Proceedings of
the 16th Annual
(International) Conference of the

Society for Special Functions and their Applications

SSFA

EDITOR
A.K. Agarwal

Bikaner (India)
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Preface

This Conference Proceedings volume contains the written versions of most of the contributions presented during the 16th Annual 'International Conference on Special Functions and Applications' (ICSFA-2017), and symposium on 'Application of Mathematical Sciences in Engineering Problems' organized by the Department of Mathematics, Government College of Engineering and Technology, Bikaner, Rajasthan (Presently upgraded to University College of Bikaner Technical University, Bikaner) during November 02--04, 2017. The Conference provided a platform for discussing recent developments in a wide variety of topics including Special functions, Lie Theory, Orthogonal Polynomials, q-series, Fractional Calculus, Number Theory, Mathematical Physics, Combinatorics and Statistics. Apart from this, there was a symposium on "Application of Mathematical Sciences in the Engineering Problems". The Chief Guest of the inaugural function was Prof. Tom Koornwinder, University of Amsterdam, Amsterdam, Netherlands, the Guest of Honour was Prof. D. Shringhi, Member Secretary and Principal, Govt. Engineering College Bikaner and Chairperson was Dr. S.K. Bansal, Principal, C.E.T. Bikaner. The felicitation ceremony of Prof. N. K. Thakare, Patron, SSFA was also the part of the inaugural ceremony. The Special Guest of the valedictory function was Prof. Tibor K. Pogany, University of Rijeka, Rijeka, Croatia and Prof. M.A. Pathan, Ex-Chairman, Aligarh Muslim University, Alligarh, the Guest of Honour was Prof. S. Ponnusamy, ISI, Chennai Centre and IIT Madras, Chennai.

This volume includes the lectures delivered during the 'International Conference on Special Functions and their Applications' (ICSFA-2017). We accomplish this task by collecting research papers which are concerned with rather diversified areas which will give good insight into new researches in the new century. These peer-refereed articles showcase the latest developments and trends in classical special functions, highlighting the cross-fertilization of new techniques and ideas with the existing ones. The sixteenth volume of the proceedings of this conference is a very clear impression of the present state of the art in classical and modern aspects of special functions and allied topics. These were the issues addressed by the papers presented at the conference. The level of interest in the subject matter of the conference was maintained from previous events and over 100 suitable papers were submitted for presentation at the conference. This required the programme to be organised in two parallel sessions, each on a specific theme, to provide each paper with sufficient time for presentation and to accommodate all of them within the overall time allocated.

Special functions and their applications have been one of the major fields of research in India in the last 100 years, since they constitute a reasonably big class of mathematicians including S. Ramanujan working in analysis, number theory, partition theory, combinatorics and related areas. There are very extensive applications of special functions. SSFA was established in 1997 as an independent association of Academics and Researchers. Its mission is to act as a forum where Academics and Researchers from all over the world can meet in order to exchange ideas on their research, and to discuss future developments in their disciplines. SSFA is the global honor society of mathematicians, statisticians, physicists and engineers that recognizes scientific achievements in the field of special functions and their applications.

The objects of the Society are to promote research in mathematics and mathematical sciences in general and special functions in particular. The society has been trying hard for the development of teaching and research in special functions and their applications. The society is dedicated to the advancement of science and engineering through outstanding programs and services delivered in a collegial and supportive environment. Each year, society members, experts and bright research scholars gather to interact with one another at the SSFA-annual conferences held at different places in India.

All those who are directly or even remotely concerned with special functions, number theory, partition theory and their applications are aware of the significant contributions made by Professor R.P. Agarwal. SSFA would not have been successful had it not been for the efforts of its members and leaders like Professor R.P. Agarwal, founder member and Patron of the

SSFA who initiated planning activities of the society all over the country. We lost a top researcher and one of our brightest jewels in the fields of special functions, number theory and partition theory with the passing away on February 9, 2008, of Professor R.P. Agarwal. To perpetuate the initiatives and efforts taken up by Professor Agarwal, several of his students, colleagues and admirers thought it fit to organize a lecture in his memory. In his honor, society is organizing R.P. Agarwal memorial lecture every year. During the tenth annual international conference at Jodhpur in 2011, the first R.P. Agarwal memorial lecture was delivered by Professor George E. Andrews, U.S.A. The second R.P. Agarwal memorial lecture was delivered by Professor H.L. Manocha from U.S.A., during the eleventh international conference at Surat in 2012. For the 2013, R.P. Agarwal memorial lecture Professor A.M. Mathai, Emeritus Professor of Mathematics and Statistics, McGill University, Canada and Director, Centre for Mathematical and Statistical Sciences, Peechi, Kerala, India was the lecturer during the twelfth international conference held at Malviya National Institute of Technology, Jaipur, Rajasthan. The 14th annual International Conference on Special Functions and their Applications (ICSFA-2015) was organized by the Amity University,NOIDA,U.P. The fourth R.P. Agarwal memorial lecture was delivered by Prof. S. Kanemitsu from Kindai University, Japan. He is a well known Japanese mathematician and educator. His achievements include research in Symmetry of special functions.

The 15th annual International Conference on Special Functions and their Applications (ICSFA-2016) was organized by the Department of Applied Science and Humanities, Faculty of Engineering and technology, Jamia Millia Islamia, New Delhi from Sept 9 to 11, 2016.During this conference the fifth R.P. Agarwal memorial lecture was delivered by Prof. Michel Waldschmidt, University of Paris, France. Chief guest of the inaugural function was also Prof. Michel Waldschmidt and Guests of Honor were Prof. S. Kanemitsu from Kindai University, Japan and Prof. A.M. Mathai from Mc.Gill University, Canada. Waldschmidt was educated at Lyce Henri Poincar and the University of Nancy until 1968. In 1972 he defended his thesis, titled Independance algebrique de nombres transcendants (Algebraic independence of transcendental numbers) and directed by Jean

Fresnel, the University of Bordeaux, where he was research associate of CNRS in 1971-2. He was then a lecturer at Paris-Sud 11 University in 1972-3, then a lecturer at the University of Paris VI (Pierre et Marie Curie), where he is Professor since 1973. Waldschmidt was also a visiting professor at various places including the cole normale suprieure. He is a member of the Institut de mathmatiques de Jussieu. Michel Waldschmidt is an expert in the theory of transcendental numbers and diophantine approximations. He was awarded the Albert Chtelet Prize in 1974, the CNRS Silver Medal in 1978, the Marquet Prize of Academy of Sciences in 1980 and the Special Award of the Hardy-Ramanujan Society in 1986. From 2001 to 2004 he was president of the Mathematical Society of France. He is a member of several mathematical societies, including the EMS, the AMS and Ramanujan Mathematical Society.

The 16th annual International Conference on Special Functions and their Applications (ICSFA-2017) was organized by the Department of Mathematics, Government College of Engineering and Technology,Bikaner,Rajasthan. During this conference the sixth R.P. Agarwal memorial lecture was delivered by Prof. Tom Koornwinder, University of Amsterdam, Amsterdam, Netherlands.The title of his lecture was Dual addition formula for continuous q-ultraspherical polynomials. Tom Koornwinder (b. 1943 in Rotterdam, The Netherlands)Professor Emeritus in the Kortewegde Vries Institute for Mathematics at the University of Amsterdam, The Netherlands. During 1968-1992 he was a Researcher in the Centrum Wiskunde Informatica (CWI), Amsterdam. Koornwinder has published numerous papers on special functions, harmonic analysis, Lie groups, quantum groups, computer algebra, and their interrelations, including an interpretation of Askey-Wilson polynomials on quantum SU(2), and a five-parameter extension (the Macdonald Koornwinder polynomials) of Macdonalds polynomials for root systems BC. Books for which he has been editor or coeditor include Special Functions: Group Theoretical Aspects and Applications (with R. A. Askey and W. Schempp), published by Reidel in 1984, and Wavelets: mathematical preliminaries, published by World Scientific in 1993. Koornwinder has been active as an officer in the SIAM Activity Group on

Special Functions and Orthogonal Polynomials. Currently he is on the editorial board for Constructive Approximation, and is editor for the volume on Multivariable Special Functions in the ongoing Askey-Bateman book project.

The contributors to the proceedings of the conference touch on several topics of special functions and their applications. Their articles covers a number of lectures on a variety of areas and topics ranging from Dual addition formula for continuous q-ultraspherical polynomials by T. Koorwinder, Y-Bessel sampling series of $L(2) stochastic processes by Tibor K. Pogany, On Generalized Kratzel Integrals by A.M. Mathai, Generalized Rathie-Swamee distributions and applications by Prabhata K. Swamee, Ronald T. Nojosa and Pushpa N. Rathie, Advanced Special Functions Associated With Lie And Witt Algebras by M.A. Pathan, On Bohr's inequality and Beyond by S. Ponnusamy and so on.

We would like to thank the contributors of the articles and referees for their prompt action in refereeing process. We are also thankful to all members of the National and International Advisory Committee and all members of the local organizing committee whose enthusiastic involvement contributed greatly to the grand success of the international conference. All participants were impressed by the quality of hosting institution, the great hospitality, and for the efforts to make this conference into a success. Also, for this conference, we acknowledge the financial support from TEQIP-II and Government College of Engineering and Technology, Bikaner, Rajasthan (Presently upgraded to University College of Bikaner Technical University, Bikaner.)

All the articles have been Latexed. In addition to this, Dr. N.U. Khan has taken care of all the tedious detailed drudgery that occur in process of preparing a proceedings manuscript. We thank him for his generous help.

October, 2018

A.K. Agarwal
M.A. Pathan
R.K.Parmar

A brief report of International Conference on Special Functions and their Applications (ICSFA - 2017)

The 16th Annual International Conference on Special Functions and their Applications (ICSFA-2017), and symposium on application of Mathematical Sciences in Engineering problems was organized by the Department of Mathematics, Govt. College of Engineering and Technology during November 02-04, 2017.

The inaugural ceremony was presided over by Prof. A. K. Agarwal, the President of the society of Special Functions and their Applications. The chief guest of the inaugural function was Prof. Tom Koornwinder, University of Amsterdam, Amsterdam, The Netherlands and the Guest of Honour was Prof. Dinesh Shringhi, Member Secretary and Principal, Govt. Engineering College Bikaner. The felicitation ceremony of Prof. N. K. Thakare, Patron, SSFA was also the part of the inaugural ceremony.

The welcome address was given by Dr. S. K. Bansal, Principal, G.C.E.T, Bikaner and Chairperson of ICSFA-2017, followed by an address of Dr. Rakesh Kumar Parmar, Convener, ICSFA-2017. After that a report was presented by Prof. M. A. Pathan, General Secretary, SSFA.

The conference was declared open with the release of the Souvenir of ICSFA-2017. The chief guest later addressed the gathering and laid stress on the importance of the mathematics in all disciplines of science and engineering as well as other disciplines. After chief guest's address guest of honour shared his views regarding this international conference. At the end of the inaugural ceremony vote of thanks was given by Dr. Pratibha Choudhary, Organizing Secretary, ICSFA-2017.

The aim of this conference ICSFA-2017 was to provide common platform for interaction, exchange of ideas and latest development in the field of Special Functions and various related fields of the Applied Mathematics. The day to day activities during the conference were designed to be interactive, involving sessions like Plenary lectures, Invited Talks, Paper Presentation Session covering a wide range of topics including Special functions, Lie Theory, Orthogonal Polynomials, q- Theory, Fractional Calculus, Number Theory, Cryptography, Combinatorics etc. Apart from this, there was a symposium on “Application of Mathematical Sciences in the Engineering Problems”.

