).*

The following proposition extends the result in ([vD], lemma 5.1) to a larger parameter set.

Proposition 4.6 *Fix $\lambda, \mu \in P^+$ with $\mu < \lambda$. Then*

$$a_\mu^{AW}(a,b,c,d;q,t) < a_\lambda^{AW}(a,b,c,d;q,t).$$

for all $a,b,c,d,t \in \mathbf{C}$ satisfying $abcd \in [-q,1)$ and $t \in (0,1)$.

Proof: According to proposition 4.4, there exists a sequence $\lambda = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(j)} = \mu$ in P^+ such that $\lambda^{(i-1)} - \lambda^{(i)} \in \tilde{R}^+$ for $i = 1, \ldots, j$. Since

$$a_\lambda^{AW} - a_\mu^{AW} = \sum_{i=1}^{j} \left(a_{\lambda^{(i-1)}}^{AW} - a_{\lambda^{(i)}}^{AW} \right),$$

it will be sufficient to prove the proposition for $\mu < \lambda$ with $\lambda - \mu \in \tilde{R}^+$.
Case (1) Suppose $\lambda - \mu = e_i$ for some $i \in \{1, \ldots, n\}$, so $\lambda_j = \mu_j$ for $j \neq i$ and $\lambda_i = \mu_i + 1 \geq 1$. Then

$$a_\lambda^{AW}(a,b,c,d;q,t) - a_\mu^{AW}(a,b,c,d;q,t) = \left(-q^{-2} abcd t^{2(n-i)} q^{\lambda_i} + q^{-\lambda_i} \right) (1-q) t^{i-1},$$

so in this case we have that $a_\lambda^{AW}(a,b,c,d;q,t) > a_\mu^{AW}(a,b,c,d;q,t)$ for all $t \in (0,1)$ if $abcd < 1$.
Case (2): Suppose $\lambda - \mu = e_i - e_j$ for certain $1 \leq i < j \leq n$, so $\lambda_i = \mu_i + 1$, $\lambda_j = \mu_j - 1$ and $\lambda_k = \mu_k$ for $k \neq i,j$. A calculation gives that

$$a_\lambda^{AW}(a,b,c,d;q,t) - a_\mu^{AW}(a,b,c,d;q,t) = (1-q) t^{i-1} q^{-\mu_i - 1} \left(1 - t^{j-i} q^{\mu_i - \mu_j + 1} \right) \times \left(1 + abcd t^{2n-i-j} q^{\mu_i + \mu_j - 1} \right).$$

Since $i < j$ and $\mu_i \geq \mu_j$, we have that $a_\lambda^{AW}(a,b,c,d;q,t) - a_\mu^{AW}(a,b,c,d;q,t) > 0$ for all $t \in (0,1)$ if $abcd \geq -q$. $\square$

As an immediate consequence, we have

Proposition 4.7 *Fix $\lambda, \mu \in P^+$ with $\mu < \lambda$.*
Then

$$a_\mu^{B,L}(a,b;q,t) < a_\lambda^{B,L}(a,b;q,t)$$

for all $a,b,t \in \mathbf{C}$ satisfying $ab \in \left[-q^{-1}, q^{-2}\right)$ and $t \in (0,1)$.

Comparing the eigenvalues related by the partial order in the case of generalized Jacobi polynomials, gives:

Proposition 4.8 *Fix $\lambda, \mu \in P^+$ with $\mu < \lambda$. Then*

$$a^J_\mu(\alpha, \beta; \tau) < a^J_\lambda(\alpha, \beta; \tau)$$

for all $(\alpha, \beta, \tau) \in V_J$.

Proof: There exists a sequence $\lambda = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(j)} = \mu$ in P^+ such that $\lambda^{(i-1)} - \lambda^{(i)} \in \tilde{R}^+$ for all $i = 1, \ldots, j$ (proposition 4.4). Since

$$\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle = \sum_{i=1}^{j} \left(\langle \lambda^{(i-1)}, \lambda^{(i-1)} + 2\rho \rangle - \langle \lambda^{(i)}, \lambda^{(i)} + 2\rho \rangle \right),$$

it is sufficient to prove the proposition for $\mu, \lambda \in P^+$ with $\lambda - \mu \in \tilde{R}^+$.
Case (1) $\lambda - \mu = e_i$, so $\lambda_i = \mu_i + 1 \geq 1$ and $\lambda_k = \mu_k$ for $k \neq i$. Then

$$\langle \lambda - \mu, \lambda + \mu \rangle = 2\lambda_i - 1 \geq 1,$$

and (4.9), together with the definition of V_J, implies

$$\langle \lambda - \mu, 2\rho(\alpha, \beta, \tau) \rangle = \alpha + \beta + 1 + 2(n - i)\tau > -1.$$

So $\langle \lambda, \lambda + 2\rho(\alpha, \beta, \tau) \rangle - \langle \mu, \mu + 2\rho(\alpha, \beta, \tau) \rangle = \langle \lambda - \mu, \lambda + \mu + 2\rho(\alpha, \beta, \tau) \rangle > 0$.
Case (2) $\lambda - \mu = e_i - e_j$ for certain $1 \leq i < j \leq n$. Then $\lambda_i = \mu_i + 1 \geq \mu_j + 1 = \lambda_j + 2$ and $\lambda_k = \mu_k$ for $k \neq i, j$. Then we have that

$$\langle \lambda - \mu, \lambda + \mu \rangle = 2(\lambda_i - \lambda_j) - 2 \geq 2,$$

and (4.9) implies

$$\langle \lambda - \mu, 2\rho(\alpha, \beta, \tau) \rangle = 2(j - i)\tau > 0,$$

since $\tau > 0$. So $\langle \lambda, \lambda + 2\rho(\alpha, \beta, \tau) \rangle - \langle \mu, \mu + 2\rho(\alpha, \beta, \tau) \rangle > 2$. □

5 The limit transitions

For $(a, b) \in \mathbf{C}^2$ and $\lambda \in P^+$, define

$$J_\lambda(a, b) := \{t \in (0, 1) \,/\, a^{B,L}_\lambda(a, b; q, t) \neq a^{B,L}_\mu(a, b; q, t) \text{ for all } \mu < \lambda\}. \tag{5.1}$$

In this section, we will generalize the limit transitions from Askey-Wilson polynomials to big and little q-Jacobi polynomials and a limit transition from big q-Jacobi polynomials to little q-Jacobi polynomials ((3.20), (3.21) and (3.22)) to the multivariable case (BC_n) for parameter values a, b, c, d, t with $(a, b, c, d) \in V^q_B$ resp. $(a, b) \in V^q_L$ and $t \in J_\lambda(a, b)$. Furthermore, we will generalize the limit transition from the Askey-Wilson polynomials to the Jacobi polynomials (3.23) and the limit transition from big resp. little q-Jacobi polynomials to the Jacobi polynomials ((3.24) resp. (3.25)) to the multivariable case for the full parameter domain.

Denote $|\lambda| := \sum_{i=1}^n \lambda_i$ for $\lambda \in P^+$ and $c_1 + c_2 x := (c_1 + c_2 x_1, \ldots, c_1 + c_2 x_n)$ resp. $x^{-1} := (x_1^{-1} \ldots, x_n^{-1})$ for $c_1, c_2 \in \mathbf{C}$ and $x = (x_1, \ldots, x_n)$. The limit transitions are given by

Theorem 5.1 *Fix $\lambda \in P^+$.*
(1) *Fix $q \in (0,1)$. Suppose that $(a,b,c,d) \in V_B^q$ and $t \in J_\lambda(a,b)$, then*

$$\lim_{\epsilon \to 0} \left(\frac{\epsilon (cd)^{\frac{1}{2}}}{q^{\frac{1}{2}}} \right)^{|\lambda|} P_\lambda^{AW} \left(\frac{q^{\frac{1}{2}} x}{\epsilon (cd)^{\frac{1}{2}}}; \epsilon q^{\frac{1}{2}} a (d/c)^{\frac{1}{2}}, \epsilon^{-1} q^{\frac{1}{2}} (c/d)^{\frac{1}{2}}, -\epsilon^{-1} q^{\frac{1}{2}} (d/c)^{\frac{1}{2}}, -\epsilon q^{\frac{1}{2}} b (c/d)^{\frac{1}{2}}; q, t \right) = P_\lambda^B(x; a, b, c, d; q, t). \quad (5.2)$$