The academic sessions of the conference began with the presidential address by Prof. A. K. Agarwal, President of SSFA on “Combinatorics of mock theta functions” in the Swaran Mahal which was chaired by Prof. N. K. Thakare.

In the second session of this conference, the first plenary lecture was delivered by Prof. Tom Koornwinder, University of Amsterdam, in the honor of Prof. R. P. Agarwal on the topic "Bispectrality and dual addition formulas". This session was chaired by Prof. A. K. Agarwal.

In the third Session, Prof. A. M. Mathai, Mc Gill University, Canada, Prof. Kalyan Chakraborty, Harish Chandra Research Institute, Allahabad and S. Ponnusamy, Indian Statistical Institute, Chennai delivered their invited talks on Ultra Gamma Integral, An Analogue of Wilton's Formula and Some Applications and On Bohr's inequality and beyond respectively. This session was chaired by Prof. Tom Koornwinder.

The 4th and 5th sessions were devoted to Paper presentations for A.K. Agarwal Best Publication Award Presentation, Aruna Gupta Prize and M. I. Qureshi Prize for the best paper presentation. These presentations were held in the evening of first day of the conference. These sessions were chaired by Prof. A. M. Mathai, Prof. P. K. Banerji, Prof. Kalyan Chakraborti and Dr. Ajay Shukla.

Second Day

The second day of the conference (03/11/2017) included, two Sessions of invited talks, one session of symposium and two sessions of paper presentations were held. Very informative and innovative talks were delivered in session 6th in which Prof M. A. Pathan, Centre for Mathematical and Statistical Sciences, Kerala, Prof. Tibor K. Pogany, Obuda University, Hungary, Prof. P. K. Swamee, North Cap University, Gurgaon and Prof. A. K. Shukla, SVNIT, Surat delivered their talks on Advanced Special Functions associated with Lie and Witt Algebras , Y - Bessel sampling series of L^2 stochastic processes, Generalized Rathie-Swamee distribution and applications and Some generalizations of Mittag-Leffler Function respectively. This session was chaired by Prof. A. M. Mathai.

In session 7th invited talks were delivered. Dr. R. Jana, SVNIT, Surat and Prof. Subuhi Khan, AMU, Aligarh delivered their invited talks On Generalized Pochhammer symbols and Lie-algebraic approach to 1-parameter 2D-Hermite polynomials, respectively. The session was chaired by Prof. S. Ponnusamy.

Session 8th and 9th were conducted as parallel sessions. In these sessions various research papers were presented by the researchers. Sessions 8th and 9th were chaired by Prof. Rashmi Jain and Prof. Subuhi Khan respectively.

Session 10th and 11th were conducted as parallel sessions. In session 10th , a symposium on Applications of Mathematical Sciences in Engineering Problems was organized which was chaired by Dr. Deepak Bansal. A total of 12 papers were presented in this session whereas in session 11th , invited talks are delivered by the experts. Prof. S. N. Fathima, Pondicherry University, Pondicherry, Prof. Azizhul Hoque, HRI, Allahabad and Prof. Ritu Agarwal, MNIT, Jaipur delivered their talks during 11th session on The Partition Function and Modular Forms, On The Class Numbers of Imaginary Quadratic Fields and Ruscheweyh derivatives of fractional order and its applications , respectively.

Session 12th and 13th were also conducted as parallel sessions. In both sessions different research papers were presented by various researchers. These sessions were chaired by Prof. J. K. Prajapat and Prof. R. K. Jana.

Third Day

On third day of the conference (04/11/2017), two sessions of invited talks, one session of paper presentation, general body meeting of SSFA and valedictory function were held. Expert and interested talks were given by Prof. S. N, Singh, Purvanchal University, Jaunpur, Dr. Manoj Sharma, Dr. Fatemah Ayatollah Zadeh Shirazi, Tehran University, Iran, Dr. J. K. Prajapat, Central University of Rajasthan, Ajmer, Dr. Raj Kumar Mistri, HRI, Allahabad, Dr. Hemant Kumar, DAV College, Kanpur, Dr. N.S. Solanki, CET, Bikaner, Dr. Shashi Kant, Govt. Dungar College, Bikaner, Dr. Rakesh K Parmar, G.C.E.T. Bikaner in sessions 14th and 15th. These sessions were chaired by Prof. M. A. Pathan and Prof. Ajay Shukla ,respectively.

Various research papers were presented in session 16th by researchers. This session was chaired by Dr. Manoj Sharma.

After the completion of session 16th the general body meeting of SSFA was held. In this meeting new and innovative ideas in the field of special functions were discussed and new dimensions were explored. There after valedictory function was organized.

In valedictory function Prof. Tibor K. Pogany, Obuda University, Hungary was the Chief guest while Prof. S. Ponnusamy, ISI Chennai was the guest of honour. Prof. Tom Koornwinder was offered an honorary Life-membership of the Society for Special Functions and their Applications (SSFA) during the function. Various awards were also declared in this ceremony. At the end of the conference Dr. Deepak Bansal, co- convener, ICSFA-2017 proposed the vote of thanks. Also special thanks to Dr. Kanika Solanki , Joint Organizing Secretary, ICSFA-2017 for doing the great job in conference.

More than 100 delegates from all over country and abroad attended this conference. 28 invited talks and more than 70 research papers presented at the conference bear testimony to the diverse topics which were covered.

Dr. Rakesh Kumar Parmar

Convener of ICSFA - 2017

Presidential Address*

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Dr. S. K. Bansal- Principal, Government College of Engineering & Technology, Bikaner and Chairperson of the International Conference on Special Functions and their Applications (ICSFA-2017), Prof. Tom Koornwinder - Chief Guest, Prof. Dinesh Shringhi - Guest of Honour, Prof. N. K. Thakare - Patron, Society for Special Functions and their Applications (SSFA), Prof.M.A. Pathan - Secretary, SSFA, Dr. R. K. Parmar - Convener, ICSFA - 2017, other dignitaries on the dais and off the dais, Ladies and Gentlemen!

On behalf of SSFA and on my personal behalf, I would first like to thank the organizers for hosting this 16th Annual Conference of SSFA. I wish all the best to the Organizing Committee, particularly its convener Dr.R.K.Parmar for making substantial efforts necessary to bring this event in to being. Rajasthan and UP are known for researches in special functions. In Rajasthan, Jaipur and Jodhpur have contributed significantly for the development to special functions. It is good to see that new centres are coming up. We are happy to be here in this old city of Bikaner, known for its Junagarh fort throughout the country.

Friends, during the next three days, there will be several brain-storming academic sessions in which various aspects of special functions such as orthogonal polynomials, Lietheory, hypergeometric functions, fractional calculus, number theory, combinatorics etc. will

*Society for Special Functions and their Applications, 16th Annual Conference, Government College of Engineering & Technology, Bikaner, November 02-04, 2017.

be discussed. As a part of the academic program, a special symposium on “Applications of Mathematical Sciences in Engineering Problems” will also be held. About my own recent work on mock theta functions, I will talk in the technical address scheduled just after this inaugural function. In this general address, I take the opportunity to make some general remarks about mathematics.

I begin with a recently published book. The title is : “The Math Myth : And Other STEM Delusions" (STEM is an acronym and stands for Science, Technology, Engineering and Mathematics), written by Andrew Hacker. He is Professor Emeritus in the Department of Political Science at Queens College in New York. He did his graduation from Oxford University and Ph.D. from Princeton University. In this book he writes: "Abstract math is scary, damaging and should be optional in American education". About algebra he writes: “Far from being a pipe line to success, it is a barrier that ends up supporting opportunities, stiing creativity, and denying society a wealth of varied talents". People like Hacker are proposing to eliminate subjects like algebra from the curriculum.

It is unfortunate that mathematics is not well understood by large portions of the general public. There are too many people who think that mathematics is a dead science in the sense that everything that needed to be discovered has already been discovered. They doubt the relevance of mathematics to our daily lives in view of the advances made in computer science. The mathematics community has to eliminate these false perceptions and increase appreciation for mathematics and for its fundamental role in culture and society. To make my point clear, I would like to give an example. About four decades ago when I was doing my Ph.D. at IIT, Delhi, I used to run to the library every month to look at the latest printed volume of Math Reviews. Then we started using Math Sci Net. Now people of ten by pass Math Sci Net also when trying to get information. They collaborate through blogs. Students and teachers both use their tablets, cell phones and laptops. These are all the wonders of a technology called Information

Technology (IT). It is one of the most useful technologies these days. Enormous Amount of data can be communicated from one part of the world to the other in no time. Many of us know that this technology uses digital communication in which satellites play a key role. But not many people know that at the back of this technology the driving force is coding theory. This is an area in which finite fields which are essentially parts of abstract algebra are studied. If people like Prof. Andrew Hacker propose to eliminate algebra from the curriculum, I would say: it is their ignorance of the subject only. Perhaps, we have not done a good job presenting our field to the public. We have to develop a broader awareness of mathematical culture. We have to promote an understanding of the fundamental role of mathematics in the world.

With these words, I conclude by wishing the conference a grand success so that the organizers may feel fully contended.

Thank you very much.

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DUAL ADDITION FORMULA FOR CONTINOUS q-ULTRSPHERICAL POLYNOMIALS

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Abstract: We settle the dual addition formula for continuous q-ultraspherical polynomials as an expansion in terms of special q-Racah polynomials for which the constant term is given by the linearization formula for the continuous q-ultraspherical polynomials. In a second proof we derive the dual addition formula from the Rahman-Verma addition formula for these polynomials by using the self-duality of the polynomials. The paper starts with a tutorial on duality properties of orthogonal polynomials in the (q-)Askey scheme.

Keywords and Phrases: Duality, Askey scheme, q-Askey scheme, continuous q-ultaspherical polynomials, addition formula, dual addition formula

Mathematics subject Classification: Primary 33C45, 33.D45.

1 Introduction

This paper elaborates on the notions of *addition formula* and *duality* in connection with special orthogonal polynomials. As a natural continuation of our recent derivation [22] of the dual addition formula for ultraspherical polynomials we now derive the dual addition formula for continuous q-ultraspherical polynomials. We give two different proofs. The first proof is a perfect q-analogue of the derivation in [22]. Every step of the proof yields in the limit for $q \to 1$ the corresponding step in [22]. The second proof exploits the self-duality of the continuous q-ultraspherical polynomials. Then the dual addition formula easily follows from the known addition formula [29] for these polynomials.

Addition formulas are closely related to product formulas. For instance, the *addition formula for Legendre polynomials* [28, (18.18.9)]

$$P_n(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)=P_n(\cos\theta_1)P_n(\cos\theta_2)$$
$$+2\sum_{k=1}^{n}\frac{(n-k)!\,(n+k)!}{2^{2k}(n!)^2}\,(\sin\theta_1)^kP_{n-k}^{(k,k)}(\cos\theta_1)\,(\sin\theta_2)^kP_{n-k}^{(k,k)}(\cos\theta_2)\cos(k\phi) \tag{1.1}$$

gives the Fourier-cosine expansion of the left-hand side as a function of ϕ. Integration with respect to ϕ over $[0,\pi]$ gives the constant term in this expansion, which is the *product formula for Legendre polynomials* [28, (18.17.6)]

$$P_n(\cos\theta_1)P_n(\cos\theta_2)=\frac{1}{\pi}\int_0^\pi P_n(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)\,d\phi. \tag{1.2}$$

Two formulas involving Legendre polynomials $P_n(x)$ (or more generally some orthogonal polynomials $p_n(x)$) are called *dual* to each other if the roles of n and x in the second formula are interchanged in comparison with the first formula. The formula dual to the product formula (1.2) is the *linearization formula*, which expands the product $P_\ell(x)P_m(x)$ in terms of Legendre polynomials $P_k(x)$. This expansion is a sum running from $k=|\ell-m|$ to $k=\ell+m$, where only terms with $\ell+m-k$ even will occur since $P_n(-x)=(-1)^nP_n(x)$. The linearization formula for Legendre polynomials is explicitly known (see [28, (18.18.22)] for $\lambda=\frac12$ together with [28, (18.7.9)], and see after (1.4) for the shifted factorial $(a)_n$):

$$P_\ell(x)P_m(x)=\\ \sum_{j=0}^{\min(\ell,m)}\frac{(2(\ell+m-2j)+1)(\frac12)_j(\frac12)_{\ell-j}(\frac12)_{m-j}(\ell+m-j)!}{j!\,(\ell-j)!\,(m-j)!\,(\frac32)_{\ell+m-j}}\,P_{\ell+m-2j}(x). \tag{1.3}$$

Richard Askey, in his lectures at conferences, often raised the problem to find an addition type formula associated with (1.3) in a similar way as the addition formula (1.1) is associated with the product formula (1.2). The author finally solved this in [22] by recognizing the coefficient of $P_{\ell+m-2j}(x)$ in (1.3) as the weight of a special Racah polynomial [18, (9.2.1)] depending on j, and then finding the expansion of $P_{\ell+m-2j}(x)$ in terms of these Racah polynomials. More generally, the same idea worked out well in [22] for ultraspherical polynonmials.

While (1.1), (1.2), (1.3), and their generalizations to ultraspherical polynomials, are formulas established long ago and staying within the realm of classical orthogonal polynomials, it is remarkable that the dual addition formula steps out from this and needs Racah polynomials, which live high up in the *Askey scheme* (see Figure 1 in §2.1). Parallel to the Askey scheme there is the much larger q-Askey scheme[1]. Families of orthogonal polynomials in the Askey scheme are limit cases of families in the q-Askey scheme. The continuous q-ultraspherical polynomials form the family which is the q-analogue of the ultraspherical polynomials. Moreover, the q-analogues of (1.1), (1.2) and (1.3) for these polynomials are available in the literature. The continuous q-ultraspherical polynomials also

[1]See `http://homepage.tudelft.nl/11r49/pictures/large/q-AskeyScheme.jpg`

have the property of *self-duality*, which is lost in the limit to $q = 1$. This notion means that, for a suitable function σ, an orthogonal polynomial $p_n(x)$ has the property that $p_n(\sigma(m)) = p_m(\sigma(n))$ $(m, n = 0, 1, \ldots)$. With all this material available it becomes a smooth, although nontrivial job to derive the dual addition formula for these polynomials.

This paper is based on the *R. P. Agarwal Memorial Lecture*, which the author delivered on November 2, 2017 during the conference ICSFA-2017 held in Bikaner, Rajasthan, India. With pleasure I remember to have met Prof. Agarwal during the workshop on Special Functions and Differential Equations held at the Institute of Mathematical Sciences in Chennai, January 1997, where he delivered the opening address [1]. I cannot resist to quote from it the following wise words, close to the end of the article:

"*I think that I have taken enough time and I close my discourse- with a word of caution and advice to the research workers in the area of special functions and also those who use them in physical problems. The corner stones of classical analysis are Oelegance, simplicity, beauty and perfection.O Let them not be lost in your work. Any analytical generalization of a special function, only for the sake of a generalization by adding a few terms or parameters here and there, leads us nowhere. All research work should be meaningful and aim at developing a quality technique or have a bearing in some allied discipline.*"

The contents of the paper are as follows. Section 2, of tutorial nature, discusses the notion of duality in special functions. Section 3 gives the necessary preliminaries about special classes of orthogonal polynomials which will occur later in the paper. Next, Section 4 summarizes the results about the addition formula and dual addition formula for ultraspherical polynomials. The new results of the paper appear in Section 5. It contains the two proofs of the dual addition formula for continuous q-ultraspherical polynomials.

Note For definition and notation of (q-)shifted factorials and (q-)hypergeometric series see [13, §1.2]. In the $q = 1$ case we will mostly meet terminating hypergeometric series

$$ {}_rF_s\left(\begin{matrix} -n, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; z\right) := \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(a_2, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k} z^k. \tag{1.4} $$

Here $(b_1, \ldots, b_s)_k := (b_1)_k \ldots (b_s)_k$ and $(b)_k := b(b + 1) \ldots (b + k - 1)$ is the *Pochhammer symbol* or *shifted factorial.* In (1.4) we even allow that $b_i = -N$ for some i with N integer $\geq n$. There is no problem because the sum on the right terminates at $k = n \leq N$.

In the q-case we will always assume that $0<q<1$. We will only meet terminating q-hypergeometric series of the form

$$ {}_{s+1}\phi_s\left(\begin{matrix} q^{-n}, a_2, \ldots, a_{s+1} \\ b_1, \ldots, b_s \end{matrix}; q, z\right) := \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} \frac{(a_2,\ldots,a_{s+1};q)_k}{(b_1,\ldots,b_s;q)_k} z^k. \tag{1.5} $$

Here $(b_1,\ldots,b_s;q)_k := (b_1;q)_k \ldots (b_s;q)_k$ and $(b;q)_k := (1-b)(1-qb)\ldots(1-q^{k-1}b)$ is the *q-Pochhammer symbol* or *q-shifted factorial.* In (1.5) we even allow that $b_i = q^N$ for some i with N integer $\ge n$.
For formulas on orthogonal polynomials in the (q-)Askey scheme we will often refer to Chapters 9 and 14 in [18]. Almost all of these formulas, with different numbering, are available in open access[2].

2 The notion of duality in special functions

With respect to a (positive) measure μ on $\mathbb{R}$ with support containing infinitely many points we can define *orthogonal polynomials* (OP's) p_n $(n=0,1,2,\ldots)$, unique up to nonzero real constant factors, as (real-valued) polynomials p_n of degree n such that

$$ \int_{\mathbb{R}} p_m(x)\, p_n(x)\, d\mu(x) = 0 \qquad (m,n \ne 0). $$

Then the polynomials p_n satisfy a *three-term recurrence relation*

$$ x\, p_n(x) = A_n\, p_{n+1}(x) + B_n\, p_n(x) + C_n\, p_{n-1}(x) \qquad (n=0,1,2,\ldots), \tag{2.1} $$

where the term $C_n\, p_{n-1}(x)$ is omitted if $n=0$, and where A_n, B_n, C_n are real and

$$ A_{n-1} C_n > 0 \qquad (n=1,2,\ldots). \tag{2.2} $$

By *Favard's theorem* [12] we can conversely say that if $p_0(x)$ is a nonzero real constant, and the $p_n(x)$ $(n=0,1,2,\ldots)$ are generated by (2.1) for certain real A_n, B_n, C_n which satisfy (2.2), then the p_n are OP's with respect to a certain measure μ on $\mathbb{R}$.

With A_n, B_n, C_n as in (2.1) define a *Jacobi operator* M, acting on infinite sequences $\{g(n)\}_{n=0}^{\infty}$, by

$$ (Mg)(n) = M_n\big(g(n)\big) := A_n\, g(n+1) + B_n\, g(n) + C_n\, g(n-1) \qquad (n=0,1,2,\ldots), $$

[2] See `http://aw.twi.tudelft.nl/~koekoek/askey/`

where the term $C_n\, g(n-1)$ is omitted if $n=0$. Then (2.1) can be rewritten as the eigenvalue equation

$$M_n\big(p_n(x)\big) = x\, p_n(x) \qquad (n=0,1,2,\ldots). \tag{2.3}$$

One might say that the study of a system of OP's p_n turns down to the spectral theory and harmonic analysis associated with the operator M. From this perspective one can wonder if the polynomials p_n satisfy some *dual* eigenvalue equation

$$(Lp_n)(x) = \lambda_n\, p_n(x) \tag{2.4}$$

for $n=0,1,2,\ldots$, where L is some linear operator acting on the space of polynomials. We will consider varioua types of operators L together with the corresponding OP's, first in the Askey scheme and next in the q-Askey scheme.

2.1 The Askey scheme

Classical OP's *Bochner's theorem* [9] classifies the second order differentai operators L together with the OP's p_n such that (2.4) holds for certain eigenvalues λ_n. The resulting *classical orthogonal polynomials* are essentially the polynomials listed in the table below. Here $d\mu(x) = w(x)\,dx$ on (a,b) and the closure of that interval is the support of μ. Furthermore, $w_1(x)$ occurs in the formula for L to be given after the table.

name	$p_n(x)$	$w(x)$	$\frac{w_1(x)}{w(x)}$	(a,b)	constraints	λ_n
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$1(1-x)^\alpha(1+x)^\beta$	$1-x^2$	$(-1,1)$	$\alpha,\beta>-1$	$-n(n+\alpha+\beta+1)$
Laguerre	$L_n^{(\alpha)}(x)$	$x^\alpha e^{-x}$	x	$(0,\infty)$	$\alpha>-1$	$-n$
Hermite	$H_n(x)$	e^{-x^2}	1	$(-\infty,\infty)$		$-2n$

Then

$$(Lf)(x) = w(x)^{-1}\frac{d}{dx}\big(w_1(x)\, f'(x)\big).$$

For these classical OP's the duality goes much further than the two dual eigenvalue equations (2.3) and (2.4). In particular for Jacobi polynomials it is true to a large extent that every formula or property involving n and x has a dual formula or property where the roles of n and x are interchanged. We call this the *duality principle*. If the partner formula or property is not yet known then it is usually a good open problem to find it (but one should be warned that there are examples where the duality fails).

The Jacobi, Laguerre and Hermite families are connected by limit transitions, as is already suggested by limit transitions for their (rescaled) weight functions:

- Jacobi → Laguerre: $x^\alpha(1-\beta^{-1}x)^\beta \to x^\alpha e^{-x}$ as $\beta\to\infty$;

- Jacobi → Hermite: $\big(1-\alpha^{-1}x^2\big)^\alpha \to e^{-x^2}$ as $\alpha\to\infty$;
- Laguerre → Hermite: $(e/\alpha)^\alpha\big((2\alpha)^{\frac12}x+\alpha\big)^\alpha e^{-(2\alpha)^{\frac12}x-\alpha} \to e^{-x^2}$ as $\alpha\to\infty$.

Formulas and properties of the three families can be expected to be connected under these limits. Although this is not always the case, this *limit principle* is again a good source of open problems.