(2) *Fix $q \in (0,1)$. Suppose that $(a,b) \in V_L^q$ and $t \in J_\lambda(a,b)$, then*

$$\lim_{\epsilon \to 0} \left(\frac{\epsilon}{q^{\frac{1}{2}}} \right)^{|\lambda|} P_\lambda^{AW} \left(\frac{q^{\frac{1}{2}} x}{\epsilon}; \epsilon q^{\frac{1}{2}} b, \epsilon^{-1} q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}} a; q, t \right) = P_\lambda^L(x; a, b; q, t). \quad (5.3)$$

(3) *Fix $q \in (0,1)$. Suppose that $(a,b) \in V_L^q$ and $t \in J_\lambda(a,b)$, then*

$$\lim_{d \downarrow 0} P_\lambda^B(x; b, a, 1, d; q, t) = P_\lambda^L(x; a, b; q, t). \quad (5.4)$$

(4) *Let $(\alpha, \beta, \tau) \in V_J$ and $c, d \in \mathbf{C}$ such that $c, d \neq 0$, $c^2 \neq d$, $d \neq 1$. Then*

$$\lim_{q \uparrow 1} P_\lambda^{AW} \left(x; c, \frac{q^{\alpha+1}}{c}, \frac{q^{\beta+1} d}{c}, \frac{c}{d}; q, q^\tau \right) = k_{|\lambda|}^{c,d} P_\lambda^J \left(\frac{1 + c^2 - c(x + x^{-1})}{(1-d)(1-c^2/d)}; \alpha, \beta; \tau \right) \quad (5.5)$$

with $k_m^{c,d} := \left(\frac{(d-1)(1-c^2/d)}{c} \right)^m$.
(5) *Let $(\alpha, \beta, \tau) \in V_J$ and $c, d > 0$, then*

$$\lim_{q \uparrow 1} P_\lambda^B(x; q^\alpha, q^\beta, c, d; q, q^\tau) = (-1)^{|\lambda|} (c+d)^{|\lambda|} P_\lambda^J \left(\frac{c - x}{c + d}; \alpha, \beta; \tau \right). \quad (5.6)$$

(6) *Suppose that $(\alpha, \beta, \tau) \in V_J$, then*

$$\lim_{q \uparrow 1} P_\lambda^L \left(x; q^\alpha, q^\beta; q, q^\tau \right) = P_\lambda^J(x; \alpha, \beta; \tau). \quad (5.7)$$

The limits are pointwise limits in the following sense:
Denote $\tilde{P} := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{Z}^n \,/\, \lambda_1 \geq \ldots \geq \lambda_n\}$. We will see later that the rescaled Askey-Wilson polynomials on the left hand side of (5.2) and (5.3) can be written as symmetric Laurent polynomials of the form

$\sum_{\mu\in\tilde{P}} d_\mu m_\mu(x)$. All the other functions occuring in theorem 5.1 are symmetric polynomials of the form $\sum_{\mu\in P^+} d_\mu m_\mu(x)$. We will say that a symmetric (Laurent) polynomial $p(s,x)=\sum_{\mu\in\tilde{P}} d_\mu(s)m_\mu(x)$ tends to the symmetric (Laurent) polynomial $p(x)=\sum_{\mu\in\tilde{P}} d_\mu m_\mu(x)$ for s tending to zero, if for every fixed $(x_1,\ldots,x_n)\in(\mathbf{C}\backslash\{0\})^n$,

$$\lim_{s\to 0} p(s,x)=p(x).$$

If $\tilde{P}(s):=\{\mu\in\tilde{P}\,/\,d_\mu(s)\neq 0\}$ is contained in a finite subset $J\subset\tilde{P}$ for all s in an open neighbourhood of 0, then $\lim_{s\to 0} p(s,x)=p(x)$ if and only if $\lim_{s\to 0} d_\mu(s)=d_\mu$ for all $\mu\in\tilde{P}$.

Note that for the limit transitions (5.2), (5.3) and (5.5), a definition of BC_n type Askey-Wilson polynomials for more general parameter values is needed. We will introduce this definition later on.

Remark 5.2 *The first three limit transitions are especially valid for the parameter values a,b,c,d and t satisfying $\frac{-1}{q}\leq ab<\frac{1}{q^2}$, $c,d>0$ and $t\in(0,1)$, in view of proposition 4.7. Note that these conditions are independent of $\lambda\in P^+$.*

In view of limit transition (5.6) resp. (5.5), we will first look what happens with the second order differential operator $D^{\alpha,\beta}_{J,\tau}$ under the change of variables $x_i=\frac{c-y_i}{c+d}$ $(i=1,\ldots,n)$ resp. $x_i=\frac{1+c^2-c(y_i+y_i^{-1})}{(1-d)(1-c^2/d)}$ $(i=1,\ldots,n)$.

Under the change of variables $x_i=\frac{c-y_i}{c+d}$ $(i=1,\ldots,n)$ for $c+d\neq 0$, the second order differential operator $D^{\alpha,\beta}_{J,\tau}$ becomes

$$D^{\alpha,\beta,c,d}_{B,J,\tau}:=\sum_{j=1}^n\big((y_j-c)(y_j+d)\partial_j^2+\left((2+\alpha+\beta)y_j+d(\alpha+1)-c(\beta+1)\right)\partial_j$$
$$+2\tau(y_j-c)(y_j+d)\Delta(y)^{-1}\left(\partial_j\Delta\right)(y)\partial_j\big).\quad(5.8)$$

We have

$$m_\lambda(c_1+c_2y_1,\ldots,c_1+c_2y_n)=c_2^{|\lambda|}m_\lambda(y)+\sum_{\mu<\lambda}b_{\lambda,\mu}(c_1,c_2)m_\mu(y)\qquad(5.9)$$

with $b_{\lambda,\mu}(c_1,c_2)$ a polynomial expression in c_1 and c_2, of degree $|\lambda|-|\mu|$ in c_1 and of degree $|\mu|$ in c_2 for $c_1,c_2\in\mathbf{C}$. Hence

$$\left(D^{\alpha,\beta}_{J,\tau}m_\lambda\right)\left(\frac{c-y}{c+d}\right)=(-(c+d))^{-|\lambda|}\left(D^{\alpha,\beta,c,d}_{B,J,\tau}m_\lambda\right)(y)+$$
$$\sum_{\mu<\lambda}b_{\lambda,\mu}\left(\frac{c}{(c+d)},\frac{-1}{(c+d)}\right)\left(D^{\alpha,\beta,c,d}_{B,J,\tau}m_\mu\right)(y).\qquad(5.10)$$

By proposition 4.3 and by application of (5.9), the left hand side of (5.10) can be rewritten as

$$a^J_\lambda(\alpha,\beta;\tau)\left(-(c+d)\right)^{-|\lambda|}m_\lambda(y)+\sum_{\mu<\lambda}d_{\lambda,\mu}(\alpha,\beta,c,d;\tau)m_\mu(y)\qquad(5.11)$$

for certain $d_{\lambda,\mu}(\alpha,\beta,c,d;\tau)\in\mathbf{C}$ with $(c+d)^{|\mu|}d_{\lambda,\mu}(\alpha,\beta,c,d;\tau)\in\mathbf{C}$ depending polynomially on α,β,c,d and τ. By complete induction with respect to λ it follows from (5.10) and (5.11) that

$$D^{\alpha,\beta,c,d}_{B,J,\tau}m_\lambda=\sum_{\mu\le\lambda}c^{B,J}_{\lambda,\mu}(\alpha,\beta,c,d;\tau)m_\mu \tag{5.12}$$

with $c^{B,J}_{\lambda,\mu}(\alpha,\beta,c,d;\tau)$ depending polynomially on α,β,τ,c,d, and with leading term $c^{B,J}_{\lambda,\lambda}(\alpha,\beta,c,d;\tau)=a^J_\lambda(\alpha,\beta;\tau)$ independent of c and d (a^J_λ given by (4.10)).

Note that

$$\Delta(y)^{-1}\left(\partial_j\Delta\right)(y)=\sum_{k\ne j}\frac{1}{y_j-y_k}.$$

Define

$$\tilde\Delta(y):=\prod_{i=1}^n y_i^{1-n}\prod_{1\le k<l\le n}(y_k-y_l)(y_ky_l-1), \tag{5.13}$$

then we have that

$$\tilde\Delta(y)^{-1}\left(\partial_j\tilde\Delta\right)(y)=\frac{(y_j^2-1)}{y_j^2}\sum_{k\ne j}\frac{1}{y_j+y_j^{-1}-y_k-y_k^{-1}}.$$

With the help of this formula, one deduces that under the change of variables $x_i=\frac{1+c^2-c(y_i+y_i^{-1})}{(1-d)(1-c^2/d)}$, the second order differential operator $D^{\alpha,\beta}_{J,\tau}$ becomes

$$\begin{aligned}D^{\alpha,\beta,c,d}_{AW,J,\tau}=\sum_{j=1}^n\Bigg(&\frac{y_j^2(1-cy_j)(1-y_j/c)(1-y_jc/d)(1-y_jd/c)}{(1-y_j^2)^2}\partial_j^2\\&+2\frac{y_j(1-cy_j)(1-y_j/c)(1-y_jc/d)(1-y_jd/c)}{(1-y_j^2)^3}\partial_j\\&-y_j\frac{((\alpha+1)(1-y_jc/d)(1-y_jd/c)+(\beta+1)(1-cy_j)(1-y_j/c))}{(1-y_j^2)}\partial_j\\&+2\tau\frac{y_j^2(1-cy_j)(1-y_j/c)(1-y_jc/d)(1-y_jd/c)}{(1-y_j^2)^2}\tilde\Delta(y)^{-1}\left(\partial_j\tilde\Delta\right)(y)\partial_j\Bigg).\end{aligned}$$

Note that

$$m_\lambda(y+y^{-1})=\tilde m_\lambda(y)+\sum_{\mu<\lambda}c_{\lambda,\mu}\tilde m_\mu(y), \tag{5.14}$$

for certain $c_{\lambda,\mu}\in\mathbf{C}$, hence we have by (5.9) that

$$\begin{aligned}m_\lambda\left(\frac{1+c^2-c(y+y^{-1})}{(1-d)(1-c^2/d)}\right)=\left(\frac{-c}{(1-d)(1-c^2/d)}\right)^{|\lambda|}\tilde m_\lambda(y)+&\\+\sum_{\mu<\lambda}\tilde b_{\lambda,\mu}(c,d)\tilde m_\mu(y)&\end{aligned} \tag{5.15}$$

for certain $\tilde{b}_{\lambda,\mu}(c,d) \in \mathbf{C}$ with $\big((1-d)(1-c^2/d)\big)^{|\lambda|}\,\tilde{b}_{\lambda,\mu}(c,d)$ depending polynomially on c and d. It follows now from (4.3), with similar arguments as for the proof of (5.12), that

$$D^{\alpha,\beta,c,d}_{AW,J,\tau}\tilde{m}_\lambda = \sum_{\mu\le\lambda} c^{AW,J}_{\lambda,\mu}(\alpha,\beta,c,d;\tau)\tilde{m}_\mu$$

for $c,d \neq 0$, $c^2 \neq d$, $d \neq 1$ with constants $c^{AW,J}_{\lambda,\mu}$ depending polynomially on α,β and τ. The leading term $c^{AW,J}_{\lambda,\lambda}(\alpha,\beta,c,d;\tau) = a^J_\lambda(\alpha,\beta;\tau)$ is independent of c and d (a^J_λ given by (4.10)). The behaviour of the second order q-difference operators under the limit transitions is given by

Proposition 5.3 *Fix $\lambda \in P^+$.*
(1) *Fix $(a,b,c,d) \in \mathbf{C}^4$ and $q \in (0,1)$, then for all $\mu \le \lambda$ we have that*

$$\tilde{c}^{AW,B}_{\lambda,\mu}(t,\epsilon) := \left(\frac{\epsilon\,(cd)^{\frac{1}{2}}}{q^{\frac{1}{2}}}\right)^{|\lambda|-|\mu|} c^{AW}_{\lambda,\mu}\Big(\epsilon q^{\frac{1}{2}}a\,(d/c)^{\frac{1}{2}}, \epsilon^{-1}q^{\frac{1}{2}}\,(c/d)^{\frac{1}{2}}, \\ -\epsilon^{-1}q^{\frac{1}{2}}\,(d/c)^{\frac{1}{2}}, -\epsilon q^{\frac{1}{2}}b\,(c/d)^{\frac{1}{2}}; q,t\Big) \qquad (5.16)$$

depends polynomially on t and ϵ, and the zero order term with respect to the variable ϵ is $c^B_{\lambda,\mu}(a,b,c,d;q,t)$.
(2) *Fix $(a,b) \in \mathbf{C}^2$ and $q \in (0,1)$, then for all $\mu \le \lambda$ we have that*

$$\tilde{c}^{AW,L}_{\lambda,\mu}(t,\epsilon) := \left(\frac{\epsilon}{q^{\frac{1}{2}}}\right)^{|\lambda|-|\mu|} c^{AW}_{\lambda,\mu}\Big(\epsilon q^{\frac{1}{2}}b, \epsilon^{-1}q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}a; q,t\Big) \qquad (5.17)$$

depends polynomially on t and ϵ, and the zero order term with respect to the variable ϵ is $c^L_{\lambda,\mu}(a,b;q,t)$.
(3) *Fix $(a,b) \in \mathbf{C}^2$ and $q \in (0,1)$, then for all $\mu \le \lambda$ we have that $c^B_{\lambda,\mu}(b,a,1,d;q,t)$ depends polynomially on t and d, and the zero order term with respect to the variable d is $c^L_{\lambda,\mu}(a,b;q,t)$.*
(4) *Fix $(\alpha,\beta,\tau) \in V_J$ and $c,d \in \mathbf{C}$ such that $c,d \neq 0$, $c^2 \neq d$ and $d \neq 1$, then for all $\mu \le \lambda$ we have that*

$$\lim_{q\uparrow 1}\frac{c^{AW}_{\lambda,\mu}(c, q^{\alpha+1}/c, q^{\beta+1}d/c, c/d; q; q^\tau)}{(1-q)^2} = c^{AW,J}_{\lambda,\mu}(\alpha,\beta,c,d;\tau).$$

(5) *Fix $(\alpha,\beta,\tau) \in V_J$ and $c,d > 0$, then for all $\mu \le \lambda$ we have that*

$$\lim_{q\uparrow 1}\frac{c^B_{\lambda,\mu}(q^\alpha, q^\beta, c, d; q, q^\tau)}{(1-q)^2} = c^{B,J}_{\lambda,\mu}(\alpha,\beta,c,d;\tau).$$

(6) *Fix $(\alpha,\beta,\tau) \in V_J$, then for all $\mu \le \lambda$ we have that*

$$\lim_{q\uparrow 1}\frac{c^L_{\lambda,\mu}(q^\alpha, q^\beta; q, q^\tau)}{(1-q)^2} = c^J_{\lambda,\mu}(\alpha,\beta;\tau).$$

Proof: We first prove **(2)**. Fix $(a,b) \in \mathbf{C}^2$. Proposition 4.1 implies that $\tilde{c}^{AW,L}_{\lambda,\mu}(t,\epsilon)$ depends polynomially on t, and Laurent polynomially on ϵ. So it is sufficient to prove that for arbitrary fixed $t \in \mathbf{C}$,

$$\lim_{\epsilon\to 0} \tilde{c}^{AW,L}_{\lambda,\mu}(t,\epsilon) = c^L_{\lambda,\mu}(t).$$

So fix $t \in \mathbf{C}$. Proposition 4.1 gives for fixed $a,b,t \in \mathbf{C}$ and $0 \neq \epsilon \in \mathbf{R}$ that

$$\left(\frac{\epsilon}{q^{\frac{1}{2}}}\right)^{|\lambda|} \left(D^{(\epsilon q^{1/2}b,\epsilon^{-1}q^{1/2},-q^{1/2},-q^{1/2}a)}_{AW,q,t} \tilde{m}_\lambda\right) \left(\frac{q^{\frac{1}{2}}x}{\epsilon}\right) \tag{5.18}$$

is equal to

$$\sum_{\mu\leq\lambda} \tilde{c}^{AW,L}_{\lambda,\mu}(t,\epsilon) \left(\frac{\epsilon}{q^{\frac{1}{2}}}\right)^{|\mu|} \tilde{m}_\mu \left(\frac{q^{\frac{1}{2}}x}{\epsilon}\right). \tag{5.19}$$