Discrete analogues of classical OP's Let L be a second order difference operator:

$$(Lf)(x) := a(x)\,f(x+1)+b(x)\,f(x)+c(x)\,f(x-1). \tag{2.5}$$

Here as solutions of (2.4) we will also allow OP's $\{p_n\}_{n=0}^N$ for some *finite* $N\ge 0$, which will be orthogonal with respect to positive weights w_k $(k=0,1,\ldots,N)$ on a finite set of points x_k $(k=0,1,\ldots,N)$:

$$\sum_{k=0}^N p_m(x_k)\,p_n(x_k)\,w_k = 0 \qquad (m,n=0,1,\ldots,N;\ m\ne n).$$

If such a finite system of OP's satisfies (2.4) for $n=0,1,\ldots,N$ with L of the form (2.5) then the highest n for which the recurrence relation (2.1) holds is $n=N$, where the zeros of p_{N+1} are precisely the points $x_0,x_1,\ldots,x_N$.

The classification of OP's satisfying (2.4) with L of the form (2.5) (first done by O. Lancaster, 1941, see [2]) yields the four families of Hahn, Krawtchouk, Meixner and Charlier polynomials, of which Hahn and Krawtchouk are finite systems, and Meixner and Charlier infinite systems with respect to weights on countably infinite sets.

Krawtchouk polynomials [18, (9.11.1)] are given by

$$K_n(x;p,N) := {}_2F_1\left(\begin{matrix}-n,-x\\-N\end{matrix};p^{-1}\right) \quad (n=0,1,2,\ldots,N). \tag{2.6}$$

They satsify the orthogonality relation

$$\sum_{x=0}^N (K_m\,K_n\,w)(x;p,N) = \frac{(1-p)^N}{w(n;p,N)}\,\delta_{m,n}$$

with weights

$$w(x;p,N) := \binom{N}{x} p^x(1-p)^{N-x} \quad (0<p<1).$$

By (2.6) they are *self-dual*:

$$K_n(x;p,N) = K_x(n;p,N) \qquad (n,x = 0,1,\ldots,N).$$

The three-term recurrence relation (2.3) immediately implies a dual equation (2.4) for such OP's.

The four just mentioned families of discrete OP's are also connected by limit relations. Moreover, the classical OP's can be obtained as limit cases of them, but not conversely. For instance, *Hahn polynomials* [18, (9.5.1)] are given by

$$Q_n(x;\alpha,\beta,N) := {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}; 1\right) \qquad (n = 0,1,\ldots,N) \quad (2.7)$$

and they satisfy the orthogonality relation

$$\sum_{x=0}^{N} (Q_m Q_n w)(x;\alpha,\beta,N) = 0 \qquad (m,n = 0,1,\ldots,N;\ m \neq n;\ \alpha,\beta > -1)$$

with weights

$$w(x;\alpha,\beta,N) := \frac{(\alpha+1)_x\,(\beta+1)_{N-x}}{x!\,(N-x)!}\,.$$

Then by (2.7) (rescaled) Hahn polynomials tend to (shifted) Jacobi polynomials:

$$\lim_{N\to\infty} Q_n(Nx;\alpha,\beta,N) = {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; x\right) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}\,. \quad (2.8)$$

Continuous versions of Hahn and Meixner polynomials

A variant of the difference operator (2.5) is the operator

$$(Lf)(x) := A(x)\,f(x+i) + B(x)\,f(x) + \overline{A(x)}\,f(x-i) \qquad (x \in \mathbb{R}), \qquad (2.9)$$

where $B(x)$ is real-valued. Then further OP's satisfying (2.4) are the continuous Hahn polynomials and the Meixner-Pollaczek polynomials [18, Ch. 9].

Insertion of a quadratic argument

For an operator $\widetilde{L}$ and some polynomial σ of degree 2 we can define an operator L by

$$(Lf)\big(\sigma(x)\big) := \widetilde{L}_x\Big(f\big(\sigma(x)\big)\Big), \qquad (2.10)$$

Now we look for OP's satisfying (2.4) where $\widetilde{L}$ is of type (2.5) or (2.9). So

$$\widetilde{L}_x\Big(p_n\big(\sigma(x)\big)\Big) = \lambda_n\,p_n\big(\sigma(x)\big). \qquad (2.11)$$

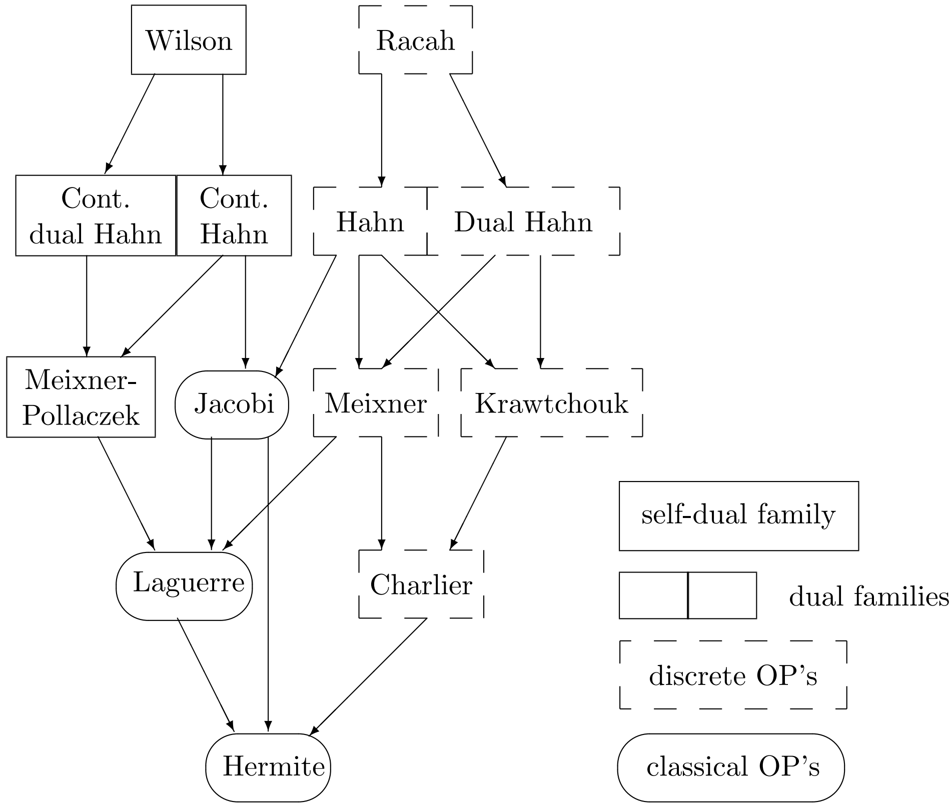

Figure 1: The Askey scheme

The resulting OP's are the Racah polynomials and dual Hahn polynomials for (2.11) with $\widetilde{L}$ of type (2.5), and Wilson polynomials and continuous dual Hahn polynomials for (2.11) with $\widetilde{L}$ of type (2.9), see again [18, Ch. 9].

The OP's satisfying (2.4) in the cases discussed until now form together the *Askey scheme*, see Figure 1. The arrows denote limit transitions.

In the Askey scheme we emphasize the self-dual families: Racah, Meixner, Krawtchouk and Charlier for the OP's with discrete orthogonality measure, and Wilson and Meixner-Pollaczek for the OP's with non-discrete orthogonality measure. We already met perfect self-duality for the Krawtchouk polynomials, which is also the case for Meixner and Charlier polynomials. For the Racah polynomials the dual OP's are still Racah polynomials, but with different values of the parameters:

$$\begin{aligned} R_n\big(x(x+\delta-N);\alpha,\beta,-N-1,\delta\big) &:= {}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\delta-N\\ \alpha+1,\beta+\delta+1,-N\end{matrix};1\right)\\ &= R_x(n(n+\alpha+\beta+1);-N-1,\delta,\alpha,\beta) \qquad (n,x=0,1,\ldots,N). \end{aligned}$$

The orthogonality relation for these Racah polynomials involves a weighted sum of terms $(R_m R_n)\big(x(x+\delta-N);\alpha,\beta,-N-1,\delta\big)$ over $x=0,1,\ldots N$, see also §3.2.

For Wilson polynomials we have also self-duality with a change of parameters but the self-duality is not perfect, i.e., not related to the orthogonality relation:

$$\begin{aligned}\text{const.}\, W_n(x^2;a,b,c,d) &:= {}_4F_3\left(\begin{matrix}-n,n+a+b+c+d-1,a+ix,a-ix\\ a+b,a+c,a+d\end{matrix};1\right)\\ &= \text{const.}\, W_{-ix-a}\Big(\big(i(n+a')\big)^2;a',b',c',d'\Big),\end{aligned}\tag{2.12}$$

where $a'=\frac{1}{2}(a+b+c+d-1)$, $a'+b'=a+b$, $a'+c'=a+c$, $a'+d'=a+d$. The duality (2.12) holds for $-ix-a=0,1,2,\ldots$, while the orthogonality relation for the Wilson polynomials involves a weighted integral of $(W_mW_n)(x^2;a,b,c,d)$ over $x\in[0,\infty)$.

As indicated in Figure 1, the dual Hahn polynomials

$$R_n\big(x(x+\alpha+\beta+1);\alpha,\beta,N\big) := {}_3F_2\left(\begin{matrix}-n,-x,x+\alpha+\beta+1\\ \alpha+1,-N\end{matrix};1\right)\quad (n=0,1,\ldots,N)$$

are dual to the Hahn polynomials (2.7):

$$Q_n(x;\alpha,\beta,N) = R_x\big(n(n+\alpha+\beta+1);\alpha,\beta,N\big)\qquad (n,x=0,1,\ldots,N).$$

The duality is perfect: the dual orthogonality relation for the Hahn polynomials is the orthogonality relation for the dual Hahn polynomials, and conversely. There is a similar, but non-perfect duality between continuous Hahn and continuous dual Hahn.

The classical OP's are in two senses exceptional within the Askey scheme. First, they are the only families which are not self-dual or dual to another family of OP's. Second, they are the only continuous families which are not related by analytic continuation to a discrete family.

With the arrows in the Askey scheme given it can be taken as a leading principle to link also the formulas and properties of the various families in the Askey scheme by these arrows. In particular, if one has some formula or property for a family lower in the Askey scheme, say for Jacobi, then one may look for the corresponding formula or property higher up, and try to find it if it is not yet known. In particular, if one could find the result on the highest Racah or Wilson level, which is self-dual then, going down along the arrows, one might also obtain two mutually dual results in the Jacobi case.

2.2 The q-Askey scheme

The families of OP's in the q-Askey scheme[3] [18, Ch. 14] result from the classification [16], [15], [17], [30] of OP's satisfying (2.4), where L is defined in terms of the operator $\widetilde{L}$ and the function σ by (2.10), where $\widetilde{L}$ is of type (2.5) or (2.9), and where $\sigma(x) = q^x$ or equal to a quadratic polynomial in q^x. This choice of $\sigma(x)$ is the new feature deviating from what we discussed about the Askey scheme. And here q enters, with $0 < q < 1$ always assumed. The q-Askey scheme is considerably larger than the Askey scheme, but many features of the Askey scheme return here, in particular it has arrows denoting limit relations. Moreover, the q-Askey scheme is quite parallel to the Askey scheme in the sense that OP's in th q-Askey scheme, after suitable rescaling, tend to OP's in the Askey scheme as $q \uparrow 1$. Parallel to Wilson and Racah polynomials at the top of the Askey scheme there are Askey-Wilson polynomials [7] and q-Racah polynomials at the top of the q-Askey scheme. These are again self-dual families, with the self-duality for q-Racah being perfect.