Let for $\nu \in \tilde{P}$, $\tilde{c}_\nu(\epsilon)$ be the coefficient of $m_\nu(x) := \sum_{\rho\in\mathcal{S}\nu} x^\rho$ in (5.19). This makes sense, since $\tilde{m}_\mu\left(q^{\frac{1}{2}}x/\epsilon\right)$ is a symmetric Laurent polynomial for all $\mu \in P^+$. In fact, we have for $\mu \in P^+$ that

$$\left(\frac{\epsilon}{q^{\frac{1}{2}}}\right)^{|\mu|} \tilde{m}_\mu \left(\frac{q^{\frac{1}{2}}x}{\epsilon}\right) = m_\mu(x) + r(x;\epsilon) \tag{5.20}$$

with $r(x;\epsilon)$ a sum of monomials $m_\nu(x)$ with $\nu \in \tilde{P}$, $\nu_n < 0$ and $|\nu| < |\mu|$, with coefficient given by a polynomial expression in ϵ, homogeneous of degree $|\mu| - |\nu|$. Combining this with the following two limits (cf. [S1])

$$\lim_{\epsilon\to 0} \psi^{AW}_i \left(\frac{q^{\frac{1}{2}}x}{\epsilon}; \epsilon q^{\frac{1}{2}}b, \epsilon^{-1}q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}a; q,t\right) = \psi^L_i(x;a,b;q,t), \tag{5.21}$$

$$\lim_{\epsilon\to 0} \phi^{AW}_i \left(\frac{q^{\frac{1}{2}}x}{\epsilon}; \epsilon q^{\frac{1}{2}}b, \epsilon^{-1}q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}a; q,t\right) = \phi^L_i(x;a,b;q,t), \tag{5.22}$$

and with proposition 4.2, resp. formula (5.19) gives for $\nu \in \tilde{P}$ with $\nu_n < 0$ that

$$\lim_{\epsilon\to 0} \tilde{c}_\nu(\epsilon) = 0,$$

and for $\mu \in P^+$ that

$$\lim_{\epsilon\to 0} \tilde{c}_\mu(\epsilon) = \lim_{\epsilon\to 0} \tilde{c}^{AW,L}_{\lambda,\mu}(t,\epsilon) = c^L_{\lambda,\mu}(t).$$

This proves **(2)**.
(1) We have the following limits (cf. [S1]):

$$\lim_{\epsilon\to 0} \psi^{AW}_i \Bigg(\frac{q^{\frac{1}{2}}x}{\epsilon(cd)^{\frac{1}{2}}}; \epsilon q^{\frac{1}{2}}a(d/c)^{\frac{1}{2}}, \epsilon^{-1}q^{\frac{1}{2}}(c/d)^{\frac{1}{2}}, -\epsilon^{-1}q^{\frac{1}{2}}(d/c)^{\frac{1}{2}},$$
$$-\epsilon q^{\frac{1}{2}}b(c/d)^{\frac{1}{2}}; q,t\Bigg) = \psi^B_i(x;a,b,c,d;q,t) \tag{5.23}$$

$$\lim_{\epsilon\to 0}\phi_i^{AW}\left(\frac{q^{\frac12}x}{\epsilon(cd)^{\frac12}};\epsilon q^{\frac12}a(d/c)^{\frac12},\epsilon^{-1}q^{\frac12}(c/d)^{\frac12},-\epsilon^{-1}q^{\frac12}(d/c)^{\frac12},\right.$$
$$\left.-\epsilon q^{\frac12}b(c/d)^{\frac12};q,t\right)=\phi_i^B(x;a,b,c,d;q,t). \qquad (5.24)$$

The proof is now completely analogous to the proof of **(1)**.
(3) Follows directly from proposition 4.2 and formula (4.7).
(4) An arbitrary second order q-difference operator

$$D:=\sum_{i=1}^{n}\left(\pi_i^-(x)(T_{q,i}-\mathrm{Id})+\pi_i^+(x)(T_{q^{-1},i}-\mathrm{Id})\right)$$

can be rewritten in the following way

$$(Df)(x)=\sum_{i=1}^{n}\left(A_i(x)\left(T_{q^{-1},i}\left(\left(D_q^{i,-}\right)^2f\right)\right)(x)+B_i(x)\left(T_{q^{-1},i}\left(D_q^{i,-}f\right)\right)(x)\right), \qquad (5.25)$$

with $D_q^{i,-}$ the backward partial q-derivative in direction i given by

$$\left(D_q^{i,-}f\right)(x):=\frac{(f-T_{q,i}f)(x)}{(1-q)x_i}$$

and

$$A_i(x)=q^{-1}(1-q)^2x_i^2\pi_i^-(x),$$
$$B_i(x)=q^{-1}(1-q)x_i\left(\pi_i^+(x)-q\pi_i^-(x)\right).$$

Fix $(\alpha,\beta,\tau)\in V_J$ and $c,d\in\mathbf{C}$ such that $c,d\neq 0$, $c^2\neq d$ and $d\neq 1$. Rewrite D_{AW} in the form (5.25), so let

$$\pi_i^-(x)=\psi_i^{AW}(x)=\frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{(1-x_i^2)(1-qx_i^2)}t^{n-1}\tilde\Delta(x)^{-1}\left(T_{t,i}\tilde\Delta\right)(x),$$

$$\pi_i^+(x)=\phi_i^{AW}(x)=\frac{(a-x_i)(b-x_i)(c-x_i)(d-x_i)}{(1-x_i^2)(q-x_i^2)}t^{n-1}\tilde\Delta(x)^{-1}\left(T_{t^{-1},i}\tilde\Delta\right)(x),$$

with $\tilde\Delta(x)$ given by (5.13). It is immediate that

$$\lim_{q\uparrow 1}\frac{A_i(x;c,q^{\alpha+1}/c,q^{\beta+1}d/c,c/d;q,q^\tau)}{(1-q)^2}=\frac{x_i^2(1-cx_i)(1-x_i/c)(1-x_id/c)(1-x_ic/d)}{(1-x_i^2)^2}.$$

To evaluate the limit for B_i, we need the following remark. Let z be a complex variable and fix $q\in(0,1)$, then define for $u,v\in\mathbf{R}$, $D_{u,v}^q$ by

$$\left(D_{u,v}^qf\right)(z):=\frac{\left((T_{q^{-u}}-T_{q^v})f\right)(z)}{(1-q)z},$$

with $(T_s f)(z) := f(sz)$. Then $D_{u,v}$ maps $\mathbf{C}[z]$ into $\mathbf{C}[z]$, resp. $\mathbf{C}[z, z^{-1}]$ into $\mathbf{C}[z, z^{-1}]$, and

$$\lim_{q \uparrow 1} \left(D^q_{u,v} f\right)(z) = (u+v)\frac{df}{dz}(z) \quad \forall f \in \mathbf{C}[z, z^{-1}].$$

Note that $D^q_{0,1}$ is the backward partial q-derivative in the variable z. In particular, we have that

$$\lim_{q \uparrow 1} \frac{\left(T_{q^{-\tau},j}\tilde{\Delta} - T_{q^{\tau},j}\tilde{\Delta}\right)(x)}{(1-q)x_j} = 2\tau\left(\partial_j \tilde{\Delta}\right)(x).$$

A straightforward calculation gives then that

$$\begin{aligned}\lim_{q \uparrow 1} \frac{B_i(x; c, q^{\alpha+1}/c, q^{\beta+1}d/c, c/d; q, q^\tau)}{(1-q)^2} &= 2\frac{x_i(1-cx_i)(1-x_i/c)(1-x_ic/d)(1-x_id/c)}{(1-x_i^2)^3} \\ &\quad - x_i \frac{((\alpha+1)(1-x_ic/d)(1-x_id/c) + (\beta+1)(1-cx_i)(1-x_i/c))}{(1-x_i^2)} \\ &\quad + 2\tau\frac{x_i^2(1-cx_i)(1-x_i/c)(1-x_ic/d)(1-x_id/c)}{(1-x_i^2)^2}\tilde{\Delta}(x)^{-1}\left(\partial_i\tilde{\Delta}\right)(x).\end{aligned}$$

Hence we have

$$\lim_{q \uparrow 1} \frac{\left(D^{c, q^{\alpha+1}/c, q^{\beta+1}d/c, c/d}_{AW, q, q^\tau} f\right)(x)}{(1-q)^2} = \left(D^{\alpha,\beta,c,d}_{AW,J,\tau} f\right)(x) \quad \forall f \in A^W.$$

Consequently, the coefficients satisfy

$$\lim_{q \uparrow 1} \frac{c^{AW}_{\lambda,\mu}(c, q^{\alpha+1}/c, q^{\beta+1}d/c, c/d; q, q^\tau)}{(1-q)^2} = c^{AW,J}_{\lambda,\mu}(\alpha, \beta, c, d; \tau)$$

for all $\mu \leq \lambda$.