The guiding principles discussed before about formulas or properties related by duality or limit transitions now extend to the q-Askey scheme: both within the q-Askey scheme and in relation to the Askey scheme by letting $q \uparrow 1$. For instance, one can hope to find as many dual pairs of significant formulas and properties of Askey-Wilson polynomials as possible which have mutually dual limit cases for Jacobi polynomials. In fact, we realize this in Section 5 with the addition and dual addition formula by taking limits from the continuous q-ultraspherical polynomials (a self-dual one-parameter subclass of the four-parameter class of Askey-Wilson polynomials) to the ultraspherical polynomials (a one-parameter subclass of the two-parameter class of Jacobi polynomials).

One remarkable aspect of duality in the two schemes concerns the discrete OP's living there. Leonard (1982) classified all systems of OP's $p_n(x)$ with respect to weights on a countable set $\{x(m)\}$ for which there is a system of OP's $q_m(y)$ on a countable set $\{y(n)\}$ such that

$$p_n\big(x(m)\big) = q_m\big(y(n)\big).$$

His classification yields the OP's in the q-Askey scheme which are orthogonal with respect to weights on a countable set together with their limit cases for $q \uparrow 1$ and $q \downarrow -1$ (where we allow $-1 < q < 1$ in the q-Askey scheme). The $q \downarrow -1$ limit case yields the Bannai-Ito polynomials [8].

[3] See `http://homepage.tudelft.nl/11r49/pictures/large/q-AskeyScheme.jpg`

2.3 Duality for non-polynomial special functions

For Bessel functions J_α see [28, Ch. 10] and references given there. It is convenient to use a different standardization and notation:

$$\mathcal{J}_\alpha(x) := \Gamma(\alpha+1)\,(2/x)^\alpha\, J_\alpha(x).$$

Then (see [28, (10.16.9)])

$$\mathcal{J}_\alpha(x) = \sum_{k=0}^\infty \frac{(-\frac14 x^2)^k}{(\alpha+1)_k\, k!} = {}_0F_1\left(\begin{matrix} - \\ \alpha+1 \end{matrix}; -\tfrac14 x^2\right) \qquad (\alpha > -1).$$

$\mathcal{J}_\alpha$ is an even entire analytic function. Some special cases are

$$\mathcal{J}_{-1/2}(x) = \cos x, \quad \mathcal{J}_{1/2}(x) = \frac{\sin x}{x}\,. \tag{2.13}$$

The *Hankel transform* pair [28, §10.22(v)], for f in a suitable function class, is given by

$$\begin{cases} \widehat{f}(\lambda) = \displaystyle\int_0^\infty f(x)\mathcal{J}_\alpha(\lambda x)x^{2\alpha+1}\,dx, \\ f(x) = \displaystyle\frac{1}{2^{2\alpha+1}\Gamma(\alpha+1)^2}\int_0^\infty \widehat{f}(\lambda)\mathcal{J}_\alpha(\lambda x)\lambda^{2\alpha+1}\,d\lambda. \end{cases}$$

In view of (2.13) the Hankel transform contains the Fourier-cosine and Fourier-sine transform as special cases for $\alpha = \pm\frac12$.

The functions $x \mapsto \mathcal{J}_\alpha(\lambda x)$ satisfy the eigenvalue equation [28, (10.13.5)]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x}\right)\mathcal{J}_\alpha(\lambda x) = -\lambda^2\,\mathcal{J}_\alpha(\lambda x). \tag{2.14}$$

Obviously, then also

$$\left(\frac{\partial^2}{\partial \lambda^2} + \frac{2\alpha+1}{\lambda}\frac{\partial}{\partial \lambda}\right)\mathcal{J}_\alpha(\lambda x) = -x^2\,\mathcal{J}_\alpha(\lambda x). \tag{2.15}$$

The differential operator in (2.15) involves the spectral variable λ of (2.14), while the eigenvalue in (2.15) involves the x-variable in the differential operator in (2.14).

The Bessel functions and the Hankel transform are closely related to the Jacobi polynomials (2.8) and their orthogonality relation. Indeed, we have the limit formulas

$$\lim_{n\to\infty} \frac{P_n^{(\alpha,\beta)}\big(\cos(n^{-1}x)\big)}{P_n^{(\alpha,\beta)}(1)} = \mathcal{J}_\alpha(x), \qquad \lim_{\substack{\nu\to\infty \\ \nu\lambda=1,2,\ldots}} \frac{P_n^{(\alpha,\beta)}\big(\cos(\nu^{-1}x)\big)}{P_n^{(\alpha,\beta)}(1)} = \mathcal{J}_\alpha(\lambda x).$$

There are many other examples of non-polynomial special functions being limit cases of OP's in the (q-)Askey scheme, see for instance [21], [19].

In 1986 Duistermaat & Grünbaum [10] posed the question if the pair of eigenvalue equations (2.14), (2.15) could be generalized to a pair

$$\begin{aligned} L_x\big(\phi_\lambda(x)\big) &= -\lambda^2\,\phi_\lambda(x), \\ M_\lambda\big(\phi_\lambda(x)\big) &= \tau(x)\,\phi_\lambda(x) \end{aligned} \tag{2.16}$$

for suitable differential operators L_x in x and M_λ in λ and suitable functions $\phi_\lambda(x)$ solving the two equations. Here the functions $\phi_\lambda(x)$ occur as eigenfunctions in two ways: for the operator L_x with eigenvalue depending on λ and for the operator M_λ with eigenvalue depending on x. Since the occurring eigenvalues of an operator form its spectrum, a phenomenon as in (2.16) is called *bispectrality.* For the case of a second order differential operator L_x written in potential form $L_x = d^2/dx^2 - V(x)$ they classified all possibilities for (2.16). Beside the mentioned Bessel cases and a case with Airy functions (closely related to Bessel functions) they obtained two other families where M_λ is a higher than second order differential operator. These could be obtained by successive *Darboux transformations* applied to L_x in potential form. A Darboux transformation produces a new potential from a given potential $V(x)$ by a formula which involves an eigenfunction of L_x with eigenvalue 0. Their two new families get a start by the application of a Darboux transformation to the Bessel differential equation (2.14), rewritten in potential form

$$\phi_\lambda''(x) - (\alpha^2 - \tfrac14)x^{-2}\phi_\lambda(x) = -\lambda^2\phi_\lambda(x), \qquad \phi_\lambda(x) = (\lambda x)^{\alpha+\frac12}\mathcal{J}_\alpha(\lambda x).$$

Here α should be in $\mathbb{Z} + \frac12$ for a start of the first new family or in $\mathbb{Z}$ for a start of the second new family. For other values of α one would not obtain a dual eigenvalue equation with M_λ a finite order differential operator.

Just as higher order differential operators M_λ occur in (2.16), there has been a lot of work on studying OP's satisfying (2.4) with L a higher order differential operator. See a classification in [25], [24]. All occurring OP's, the so-called *Jacobi type* and *Laguerre type polynomials*, have a Jacobi or Laguerre orthogonality measure with integer values of the parameters, supplemented by mass points at one or both endpoints of the orthogonality interval. Some of the Bessel type functions in the second new class in [10] were obtained in [11] as limit cases of Laguerre type polynomials.

2.4 Some further cases of duality

The self-duality property of the family of Askey-Wikson polynomials is reflected in Zhedanov's *Askey-Wilson algebra* [31]. A larger algebraic structure is the *double affine Hecke algebra* (DAHA), introduced by Cherednik and extended by Sahi.

The related special functions are so-called *non-symmetric* special functions. They are functions in several variables and associated with root systems. Again there is a duality, both in the DAHA and for the related special functions. For the (one-variable) case of the non-symmetric Askey-Wilson polynomials this is treated in [27]. In [23] limit cases in the q-Askey scheme are also considered.

Finally we should mention the manuscript [20]. Here the author extended the duality (3.13) for continuous q-ultraspherical polynomials to Macdonald polynomials and thus obtained the so-called Pieri formula [26, §VI.6] for these polynomials.

3 Some special classes of orthogonal polynomials

3.1 Ultraspherical polynomials

Put

$$R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \tfrac12(1-x)\right), \tag{3.1}$$

where $P_n^{(\alpha,\beta)}(x)$ is a *Jacobi polynomial* [18, (9.8.1)] in usual notation. In the standardization (3.1) we have $R_n^{(\alpha,\beta)}(1) = 1$. In the special case $\alpha = \beta$ we obtain *ultraspherical* (or *Gegenbauer*) *polynomials*

$$R_n^\alpha(x) := R_n^{(\alpha,\alpha)}(x) = \frac{P_n^{(\alpha,\alpha)}(x)}{P_n^{(\alpha,\alpha)}(1)} = \frac{C_n^{(\alpha+\frac12)}(x)}{C_n^{(\alpha+\frac12)}(1)}\,, \tag{3.2}$$

where $C_n^{(\lambda)}(x)$ is an ultraspherical polynomial in usual notation [18, §9.8.1] with $C_n^{(\lambda)}(1) = (2\lambda)_n/n!$. The polynomials R_n^α satisfy the orthogonality relation

$$\int_{-1}^1 R_m^\alpha(x)\,R_n^\alpha(x)\,(1-x^2)^\alpha\,dx = h_n^\alpha\,\delta_{m,n} \qquad (\alpha > -1),$$
$$h_n^\alpha = \frac{2^{2\alpha+1}\Gamma(\alpha+1)^2}{\Gamma(2\alpha+2)}\,\frac{n+2\alpha+1}{2n+2\alpha+1}\,\frac{n!}{(2\alpha+2)_n}\,.$$

3.2 Racah polynomials

We will consider *Racah polynomials* [18, Section 9.2]

$$R_n\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big) := {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}; 1\right) \tag{3.3}$$

for $\gamma = -N-1$, where $N \in \{1,2,\ldots\}$, and for $n \in \{0,1,\ldots,N\}$. These are orthogonal polynomials on the finite quadratic set $\{x(x+\gamma+\delta+1) \mid x \in \{0,1,\ldots,N\}\}$:

$$\sum_{x=0}^{N} (R_m R_n)\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big)\, w_{\alpha,\beta,\gamma,\delta}(x) = h_{n;\alpha,\beta,\gamma,\delta}\,\delta_{m,n} \qquad (m,n \in \{0,1,\ldots,N\})$$

with

$$w_{\alpha,\beta,\gamma,\delta}(x) = \frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x(\delta+1)_x\, x!}\,\frac{\gamma+\delta+1+2x}{\gamma+\delta+1}, \tag{3.4}$$

$$\frac{h_{n;\,\alpha,\beta,\gamma,\delta}}{h_{0;\,\alpha,\beta,\gamma,\delta}} = \frac{\alpha+\beta+1}{\alpha+\beta+2n+1}\,\frac{(\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n\, n!}{(\alpha+1)_n(\alpha+\beta+1)_n(\beta+\delta+1)_n(\gamma+1)_n},$$

$$h_{0;\,\alpha,\beta,\gamma,\delta} = \sum_{x=0}^{N} w_{\alpha,\beta,\gamma,\delta}(x) = \frac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N} \qquad (\gamma = -N-1). \tag{3.5}$$

3.3 Askey-Wilson polynomials

We will use the following standardization and notation for *Askey-Wilson polynomials*:

$$R_n[z] = R_n[z;a,b,c,d \,|\, q] := {}_4\phi_3\!\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right). \tag{3.6}$$

These are symmetric Laurent polynomials of degree n in z, so they are ordinary polynomials of degree n in $x := \frac{1}{2}(z+z^{-1})$. The polynomials (3.6) are related to the Askey-Wilson polynomials $p_n(x;a,b,c,d\,|\,q)$ in usual notation [7, (1.15)], [18, (14.1.1)] by

$$R_n[z;a,b,c,d\,|\,q] = \frac{a^n}{(ab,ac,ad;q)_n}\, p_n\big(\tfrac{1}{2}(z+z^{-1});a,b,c,d\,|\,q\big). \tag{3.7}$$

If $|a|,|b|,|c|,|d| \le 1$ such that pairwise products of a,b,c,d are not equal to 1 and such that non-real parameters occur in complex conjugate pairs, then the Askey-Wilson polynomials are orthogonal with respect to a non-negative weight function on $x = \frac{1}{2}(z+z^{-1}) \in [-1,1]$. For convenience we give this orthogonality in the variable z on the unit circle, where the integrand is invariant under $z \to z^{-1}$:

$$\int_{|z|=1} R_m[z]\, R_n[z]\, w[z]\, \frac{dz}{iz} = h_n\, \delta_{m,n}, \tag{3.8}$$

where

$$w[z] = w[z;a,b,c,d;q] = \left|\frac{(z^2;q)_\infty}{(az,bz,cz,dz\,|\,q)_\infty}\right|^2, \tag{3.9}$$

$$h_0 = h_0[a,b,c,d\,|\,q] = \frac{4\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{3.10}$$

and where the explicit expression for h_n can be obtained from [18, (14.1.2)] together with (3.7).

3.4 Continuous q-ultraspherical polynomials

The continuous q-ultraspherical polynomials are a one-parameter subfamily of the Askey-Wilson polynomials. For them we will use the following standardization and notation:

$$\begin{aligned} R_n^{\beta;q}[z] = R_n^{\beta;q}\big(\tfrac12(z+z^{-1})\big) &:= R_n\big[z;q^{\frac14}\beta^{\frac12},q^{\frac34}\beta^{\frac12},-q^{\frac14}\beta^{\frac12},-q^{\frac34}\beta^{\frac12}\,|\,q\big] \\ &= {}_4\phi_3\left(\begin{matrix} q^{-n},\beta^2q^{n+1},q^{\frac14}\beta^{\frac12}z,q^{\frac14}\beta^{\frac12}z^{-1} \\ \beta q,-\beta q^{\frac12},-\beta q\end{matrix};q,q\right). \end{aligned} \tag{3.11}$$

The polynomials (3.11) are related to the continuous q-ultraspherical polynomials in usual notation [18, §14.10.1] by

$$R_n^{\beta;q}(x) = q^{\frac14 n}\beta^{\frac12 n}\,\frac{(q;q)_n}{(q\beta^2;q)_n}\,C_n\big(x;q^{\frac12}\beta\,|\,q\big).$$

The continuous q-ultraspherical polynomials with $\beta = q^\alpha$ tend to the ultraspherical polynomials (3.2) as $q\uparrow 1$:

$$\lim_{q\uparrow 1} R_n^{q^\alpha;q}(x) = R_n^\alpha(x).$$

In view of [13, (3.10.13)] we can represent $R_n^{\beta;q}$ by a different q-hypergeometric expression:

$$R_n^{\beta;q}[z] = {}_4\phi_3\left(\begin{matrix} q^{-\frac12 n},q^{\frac12 n+\frac12}\beta,q^{\frac14}\beta^{\frac12}z,q^{\frac14}\beta^{\frac12}z^{-1} \\ -q^{\frac12}\beta,(q\beta)^{\frac12},-(q\beta)^{\frac12}\end{matrix};q^{\frac12},q^{\frac12}\right). \tag{3.12}$$

In particular, for $m,n = 0,1,2,\ldots$,

$$R_n^{\beta;q}\big[q^{-\frac12 m-\frac14}\beta^{-\frac12}\big] = {}_4\phi_3\left(\begin{matrix} q^{-\frac12 n},q^{\frac12 n+\frac12}\beta,q^{-\frac12 m},q^{\frac12 m+\frac12}\beta \\ -q^{\frac12}\beta,(q\beta)^{\frac12},-(q\beta)^{\frac12}\end{matrix};q^{\frac12},q^{\frac12}\right).$$

Hence we have the *duality*

$$R_n^{\beta;q}\big[q^{-\frac12 m-\frac14}\beta^{-\frac12}\big]=R_m^{\beta;q}\big[q^{-\frac12 n-\frac14}\beta^{-\frac12}\big]\qquad(m,n=0,1,2,\ldots).\tag{3.13}$$

Note the special value

$$R_n^{\beta;q}\big[q^{\frac14}\beta^{\frac12}\big]=1$$

and the coefficient of the term of highest degree

$$R_n^{\beta;q}(x)=2^n(q^{\frac12}\beta)^{\frac12 n}\,\frac{(q^{\frac12}\beta;q)_n}{(q\beta^2;q)_n}\,x^n+\text{terms of lower degree.}\tag{3.14}$$

For $0<\beta<q^{-\frac12}$ the polynomials $R_n^{\beta;q}(x)$ are orthogonal on $[-1,1]$ with respect to the even weight function

$$w_{\beta,q}(x):=(1-x^2)^{-\frac12}\left|\frac{(e^{2i\theta};q)_\infty}{(q^{\frac12}\beta e^{2i\theta};q)_\infty}\right|^2,\quad x=\cos\theta,\tag{3.15}$$

see [18, (14.10.18)]. This weight function satisfies the recurrence

$$\begin{aligned}\frac{w_{q\beta,q}(x)}{w_{\beta,q}(x)}&=\big(1+q^{\frac12}\beta\big)^2-4q^{\frac12}\beta x^2\\&=4q^{\frac12}\beta\big(a^2-x^2\big),\qquad a=\tfrac12\big(q^{\frac14}\beta^{\frac12}+q^{-\frac14}\beta^{-\frac12}\big).\end{aligned}\tag{3.16}$$

We will need the difference formula

$$\begin{aligned}R_n^{\beta;q}(x)-R_{n-2}^{\beta;q}(x)&=\frac{4q^{-\frac12 n+\frac32}\beta}{(1+q^{\frac12}\beta)(1+q\beta)}\,\frac{1-q^{n-\frac12}\beta}{1-q\beta}\\&\quad\times\Big(x^2-\big(\tfrac12(q^{\frac14}\beta^{\frac12}+q^{-\frac14}\beta^{-\frac12})\big)^2\Big)R_{n-2}^{q\beta;q}(x)\qquad(n\ge2).\end{aligned}\tag{3.17}$$

Proof of (3.17)**.** More generally, let $w(x)=w(-x)$ be an even weight function on $[-1,1]$, let $p_n(x)=k_nx^n+\cdots$ be orthogonal polynomials on $[-1,1]$ with respect to the weight function $w(x)$, and let $q_n(x)=k_n'x^n+\cdots$ be orthogonal polynomials on $[-1,1]$ with respect to the weight function $w(x)(a^2-x^2)$ $(a\ge1)$. Assume that p_n and q_n are normalized by $p_n(a)=1=q_n(a)$. Let $n\ge2$. Then $p_n(x)-p_{n-2}(x)$ vanishes for $x=\pm a$. Hence $(p_n(x)-p_{n-2}(x))/(x^2-a^2)$ is a polynomial of degree $n-2$. It is immediately seen that x^k $(k<n-2)$ is orthogonal to this polynomial with respect to the weight function $w(x)(a^2-x^2)$ on $[-1,1]$. We conclude that

$$p_n(x)-p_{n-2}(x)=\frac{k_n}{k_{n-2}'}\,(x^2-a^2)q_{n-2}(x)\qquad(n\ge2).$$

Now specialize to the weight function (3.15) and use (3.16) and (3.14). $\square$

3.5 *q*-Racah polynomials

We will consider *q-Racah polynomials* [18, §14.2]

$$R_n(q^{-x}+\gamma\delta q^{x+1};\alpha,\beta,\gamma,\delta\,|\,q) := {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+1}\alpha\beta, q^{-x}, q^{x+1}\gamma\delta \\ q\alpha, q\beta\delta, q\gamma \end{matrix}; q, q\right) \tag{3.18}$$

for $\gamma = q^{-N-1}$, where $N \in \{1, 2, \ldots\}$, and for $n \in \{0, 1, \ldots, N\}$. They are discrete cases of the Askey-Wilson polynomials (3.6). Note that they are notated with round brackets in (3.18), while Askey-Wilson polynomials in (3.6) have straight brackets. The polynomials (3.18) are orthogonal polynomials on the finite q-quadratic set $\{q^{-x}+\gamma\delta q^{x+1} \mid x \in \{0, 1, \ldots, N\}\}$:

$$\sum_{x=0}^{N}(R_m R_n)(q^{-x}+\gamma\delta q^{x+1};\alpha,\beta,\gamma,\delta\,|\,q)\,w_{\alpha,\beta,\gamma,\delta;q}(x) = h_{n;\alpha,\beta,\gamma,\delta;q}\,\delta_{m,n} \tag{3.19}$$

with

$$w_{\alpha,\beta,\gamma,\delta;q}(x) := \frac{1-\gamma\delta q^{2x+1}}{(\alpha\beta q)^x(1-\gamma\delta q)}\,\frac{(\alpha q, \beta\delta q, \gamma q, \gamma\delta q; q)_x}{(q, \alpha^{-1}\gamma\delta q, \beta^{-1}\gamma q, \delta q; q)_x}\,, \tag{3.20}$$

$$\frac{h_{n;\alpha,\beta,\gamma,\delta;q}}{h_{0;\alpha,\beta,\gamma,\delta;q}} := \frac{(1-\alpha\beta q)(q\gamma\delta)^n}{1-\alpha\beta q^{2n+1}}\,\frac{(q, q\beta, q\alpha\beta\gamma^{-1}, q\alpha\delta^{-1}; q)_n}{(q\alpha, q\alpha\beta, q\gamma, q\beta\delta; q)_n}\,, \tag{3.21}$$

$$h_{0;\alpha,\beta,\gamma,\delta;q} := \sum_{x=0}^{N} w_{\alpha,\beta,\gamma,\delta;q}(x) = \frac{(q^2\alpha\beta, \delta^{-1}; q)_N}{(q\alpha\delta^{-1}, q\beta; q)_N} \qquad (\gamma = q^{-N-1}). \tag{3.22}$$

Clearly $R_n(1+q^{-N}\delta;\alpha,\beta,q^{-N-1},\delta\,|\,q) = 1$ while, by (3.18) and the q-Saalschütz formula [13, (1.7.2)], we can evaluate the q-Racah polynomial for $x = N$:

$$R_n(q^{-N}+\delta;\alpha,\beta,q^{-N-1},\delta\,|\,q) = \frac{(q\beta, q\alpha\delta^{-1}; q)_n}{(q\alpha, q\beta\delta; q)_n}\,\delta^n. \tag{3.23}$$

The backward shift operator equation [18, (14.2.10)] can be rewritten as

$$\begin{aligned} &w_{\alpha,\beta,\gamma,\delta;q}(x) R_n\big(q^{-x}+\gamma\delta q^{x+1};\alpha,\beta,\gamma,\delta\,|\,q\big) \\ &= \frac{1-q^2\gamma\delta}{q^{-x}-\gamma\delta q^{x+2}}\,w_{q\alpha,q\beta,q\gamma,\delta;q}(x)\,R_{n-1}\big(q^{-x}+\gamma\delta q^{x+2};q\alpha,q\beta,q\gamma,\delta\,|\,q\big) \\ &- \frac{1-q^2\gamma\delta}{q^{-x+1}-\gamma\delta q^{x+1}}\,w_{q\alpha,q\beta,q\gamma,\delta;q}(x-1)\,R_{n-1}\big(q^{-x+1}+\gamma\delta q^{x+1};q\alpha,q\beta,q\gamma,\delta\,|\,q\big). \end{aligned} \tag{3.24}$$

This holds for $x = 1, \ldots, N$ while for $x = 0$ (3.24) remains true if we put the second term on the right equal to 0. In the case $x = N$ the first term on the right is equal to zero because of (3.20), and the identity (3.24) can be checked by using (3.20) and (3.23).