(5) Fix $(\alpha, \beta, \tau) \in V_J$ and $c, d > 0$. Rewrite the second order q-difference operator D_B in the form (5.25). The A_i's and B_i's are then given by the following expressions:

$$A_i(x) = (1-q)^2 q^{-2}(qax_i - c)(qbx_i + d)\, t^{n-1}\Delta(x)^{-1}(T_{t,i}\Delta)(x),$$

$$\begin{aligned}B_i(x) = \frac{(1-q)}{q} t^{n-1} \Bigg(&\left(x_i + (d-c) - \frac{cd}{x_i}\right)\Delta(x)^{-1}\left(T_{t^{-1},i}\Delta\right)(x) \\ &- \left(q^2abx_i + (qad - qbc) - \frac{cd}{x_i}\right)\Delta(x)^{-1}(T_{t,i}\Delta)(x)\Bigg). \qquad (5.26)\end{aligned}$$

Similar calculations as for **(4)** gives then that

$$\lim_{q\uparrow 1}\frac{\left(D^{q^\alpha,q^\beta,c,d}_{B,q,q^\tau}f\right)(x)}{(1-q)^2}=\left(D^{\alpha,\beta,c,d}_{B,J,\tau}f\right)(x)\quad \forall f\in\mathcal{A}^{\mathcal{S}},$$

hence the coefficients $c^B_{\lambda,\mu}(q^\alpha,q^\beta,c,d;q,q^\tau)$ $(\mu\le\lambda)$ satisfy

$$\lim_{q\uparrow 1}\frac{c^B_{\lambda,\mu}(q^\alpha,q^\beta,c,d;q,q^\tau)}{(1-q)^2}=c^{B,J}_{\lambda,\mu}(\alpha,\beta,c,d;\tau).$$

(6) This is a special case of **(5)**. □

Next we will define the BC_n type orthogonal polynomials for more general parameter values. Fix $q\in(0,1)$. Let X denote AW,B or L. Denote $m^X_\lambda:=\tilde{m}_\lambda$ for $X=AW$ resp. $m^X_\lambda:=m_\lambda$ for $X=B$ and L.

Lemma 5.4 *Fix $(a,b,c,d)\in V_X$ (if $X=AW$) resp. $(a,b,c,d)\in V^q_X$ (if $X=B$) resp. $(a,b)\in V^q_X$ (if $X=L$). Let $\mu<\lambda$, then there exists a polynomial $p_{\lambda,\mu}\left(\{x_{\nu,\rho}\}_{\rho\le\nu\le\lambda}\right)$ (independent of X) and homogeneous of total degree $\#\{\nu\in P^+\,/\,\nu<\lambda\}$ such that the rational function in t given by*

$$\tilde{d}^X_{\lambda,\mu}(t):=\frac{p_{\lambda,\mu}\left(\{c^X_{\nu,\rho}(t)\}_{\rho\le\nu\le\lambda}\right)}{\prod_{\nu<\lambda}\left(c^X_{\lambda,\lambda}(t)-c^X_{\nu,\nu}(t)\right)}\tag{5.27}$$

is a rational extension of the function $t\mapsto d^X_{\lambda,\mu}(t):(0,1)\to\mathbf{C}$, where $d^X_{\lambda,\mu}(t)$ is the coefficient of m^X_μ in the expansion of $P^X_\lambda(t)$ with respect to the basis $\{m^X_\nu\,/\,\nu\in P^+\}$ (cf. definition 3.2, 3.3 resp. 3.4).

Proof: For the given values of a,b,c,d we have that $c^X_{\mu,\mu}(t)=a^X_\mu(t)\neq a^X_\lambda(t)=c^X_{\lambda,\lambda}(t)$ as polynomials in t if $\lambda\neq\mu$, so $\tilde{d}^X_{\lambda,\mu}(t)$ is a well defined rational function in t. In view of proposition 2.5**(a)** and (2.6), there exists a polynomial $p_{\lambda,\mu}$ such that (5.27) is the rational extension of $d^X_{\lambda,\mu}(t)$. The polynomial $p_{\lambda,\mu}$ does not depend on X and can be chosen homogeneous of total degree $\#\{\nu\in P^+\,/\,\nu<\lambda\}$ in view of remark 2.6. □

Fix $(a,b,c,d)\in\mathbf{C}^4$ (if $X=AW$ or B) or $(a,b)\in\mathbf{C}^2$ (if $X=L$) such that $abcd\notin\{1,q^{-1},q^{-2},\ldots\}$ (if $X=AW$) or $ab\notin\{q^{-2},q^{-3},\ldots\}$ (if $X=B$ or L). Then, as observed in paragraph 4, $c^X_{\lambda,\lambda}(t)=a^X_\lambda(t)\neq a^X_\mu(t)=c^X_{\mu,\mu}(t)$ as polynomials in t if $\lambda\neq\mu$. Define $\tilde{d}^X_{\lambda,\mu}$ for these values of a,b,c,d by (5.27), and define

$$\tilde{P}^X_\lambda(x;t):=m^X_\lambda(x)+\sum_{\mu<\lambda}\tilde{d}^X_{\lambda,\mu}(t)m^X_\mu(x)\tag{5.28}$$

for $t\in\cap_{\mu<\lambda}\mathrm{dom}\left(\tilde{d}^X_{\lambda,\mu}\right)$. As a consequence of lemma 5.4, proposition 2.5 and remark 2.7 we have:

Corollary 5.5 *Keep the same assumptions on a,b,c,d as in lemma 5.4. Then $(0,1)\subset dom\left(\tilde{d}^X_{\lambda,\mu}\right)$ for all $\mu<\lambda$ and $\tilde{P}^X_\lambda(\,.\,;t)=P^X_\lambda(\,.\,;t)$ for all $t\in(0,1)$.*

Hence for each $\lambda\in P^+$, the polynomial $\tilde{P}^X_\lambda$ is a well defined extension of the polynomial P^X_λ to a larger set of parameters (a,b,c,d,t). We will write $P^X_\lambda(\,.\,;t)$ instead of $\tilde{P}^X_\lambda(\,.\,;t)$. For the limit transitions (theorem 5.1), we only need the extended definition of Askey-Wilson polynomials.

Remark 5.6 *According to proposition 2.1 and 2.5, the BC_n type Askey-Wilson polynomials for general parameter values have the following properties:*

Fix $\lambda\in P^+$, and let $a,b,c,d\in\mathbf{C}$ and $t\in\mathbf{C}$ such that $a^{AW}_\lambda(a,b,c,d;q,t)\neq a^{AW}_\mu(a,b,c,d;q,t)$ for all $\mu<\lambda$, then P^{AW}_λ is an eigenfunction of D_{AW} with eigenvalue $a^{AW}_\lambda(a,b,c,d;q,t)$. Furthermore, if one can extend the inner product $\langle\,.\,,\,.\,\rangle_{AW,t}$ for more general values of (a,b,c,d) ($abcd\notin\{1,q^{-1},q^{-2},\ldots\}$), such that $\langle f,g\rangle_{AW,t}$ is continuous in t for $t\in(0,1)$ for all $f,g\in A^W$, and such that D_{AW} is selfadjoint with respect to $\langle\,.\,,\,.\,\rangle_{AW}$, then the corresponding orthogonal polynomials (Definition 3.2) are exactly the polynomials $P^{AW}_\lambda(a,b,c,d;q,t)$ as defined by (5.28), for all $t\in(0,1)$ (cf. remark 2.7).

Proof of theorem 5.1
We first prove **(2)**.
(2) Let $(a,b)\in V^q_L$ and $t\in J_\lambda(a,b)$. Note that

$$a^{AW}_\lambda(\epsilon q^{\frac{1}{2}}b,\epsilon^{-1}q^{\frac{1}{2}},-q^{\frac{1}{2}},-q^{\frac{1}{2}}a;q,t)=a^{B,L}_\lambda(a,b;q,t). \tag{5.29}$$

Hence we have, in view of (5.19), that

$$\begin{aligned}\left(\frac{\epsilon}{q^{\frac{1}{2}}}\right)^{|\lambda|}P^{AW}_\lambda\left(\frac{q^{\frac{1}{2}}x}{\epsilon};\epsilon q^{\frac{1}{2}}b,\epsilon^{-1}q^{\frac{1}{2}},-q^{\frac{1}{2}},-q^{\frac{1}{2}}a;q,t\right)&=\left(\frac{\epsilon}{q^{\frac{1}{2}}}\right)^{|\lambda|}\tilde{m}_\lambda\left(\frac{q^{\frac{1}{2}}x}{\epsilon}\right)\\&+\sum_{\mu<\lambda}\tilde{d}^{AW,L}_{\lambda,\mu}(t,\epsilon)\left(\frac{\epsilon}{q^{\frac{1}{2}}}\right)^{|\mu|}\tilde{m}_\mu\left(\frac{q^{\frac{1}{2}}x}{\epsilon}\right),\end{aligned} \tag{5.30}$$

where

$$\tilde{d}^{AW,L}_{\lambda,\mu}(t,\epsilon)=\frac{p_{\lambda,\mu}\left(\{\tilde{c}^{AW,L}_{\nu,\rho}(t,\epsilon)\}_{\rho\leq\nu\leq\lambda}\right)}{\prod_{\nu<\lambda}\left(a^{B,L}_\lambda(a,b;q,t)-a^{B,L}_\nu(a,b;q,t)\right)}. \tag{5.31}$$

The denominator on the right hand side is non-zero since $t\in J_\lambda(a,b)$. Now apply proposition 5.3**(2)** and (5.20).
(1) Note that

$$\begin{aligned}a^{AW}_\lambda&\left(\epsilon q^{\frac{1}{2}}a(d/c)^{\frac{1}{2}},\epsilon^{-1}q^{\frac{1}{2}}(c/d)^{\frac{1}{2}},-\epsilon^{-1}q^{\frac{1}{2}}(d/c)^{\frac{1}{2}},\right.\\&\left.-\epsilon q^{\frac{1}{2}}b(c/d)^{\frac{1}{2}};q,t\right)=a^{B,L}_\lambda(a,b;q,t).\end{aligned} \tag{5.32}$$

Similar arguments as for **(2)** give then the desired result.
(3) Same arguments as for **(1)** can be applied with d playing the role of ϵ, since $a_\lambda^{B,L}(a,b;q,t)$ is independent of d and symmetric in a and b.
(4) Fix $(\alpha,\beta,\tau)\in V_J$ and $c,d\in \mathbf{C}$ such that $c,d\neq 0$, $c^2\neq d$ and $d\neq 1$. Denote the right hand side of formula (5.5) by $\hat{P}_\lambda^{AW,J}(x;\alpha,\beta,c,d;\tau)$, then

$$\hat{P}_\lambda^{AW,J}(x;\alpha,\beta,c,d;\tau)=\tilde{m}_\lambda(x)+\sum_{\mu<\lambda}\hat{d}_{\lambda,\mu}^{AW,J}(\alpha,\beta,c,d;\tau)\tilde{m}_\mu(x)$$

for certain constants $\hat{d}_{\lambda,\mu}^{AW,J}(\alpha,\beta,c,d;\tau)\in\mathbf{C}$ in view of (5.15). $\hat{P}_\lambda^{AW,J}(x;\alpha,\beta,c,d;\tau)$ is an eigenfunction of $D_{AW,J,\tau}^{\alpha,\beta,c,d}$ with eigenvalue $a_\lambda^J(\alpha,\beta;\tau)$, because $P_\lambda^J(x;\alpha,\beta;\tau)$ is an eigenfunction of $D_{J,\tau}^{\alpha,\beta}$ with eigenvalue $a_\lambda^J(\alpha,\beta;\tau)$. Hence

$$\hat{P}_\lambda^{AW,J}(\,.\,;\alpha,\beta,c,d;\tau)=\left(\prod_{\mu<\lambda}\frac{D_{AW,J,\tau}^{\alpha,\beta,c,d}-a_\mu^J(\alpha,\beta;\tau)}{a_\lambda^J(\alpha,\beta;\tau)-a_\mu^J(\alpha,\beta;\tau)}\right)\tilde{m}_\lambda$$

in view of proposition 2.1 and proposition 4.8. So we have for $\mu<\lambda$ that

$$\hat{d}_{\lambda,\mu}^{AW,J}(\alpha,\beta,c,d)=\frac{p_{\lambda,\mu}\left(\{c_{\nu,\rho}^{AW,J}(\alpha,\beta,c,d;\tau)\}_{\rho\leq\nu\leq\lambda}\right)}{\prod_{\nu<\lambda}\left(a_\lambda^J(\alpha,\beta;\tau)-a_\nu^J(\alpha,\beta;\tau)\right)}$$

by proposition 2.5**(a)** and (2.6), where $p_{\lambda,\mu}$ is the same polynomial as in (5.27). Note that

$$a_\lambda^{AW}(c,q^{\alpha+1}/c,q^{\beta+1}d/c,c/d;q,q^\tau)=a_\lambda^{B,L}(q^\alpha,q^\beta;q,q^\tau),$$

and $a_\lambda^{B,L}(q^\alpha,q^\beta;q,q^\tau)>a_\mu^{B,L}(q^\alpha,q^\beta;q,q^\tau)$ for all $\lambda>\mu$ and all $q\in(0,1)$ (proposition 4.7). Since $p_{\lambda,\mu}$ is homogeneous of total degree $\#\{\nu\in P^+\,/\,\nu<\lambda\}$, we thus have by lemma 5.4 that

$$d_{\lambda,\mu}^{AW}(c,\frac{q^{\alpha+1}}{c},\frac{q^{\beta+1}d}{c},\frac{c}{d};q,q^\tau)=\frac{p_{\lambda,\mu}\left(\{\tilde{c}_{\nu,\rho}^{AW}(c,q^{\alpha+1}/c,q^{\beta+1}d/c,c/d;q,q^\tau)\}_{\rho\leq\nu\leq\lambda}\right)}{\prod_{\nu<\lambda}\left(\tilde{a}_\lambda^{B,L}(q^\alpha,q^\beta;q,q^\tau)-\tilde{a}_\nu^{B,L}(q^\alpha,q^\beta;q,q^\tau)\right)}$$

with

$$\tilde{c}_{\nu,\rho}^{AW}(c,q^{\alpha+1}/c,q^{\beta+1}d/c,c/d;q,q^\tau)=\frac{c_{\nu,\rho}^{AW}(c,q^{\alpha+1}/c,q^{\beta+1}d/c,c/d;q,q^\tau)}{(1-q)^2},$$

$$\tilde{a}_\nu^{B,L}(q^\alpha,q^\beta;q,q^\tau)=\frac{a_\nu^{B,L}(q^\alpha,q^\beta;q,q^\tau)}{(1-q)^2}.$$

We have that

$$\lim_{q\uparrow 1}\tilde{a}_\nu^{B,L}(q^\alpha,q^\beta;q,q^\tau)=a_\nu^J(\alpha,\beta;\tau)$$

for all $\nu \in P^+$, so it follows from proposition 5.3**(4)** that

$$\lim_{q\uparrow 1} d^{AW}_{\lambda,\mu}(c, q^{\alpha+1}/c, q^{\beta+1}d/c, c/d; q, q^\tau) = \hat{d}^{AW,J}_{\lambda,\mu}(\alpha, \beta, c, d; \tau)$$

for all $\mu < \lambda$.
The proof of **(5)** and **(6)** can be given in a similar way. □