Hence, for a function f on $\{0,1,\ldots,N\}$ we have

$$\sum_{x=0}^{N} w_{\alpha,\beta,\gamma,\delta;q}(x) R_n\big(q^{-x}+\gamma\delta q^{x+1};\alpha,\beta,\gamma,\delta\,|\,q\big)\,f(x) = \sum_{x=0}^{N-1} \frac{1-q^2\gamma\delta}{q^{-x}-\gamma\delta q^{x+2}} \times w_{q\alpha,q\beta,q\gamma,\delta;q}(x)\,R_{n-1}\big(q^{-x}+\gamma\delta q^{x+2};q\alpha,q\beta,q\gamma,\delta\,|\,q\big)\,\big(f(x)-f(x+1)\big). \quad (3.25)$$

4 The addition and dual addition formula for ultraspherical polynomials

4.1 The addition formula

First we discuss the product and addition formula for ultraspherical polynomials, see [4, §9.8]. The product formula reads

$$R_n^\alpha(x)R_n^\alpha(y) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac12)\Gamma(\frac12)} \int_{-1}^{1} R_n^\alpha\Big(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t\Big)(1-t^2)^{\alpha-\frac12}\,dt, \quad (4.1)$$

where $\alpha>-\frac12$. Since $R_n^\alpha\big(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t\big)$ is a polynomial of degree n in t, it will have an expansion

$$R_n^\alpha\Big(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t\Big) = \sum_{k=0}^{n} c_k R_k^{\alpha-\frac12}(t)$$

with c_k depending on x and y, and with $c_0 = R_n^\alpha(x)R_n^\alpha(y)$. This expansion is called the *addition formula* for ultraspherical polynomials, in which the dependence of c_k on x and y turns out to have a nice factorized form:

$$R_n^\alpha\Big(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t\Big) = \sum_{k=0}^{n}\binom{n}{k}\frac{\alpha+k}{\alpha+\frac12 k}\,\frac{(n+2\alpha+1)_k(2\alpha+1)_k}{2^{2k}(\alpha+1)_k^2} \times (1-x^2)^{\frac12 k}R_{n-k}^{\alpha+k}(x)\,(1-y^2)^{\frac12 k}R_{n-k}^{\alpha+k}(y)\,R_k^{\alpha-\frac12}(t). \quad (4.2)$$

This addition formula is usually ascribed to Gegenbauer [14] (1874). However, it was already stated and proved by Allé [3] in 1865.

For a better understanding of the dual addition formula and of the q-analogue of the addition formula we rewrite the product formula (4.1) in kernel form:

$$R_n^\alpha(x)R_n^\alpha(y) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac12)\Gamma(\frac12)} \int_{xy-(1-x^2)^{\frac12}(1-y^2)^{\frac12}}^{xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}} R_n^\alpha(z) \\ \times \frac{(1-x^2-y^2-z^2+2xyz)^{\alpha-\frac12}}{\big((1-x^2)(1-y^2)\big)^\alpha}\,dz \qquad (x,y\in[-1,1],\ \alpha>-\tfrac12). \quad (4.3)$$

We can now recognize $R_n^\alpha(x)R_n^\alpha(y)$ as the term c_0 in the expansion

$$R_n^\alpha(z) = \sum_{k=0}^n c_k R_k^{\alpha-\frac12}\left(\frac{z-xy}{(1-x^2)^{\frac12}(1-y^2)^{\frac12}}\right).$$

The explicit expansion is a simple rewriting of the addition formula (4.2):

$$R_n^\alpha(z) = \sum_{k=0}^n \binom{n}{k} \frac{\alpha+k}{\alpha+\frac12 k}\,\frac{(n+2\alpha+1)_k(2\alpha+1)_k}{2^{2k}(\alpha+1)_k^2} \\ \times (1-x^2)^{\frac12 k} R_{n-k}^{\alpha+k}(x)\,(1-y^2)^{\frac12 k} R_{n-k}^{\alpha+k}(y)\,R_k^{\alpha-\frac12}\left(\frac{z-xy}{(1-x^2)^{\frac12}(1-y^2)^{\frac12}}\right). \quad (4.4)$$

4.2 The dual addition formula

The linearization formula for Gegenbauer polynomials, see [5, (5.7)], can be written as

$$R_\ell^\alpha(x)R_m^\alpha(x) = \frac{\ell!\,m!}{(2\alpha+1)_\ell(2\alpha+1)_m} \sum_{j=0}^{\min(\ell,m)} \frac{\ell+m+\alpha+\frac12-2j}{\alpha+\frac12} \\ \times \frac{(\alpha+\frac12)_j(\alpha+\frac12)_{\ell-j}(\alpha+\frac12)_{m-j}(2\alpha+1)_{\ell+m-j}}{j!\,(\ell-j)!\,(m-j)!\,(\alpha+\frac32)_{\ell+m-j}}\,R_{\ell+m-2j}^\alpha(x). \quad (4.5)$$

From now on assume $\alpha > -\frac12$. Then the linearization coefficients in (4.5) are nonnegative (as they are in the degenerate case $\alpha = -\frac12$). We also assume, without loss of generality, that $\ell \ge m$. Quite analogous to the way that the addition formula (4.2) was suggested by the product formula (4.1) we may try to recognize the coefficient of $R_{l+m-2j}^\alpha(x)$ in (4.5) as a weight w_j such that possibly an explicit orthogonal system with respect to the weights w_j is known. We succeeded to identify w_j in [22, §4]. They turn out to be the weights of special Racah polynomials.

Indeed, formula (4.5) can be rewritten as

$$R_\ell^\alpha(x)R_m^\alpha(x)=\sum_{j=0}^m \frac{w_{\alpha-\frac12,\alpha-\frac12,-m-1,-\ell-\alpha-\frac12}(j)}{h_{0;\,\alpha-\frac12,\alpha-\frac12,-m-1,-\ell-\alpha-\frac12}}\,R_{\ell+m-2j}^\alpha(x)\quad(\ell\ge m).\tag{4.6}$$

Just substitute (3.4) and (3.5) in (4.6) and compare with (4.5). Formula (4.6) can be considered as giving the constant term of an expansion of $R_{\ell+m-2j}^{(\alpha,\alpha)}(x)$ as a function of j in terms of the following special case of Racah polynomials (3.3):

$$\begin{aligned}&R_n\big(j(j-\ell-m-\alpha-\tfrac12);\,\alpha-\tfrac12,\alpha-\tfrac12,-m-1,-\ell-\alpha-\tfrac12\big)\\ &\qquad={}_4F_3\left(\begin{matrix}-n,n+2\alpha,-j,j-\ell-m-\alpha-\frac12\\ \alpha+\frac12,-\ell,-m\end{matrix};1\right).\end{aligned}$$

The full expansion is the *dual addition formula for ultraspherical polynomials*:

$$\begin{aligned}R_{\ell+m-2j}^\alpha(x)&=\sum_{k=0}^m \frac{\alpha+k}{\alpha+\frac12 k}\,\frac{(-\ell)_k(-m)_k(2\alpha+1)_k}{2^{2k}(\alpha+1)_k^2\,k!}\,(x^2-1)^k\,R_{\ell-k}^{\alpha+k}(x)\,R_{m-k}^{\alpha+k}(x)\\ &\quad\times R_k\big(j(j-\ell-m-\alpha-\tfrac12);\,\alpha-\tfrac12,\alpha-\tfrac12,-m-1,-\ell-\alpha-\tfrac12\big),\end{aligned}\tag{4.7}$$

where $\ell\ge m$ and $j\in\{0,1,\ldots,m\}$. One should compare the dual addition formula (4.7) with the addition formula in the form (4.4). The rols of n,z,x,y in (4.4) are respectively played by $x,\ell+m-2j,\ell,m$ in (4.7).

5 The addition and dual addition formula for continuous q-ultraspherical polynomials

5.1 The addition formula

The q-analogue of the product formula for ultraspherical polynomials in its form (4.3) was given by Rahma & Verma [29, (1.20)]. It uses a different choice of parameter for the q-ultraspherical polynomials:

$$R_n[z]:=R_n^{q^{-\frac12}a^2;q}[z]=R_n\big[z;a,q^{\frac12}a,-a,-q^{\frac12}a\,|\,q\big],\tag{5.1}$$

where the most right part contains a special Askey-Wilson polynomial (3.6). Then the duality (3.13) takes the form

$$R_n\big[q^{-\frac12 m}a^{-1}\big]=R_m\big[q^{-\frac12 n}a^{-1}\big]\qquad(m,n=0,1,2,\ldots)$$

or, in terms of special Askey-Wilson polynomials,

$$R_n\big[q^{-\frac12 m}a^{-1};a,q^{\frac12}a,-a,-q^{\frac12}a\,|\,q\big] = R_m\big[q^{-\frac12 n}a^{-1};a,q^{\frac12}a,-a,-q^{\frac12}a\,|\,q\big] \qquad (m,n=0,1,2,\ldots). \tag{5.2}$$

In terms of the polynomials (5.1) and with usage of (3.9), (3.10) the Rahman-Verma product formula reads as follows:

$$R_n[u]\,R_n[v] = \int_{|z|=1} R_n[z]\,\frac{w[z;auv,au^{-1}v^{-1},auv^{-1},au^{-1}v\,|\,q]}{h_0(auv,au^{-1}v^{-1},auv^{-1},au^{-1}v\,|\,q)}\,\frac{dz}{iz} \qquad (|u|,|v|=1,\ 0<a<1).$$

This suggests an expansion

$$R_n[z] = \sum_{k=0}^{n} c_k\,R_k\big[z;auv,au^{-1}v^{-1},auv^{-1},au^{-1}v\,|\,q\big],$$

where the term c_0 equals $R_n[u]\,R_n[v]$. Indeed, [29, (1.24)] gives the addition formula

$$\begin{aligned} R_n\big[z;a,q^{\frac12}a,-a,-q^{\frac12}a\,|\,q\big] &= \sum_{k=0}^{n} \frac{(-1)^k q^{\frac12 k(k+1)}(q^{-n},a^2,q^na^4,q^{-1}a^4;q)_k}{(q,q^{\frac12}a^2,-q^{\frac12}a^2,-a^2;q)_k(q^{-1}a^4;q)_{2k}} \\ &\times u^{-k}(a^2u^2;q)_k\,R_{n-k}\big[u;q^{\frac12 k}a,q^{\frac12(k+1)}a,-q^{\frac12 k}a,-q^{\frac12(k+1)}a\,|\,q\big] \\ &\times v^{-k}(a^2v^2;q)_k\,R_{n-k}\big[v;q^{\frac12 k}a,q^{\frac12(k+1)}a,-q^{\frac12 k}a,-q^{\frac12(k+1)}a\,|\,q\big] \\ &\times R_k\big[z;auv,au^{-1}v^{-1},auv^{-1},au^{-1}v\,|\,q\big]. \end{aligned} \tag{5.3}$$

If in (4.4) we substitue $x=\frac12(u+u^{-1})$, $y=\frac12(v+v^{-1})$ and replace z by $\frac12(z+z^{-1})$ then it will be the limit cae for $q\uparrow 1$ of (5.3) with $a=q^{\frac12\alpha+\frac14}$.