Remark 5.7 *For limit transition (5.6), the condition $c, d > 0$ can be weakened to $c + d \neq 0$ if one uses the extended definition of the big q-Jacobi polynomials. For subsets of the set of parameter values a, b, c, d, t for which we proved limit (5.5), a proof of (5.5) was already known by looking at the behaviour of the orthogonality measure when q tends to 1. See [M1] for a three parameter subset, and [vD], prop. 4.3 for a five parameter subset. For the limit transitions from big resp. little q-Jacobi polynomials to Jacobi polynomials ((5.6) resp. (5.7)), this technique can also be applied. See the end of section 6 (prop. 6.5) for more details.*

6 Some remarks about the limit transitions

We first discuss the possibilities to extend the limit transitions (5.2), (5.3) and (5.4) to the whole parameter domain. Fix $(a, b, c, d) \in V^q_X$ (if $X = B$) or $(a, b) \in V^q_X$ (if $X = L$) and fix $q \in (0, 1)$. Then

$$\tilde{d}^{AW,X}_{\lambda,\mu}(t, \epsilon) := \frac{p_{\lambda,\mu}\left(\{\tilde{c}^{AW,X}_{\nu,\rho}(t, \epsilon)\}_{\rho\leq\nu\leq\lambda}\right)}{\prod_{\nu<\lambda}\left(a^{B,L}_\lambda(a, b; q, t) - a^{B,L}_\nu(a, b; q, t)\right)} \tag{6.1}$$

for $X = B$ resp. L and $\mu < \lambda$ depend polynomially on ϵ and rationally on t. Since $a^{B,L}_\lambda(a, b; q, t)$ is independent of c and d, we have for $\mu < \lambda$ that

$$\tilde{d}^{B,L}_{\lambda,\mu}(a, b; q; t, d) := \frac{p_{\lambda,\mu}\left(\{c^B_{\nu,\rho}(b, a, 1, d; q, t)\}_{\rho\leq\nu\leq\lambda}\right)}{\prod_{\nu<\lambda}\left(a^{B,L}_\lambda(a, b; q, t) - a^{B,L}_\nu(a, b; q, t)\right)} \tag{6.2}$$

depends polynomially on d and rationally on t. Fix $\lambda \in P^+$. We have that limit transition (5.3) holds for $t \in (0, 1)$ if and only if for all $\mu < \lambda$ the following two conditions are satisfied:
(1) $t \in \mathrm{dom}\left(\tilde{d}^{AW,L}_{\lambda,\mu}(\,.\,, \epsilon)\right)$ for ϵ sufficiently small,
(2) $\lim_{\epsilon\to 0} \tilde{d}^{AW,L}_{\lambda,\mu}(t, \epsilon) = d^L_{\lambda,\mu}(t)$.
This follows from (5.20) and (5.30). Similar remarks hold for the limit transitions (5.2) and (5.4). We have the following lemma.

Lemma 6.1 *Fix $m \in \mathbf{N}_0$. Let J be a topological space, J_0 a dense subset of J and $f_0, \ldots, f_m : J_0 \to \mathbf{C}$ continuous functions. Define the function $f : J_0 \times \mathbf{C} \to \mathbf{C}$ by*

$$f(t, \epsilon) := \sum_{k=0}^{m} \epsilon^k f_k(t) \quad t \in J_0, \epsilon \in \mathbf{C}. \tag{6.3}$$

Suppose there exist $m+1$ different points $\{\epsilon_0, \ldots, \epsilon_m\}$ such that the functions

$$t \mapsto f(t, \epsilon_i) : J_0 \to \mathbf{C} \quad (i = 0, \ldots, m)$$

can be extended to continuous functions from J to $\mathbf{C}$. Then the functions $f_0, \ldots, f_m$ can be extended to continuous functions from J to $\mathbf{C}$.

In particular we have that there is a unique extension of f to a continuous function from $J \times \mathbf{C}$ to $\mathbf{C}$ such that $f(t, 0) = f_0(t)$ for all $t \in J$ (cf. (6.3)).

Proof: We have for all $t \in J_0$ the matrix identity

$$\vec{f}(t) = A\vec{g}(t), \tag{6.4}$$

with $\vec{f}(t)$ resp. $\vec{g}(t)$ the column vector with i^{th} entry $f(t, \epsilon_i)$ resp. $f_i(t)$ ($i = 0, \ldots, m$) and A the matrix with (i, j)'th entry $(\epsilon_i)^j$ ($i, j = 0, \ldots, m$) (i the row index, j the column index). The lemma follows, since every entry of the column vector $\vec{f}(t)$ can be extended to a continuous function from J to $\mathbf{C}$ and A is invertible. $\square$

For fixed $\lambda \in P^+$ and fixed a and b, take $J = (0, 1)$, and $J_\lambda(a, b)$ for the dense subset of J ($J_\lambda(a, b)$ given by (5.1)), then in the next proposition we will apply lemma 6.1 on the functions (6.1) and (6.2), with m the highest degree of the functions as polynomials in ϵ resp. d. Note that in these situations, an algebraic proof can be given of lemma 6.1.

Proposition 6.2 *Fix $\lambda \in P^+$.*
(1) *Fix $(a, b, c, d) \in V_B^q$. Suppose that $\tilde{d}_{\lambda,\mu}^{AW,B}(t, \epsilon)$ satisfies the conditions of lemma 6.1 for all $\mu < \lambda$. Then $(0, 1) \subset dom\left(\tilde{d}_{\lambda,\mu}^{AW,B}(\,.\,; \epsilon)\right)$ for all ϵ and all $\mu < \lambda$, and limit transition (5.2) is valid for all $t \in (0, 1)$.*

(2) *Fix $(a, b) \in V_L^q$. Suppose that $\tilde{d}_{\lambda,\mu}^{AW,L}(t, \epsilon)$ satisfies the conditions of lemma 6.1 for all $\mu < \lambda$. Then $(0, 1) \subset dom\left(\tilde{d}_{\lambda,\mu}^{AW,L}(\,.\,; \epsilon)\right)$ for all ϵ and all $\mu < \lambda$, and limit transition (5.3) is valid for all $t \in (0, 1)$.*

(3) *Fix $(a, b) \in V_L^q$. Suppose that $\tilde{d}_{\lambda,\mu}^{B,L}(t, d)$ satisfies the conditions of lemma 6.1 for all $\mu < \lambda$. Then $(0, 1) \subset dom\left(\tilde{d}_{\lambda,\mu}^{B,L}(\,.\,; d)\right)$ for all d and all $\mu < \lambda$, and limit transition (5.4) is valid for all $t \in (0, 1)$.*

Proof: In view of lemma 6.1 and remark 2.7, it is sufficient to check that (5.2), (5.3) resp. (5.4) holds for $t \in J_\lambda(a,b)$, so theorem 5.1 gives the desired result. $\square$

As a corollary, we have

Theorem 6.3 *Let $t \in (0,1)$, $(a,b) \in V_L^q$ and $\lambda \in P^+$, then*

$$\lim_{d \downarrow 0} P_\lambda^B(x; b, a, 1, d; q, t) = P_\lambda^L(x; a, b; q, t). \tag{6.5}$$

Proof: Fix $(a,b) \in V_L^q$. Fix $d > 0$ such that $b > -1/dq$, then $(b,a,1,d) \in V_B^q$. Hence $\tilde{d}_{\lambda,\mu}^{B,L}(t,d) = d_{\lambda,\mu}^B(b,a,1,d;q,t)$ for all $t \in (0,1)$ and for all $\mu < \lambda$ (corollary 5.5). The big q-Jacobi polynomials

$$P_\lambda^B(x; b, a, 1, d; q, t) = m_\lambda(x) + \sum_{\mu < \lambda} d_{\lambda,\mu}^B(b,a,1,d;q,t) m_\mu(x) \quad (\lambda \in P^+)$$

are orthogonal with respect to the inner product $\langle\, .\, ,\, .\, \rangle_{B,q,t}^{b,a,1,d}$. Proposition 2.3, applied to this inner product, implies that $d_{\lambda,\mu}^B(b,a,1,d;q,t)$ depends continuously on t for $t \in (0,1)$. Hence proposition 6.2**(3)** can be applied. $\square$

Remark 6.4 *The proof of theorem 6.3 for arbitrary $t \in (0,1)$ makes essential use of the interpretation of the polynomials as orthogonal polynomials.*

In order to prove the limit transitions (5.2) and (5.3) for all $(a,b,c,d) \in V_B^q$ resp. $(a,b) \in V_L^q$ and all $t \in (0,1)$, an interpretation of BC_n type Askey-Wilson polynomials as orthogonal polynomials for more general parameter values is needed (cf. remark 5.6), so that the same argument as in theorem 6.3 can be applied.

In the one variable case, Askey-Wilson polynomials for more general values of (a,b,c,d) have an interpretation as orthogonal polynomials. The orthogonality domain consists then of a continuous part and a discrete part, the discrete part coming from residues. In the one variable case, Koornwinder showed that in the limits from Askey-Wilson polynomials to big resp. little q-Jacobi polynomials, the disrete part of the orthogonality domain of the Askey-Wilson polynomials blows up to the orthogonality domain of the big resp. little q-Jacobi polynomials, while the discrete part shrinks to $\{0\}$ (cf. [K2] p. 812).

Recently, the first author has written down the orthogonality measure for the multivariable Askey-Wilson polynomials with partly continuous, partly discrete measure and described in detail the limit transitions to big resp. little q-Jacobi polynomials for the case $t = q^k$ with $k \in \mathbf{N}$ (cf. [S2]).

Define an inner product $[\, .\, ,\, .\,]_{J,\tau}^{\alpha,\beta,c,d}$ on $\mathcal{A}^{\mathcal{S}}$ for $c,d > 0$, $(\alpha,\beta,\tau) \in V_J$ by

$$[f,g]_{J,\tau} := \frac{1}{n!} \int_{x_1=-d}^{c} \cdots \int_{x_n=-d}^{c} f(x)\overline{g(x)} w_J(x;\tau) dx, \quad f,g \in \mathcal{A}^{\mathcal{S}}, \tag{6.6}$$

with weight function $w_J(x;\alpha,\beta,c,d;\tau)$ given by

$$w_J(x;\tau) := \left(\prod_{i=1}^{n}(1-x_i/c)^{\alpha}(1+x_i/d)^{\beta}\right)|\Delta(x)|^{2\tau}.$$

The polynomials $\{P_\lambda^J\left(\frac{c-x}{c+d};\alpha,\beta;\tau\right) / \lambda \in P^+\}$ are orthogonal with respect to $[\,.\,,\,.\,]_{J,\tau}^{\alpha,\beta,c,d}$.

Proposition 6.5 *Fix $\lambda,\mu \in P^+$.*
(1) *Let $\alpha \in (0,\infty), \beta \in (0,\infty)$ and $\tau \in [1/2,\infty)$, then*

$$\lim_{q\uparrow 1}\langle m_\lambda, m_\mu\rangle_{L,q,q^\tau}^{q^\alpha,q^\beta} = \langle m_\lambda, m_\mu\rangle_{J,\tau}^{\alpha,\beta}.$$

(2) *Let $\alpha,\beta,\tau \in \mathbf{N}$ and $c,d>0$, then*

$$\lim_{q\uparrow 1}\langle m_\lambda, m_\mu\rangle_{B,q,q^\tau}^{q^\alpha,q^\beta,c,d} = [m_\lambda, m_\mu]_{J,\tau}^{\alpha,\beta,c,d}.$$

Proof: **(1)** Fix $m_1, m_2 \in \mathbf{R}$ such that $m_2 \geq m_1$ and $m_1+m_2 \geq 1$. Then

$$\lim_{q\uparrow 1}\frac{(q^{m_1}z;q)_\infty}{(q^{m_2}z;q)_\infty} = (1-z)^{m_2-m_1}$$

uniform for z on $\{z \in \mathbf{C} / |z| \leq 1\}$ (cf. [K4]). This implies for the function

$$g_{\lambda,\mu}(x;\alpha,\beta;\tau;q) := q^{\binom{n}{2}\tau} m_\lambda(x'(\tau))m_\mu(x'(\tau))v(x'(\tau);q^\alpha,q^\beta;q,q^\tau)$$

with $x'(\tau) = (x_1, q^\tau x_2, \ldots, q^{(n-1)\tau}x_n)$, and fixed $\alpha,\beta \in (0,\infty)$, $\tau \in [\frac{1}{2},\infty)$, that

$$\lim_{q\uparrow 1} g_{\lambda,\mu}(x;\alpha,\beta;\tau;q) = m_\lambda(x)m_\mu(x)v_J(x;\alpha,\beta;\tau) \tag{6.7}$$

uniformly on $V := \{(x_1,\ldots,x_n) / 1 \geq x_1 \geq \ldots \geq x_n \geq 0\}$. So we have:

$$\begin{aligned}
\lim_{q\uparrow 1}\langle m_\lambda, m_\mu\rangle_{L,q,q^\tau}^{q^\alpha,q^\beta} &= \lim_{q\uparrow 1}\int_{x_1=0}^{1}\int_{x_2=0}^{x_1}\cdots\int_{x_n=0}^{x_{n-1}} g_{\lambda,\mu}(x;\alpha,\beta;\tau;q)d_qx \\
&= \frac{1}{n!}\lim_{q\uparrow 1}\int_{x_1=0}^{1}\cdots\int_{x_n=0}^{1} m_\lambda(x)m_\mu(x)v_J(x;\alpha,\beta;\tau)d_qx \\
&= \frac{1}{n!}\int_{x_1=0}^{1}\cdots\int_{x_n=0}^{1} m_\lambda(x)m_\mu(x)v_J(x;\alpha,\beta;\tau)dx \\
&= \langle m_\lambda, m_\mu\rangle_{J,\tau}^{\alpha,\beta}.
\end{aligned}$$

For the first equality we used that

$$\int_0^\gamma h(u)d_qu := \gamma\int_0^1 h(\gamma u)d_qu \quad (\gamma \neq 0),$$

for the second equality we used the uniform convergence of the integrand (formula (6.7)) and that v_J is symmetric, and for the third equality we used that v_J is continuous on $[0,1]^n$ and the fact that (3.4) is also valid for multidimensional Jackson integrals with continuous integrands.

(2) For $\tau = k \in \mathbf{N}$, the inner product $\langle\,.\,,\,.\,\rangle_{B,q,q^k}^{a,b,c,d}$ simplifies to

$$\langle f,g\rangle_B = \frac{1}{n!}\int_{x_1=-d}^{c}\cdots\int_{x_n=-d}^{c} f(x)\overline{g(x)}w(x)d_qx,$$

with weight function

$$w(x;a,b,c,d;q,q^k) = \prod_{i=1}^{n}\frac{(qx_i/c,-qx_i/d;q)_\infty}{(qax_i/c,-qbx_i/d;q)_\infty}\Delta_k(x),$$

and

$$\Delta_k(x) := (-1)^{k\binom{n}{2}}\,q^{-\binom{k}{2}\binom{n}{2}}\prod_{l=0}^{k-1}\prod_{i\neq j}\left(x_i - q^l x_j\right)$$

(cf. [S1]). We have that

$$\lim_{q\uparrow 1} w(x;q^\alpha,q^\beta,c,d;q,q^\tau) = w_J(x;\alpha,\beta,c,d;\tau)$$

uniformly for $x \in [-d,c]^n$, if $c,d > 0$ and $\alpha,\beta,\tau \in \mathbf{N}$. Similar arguments as in **(1)** gives the desired result. $\square$

With the help of this proposition, it is also possible to prove that the BC_n type big and little q-Jacobi polynomials are q-analogues of the generalized Jacobi polynomials, with techniques very similar to the techniques used by Macdonald to investigate the limit q tends to 1 for his orthogonal polynomials associated with general root systems (cf. [M1]). See also [vD], prop. 4.3, where these techniques were used for the limit transition from Askey-Wilson polynomials to Jacobi polynomials (formula (5.5)).

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