5.2 The dual addition formula

As mentioned in [6, (4.18)], Rogers already gave the linearization formula for continuous q-ultraspherical polynomials in 1895. Here we refer for this formula to [4, (10.11.10)]. It can be written in notation (3.11) as

$$\begin{aligned} R_\ell^{\beta;q}(x)R_m^{\beta;q}(x) &= \frac{(q;q)_\ell(q;q)_m}{(q\beta^2;q)_\ell(q\beta^2;q)_m}\sum_{j=0}^{\min(\ell,m)} \frac{1-q^{\ell+m-2j+\frac12}\beta}{1-q^{\frac12}\beta}\,\frac{(q^{\frac12}\beta;q)_j}{(q;q)_j} \\ &\times \frac{(q^{\frac12}\beta;q)_{\ell-j}}{(q;q)_{\ell-j}}\,\frac{(q^{\frac12}\beta;q)_{m-j}}{(q;q)_{m-j}}\,\frac{(q\beta^2;q)_{\ell+m-j}}{(q^{\frac32}\beta;q)_{\ell+m-j}}\,(q^{\frac12}\beta)^j R_{\ell+m-2j}^{\beta;q}(x). \end{aligned} \tag{5.4}$$

By the earlier assumption $0<\beta<q^{-\frac12}$ the linearization coefficients in (5.4) are nonnegative.

From now on assume without loss of generality that $\ell\ge m$. Specialization of (3.20) and (3.22) gives

$$w_{\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}(j)=\frac{(q^{\frac12}\beta;q)_{\ell+m}}{(q\beta^2;q)_{\ell+m}}\,\frac{(q;q)_\ell}{(q^{\frac12}\beta;q)_\ell}\,\frac{(q;q)_m}{(q^{\frac12}\beta;q)_m}$$
$$\times\frac{1-q^{\ell+m-2j+\frac12}\beta}{1-q^{\frac12}\beta}\,\frac{(q^{\frac12}\beta;q)_j}{(q;q)_j}\,\frac{(q^{\frac12}\beta;q)_{\ell-j}}{(q;q)_{\ell-j}}\,\frac{(q^{\frac12}\beta;q)_{m-j}}{(q;q)_{m-j}}\,\frac{(q\beta^2;q)_{\ell+m-j}}{(q^{\frac32}\beta;q)_{\ell+m-j}}\,(q^{\frac12}\beta)^j$$

and

$$h_{0;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}=\frac{(q\beta^2;q)_\ell(q\beta^2;q)_m}{(q\beta^2;q)_{\ell+m}}\,\frac{(q^{\frac12}\beta;q)_{\ell+m}}{(q^{\frac12}\beta;q)_\ell(q^{\frac12}\beta;q)_m}\,. \tag{5.5}$$

The linearization formula (5.4) can now be seen to have the equivalent concise expression

$$R^{\beta;q}_\ell(x)R^{\beta;q}_m(x)=\sum_{j=0}^m\frac{w_{\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}(j)}{h_{0;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}}\,R^{\beta;q}_{\ell+m-2j}(x). \tag{5.6}$$

This identity can be considered as giving the constant term of an expansion of $R^{\beta;q}_{\ell+m-2j}(x)$ as a function of j in terms of q-Racah polynomials

$$R_k(q^{-j}+\beta^{-1}q^{j-\ell-m-\frac12};\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12}\mid q).$$

The general terms of this expansion will be obtained by evaluating the sum

$$S^{\beta;q}_{k,\ell,m}(x):=\sum_{j=0}^m w_{\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}(j)\,R^{\beta;q}_{\ell+m-2j}(x)$$
$$\times R_k(q^{-j}+\beta^{-1}q^{j-\ell-m-\frac12};\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12}\mid q), \tag{5.7}$$

where we still assume $l\ge m$ and where $k\in\{0,\dots,m\}$.

Theorem 5.1. *The sum* (5.7) *can be evaluated as*

$$S^{\beta;q}_{k,\ell,m}(x)=\frac{(q^{\frac12(\ell+m+1)}\beta)^k(\beta^{-1}q^{-\ell-m+\frac12};q)_k}{(-q^{\frac12}\beta,\pm q\beta;q)_k}\,(\pm q^{\frac14}\beta^{\frac12}z,\pm q^{\frac14}\beta^{\frac12}z^{-1};q^{\frac12})_k$$
$$\times\frac{(q^{2k+1}\beta^2;q)_{\ell-k}(q^{2k+1}\beta^2;q)_{m-k}(q^{k+\frac12}\beta;q)_{\ell+m-2k}}{(q^{2k+1}\beta^2;q)_{\ell+m-2k}(q^{k+\frac12}\beta;q)_{\ell-k}(q^{k+\frac12}\beta;q)_{m-k}}\,R^{q^k\beta;q}_{\ell-k}(x)R^{q^k\beta;q}_{m-k}(x). \tag{5.8}$$

Here we use the convention that $(\pm a;q)_n:=(a;q)_n(-a;q)_n$.

Proof In (5.7) put
$$f(j):=R^{\beta;q}_{\ell+m-2j}(x).$$
Then comparison of (5.7) with (3.25) gives

$$\begin{aligned}S^{\beta;q}_{k,\ell,m}(x)&=\sum_{j=0}^{m}w_{\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}(j)\\&\qquad\times R_k(q^{-j}+\beta^{-1}q^{j-\ell-m-\frac12};\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12}\,|\,q)\,f(j)\\&=\sum_{j=0}^{m-1}\frac{1-\beta^{-1}q^{-\ell-m+\frac12}}{q^{-j}-\beta^{-1}q^{-\ell-m+j+\frac12}}\,w_{\beta q^{\frac12},\beta q^{\frac12},q^{-m},\beta^{-1}q^{-\ell-\frac12};q}(j)\\&\qquad\times R_{k-1}(q^{-j}+\beta^{-1}q^{j-\ell-m+\frac12};\beta q^{\frac12},\beta q^{\frac12},q^{-m},\beta^{-1}q^{-\ell-\frac12}\,|\,q)\big(f(j)-f(j+1)\big).\end{aligned}$$

We can handle the factor $f(j)-f(j+1)$ in the right part above by using (3.17):

$$\begin{aligned}f(j)-f(j+1)&=R^{\beta;q}_{\ell+m-2j}(x)-R^{\beta;q}_{\ell+m-2j-2}(x)=4\beta^2q^{\frac12\ell+\frac12m+1}\\&\times\frac{q^{-j}-\beta^{-1}q^{-\ell-m+j+\frac12}}{(1+q^{\frac12}\beta)(1-q^2\beta^2)}\left(\tfrac14\big(q^{\frac14}\beta^{\frac12}+q^{-\frac14}\beta^{-\frac12}\big)^2-x^2\right)R^{q\beta;q}_{\ell+m-2j-2}(x).\end{aligned}$$

So, with $x=\frac12(z+z^{-1})$,

$$\begin{aligned}S^{\beta;q}_{k,\ell,m}(x)&=\frac{4\beta q^{-\frac12\ell-\frac12m+\frac32}(1-\beta q^{\ell+m-\frac12})}{(1+q^{\frac12}\beta)(1-q^2\beta^2)}\\&\quad\times\left(\tfrac14\big(q^{\frac14}\beta^{\frac12}+q^{-\frac14}\beta^{-\frac12}\big)^2-x^2\right)\sum_{j=0}^{m-1}w_{\beta q^{\frac12},\beta q^{\frac12},q^{-m},\beta^{-1}q^{-\ell-\frac12};q}(j)\\&\quad\times R_{k-1}(q^{-j}+\beta^{-1}q^{j-\ell-m+\frac12};\beta q^{\frac12},\beta q^{\frac12},q^{-m},\beta^{-1}q^{-\ell-\frac12}\,|\,q)R^{q\beta;q}_{\ell+m-2j-2}(x)\\&=\frac{q^{\frac12\ell+\frac12m+\frac12}\beta(1-\beta^{-1}q^{-l-m+\frac12})}{(1+q^{\frac12}\beta)(1-q^2\beta^2)}\,(1+q^{\frac14}\beta^{\frac12}z)(1-q^{\frac14}\beta^{\frac12}z)\\&\qquad\qquad\times(1+q^{\frac14}\beta^{\frac12}z^{-1})(1-q^{\frac14}\beta^{\frac12}z^{-1})S^{q\beta,q}_{k-1,\ell-1,m-1}(x).\end{aligned}$$

Iteration gives

$$\begin{aligned}S^{\beta;q}_{k,\ell,m}(x)&=\frac{(q^{\frac12(\ell+m+1)}\beta)^k(\beta^{-1}q^{-\ell-m+\frac12};q)_k}{(-q^{\frac12}\beta,\pm q\beta;q)_k}\\&\qquad\times(\pm q^{\frac14}\beta^{\frac12}z,\pm q^{\frac14}\beta^{\frac12}z^{-1};q^{\frac12})_k\,S^{q^k\beta;q}_{0,\ell-k,m-k}(x).\end{aligned}\tag{5.9}$$

By (5.7)

$$S^{\beta;q}_{0,\ell,m}(x)=h_{0;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}R^{\beta;q}_{\ell}(x)R^{\beta;q}_{m}(x).\tag{5.10}$$

Hence, by (5.5),

$$
\begin{aligned}
&S^{q^k\beta;q}_{0,\ell-k,m-k}(x) = h_{0;\beta q^{k-\frac12},\beta q^{k-\frac12},q^{k-m-1},\beta^{-1}q^{-\ell-\frac12};q} R^{q^k\beta;q}_{\ell-k}(x) R^{q^k\beta;q}_{m-k}(x)\\
&= \frac{(q^{2k+1}\beta^2;q)_{\ell-k}(q^{2k+1}\beta^2;q)_{m-k}}{(q^{2k+1}\beta^2;q)_{\ell+m-2k}} \, \frac{(q^{k+\frac12}\beta;q)_{\ell+m-2k}}{(q^{k+\frac12}\beta;q)_{\ell-k}(q^{k+\frac12}\beta;q)_{m-k}} R^{q^k\beta;q}_{\ell-k}(x) R^{q^k\beta;q}_{m-k}(x).
\end{aligned}
$$

Substitution of this last result in (5.9) yields (5.8). □

Theorem 5.2 (Dual addition formula)**.** *For $l \ge m$ and for $j \in \{0,\ldots,m\}$ there is the expansion*

$$
\begin{aligned}
R^{\beta;q}_{\ell+m-2j}(x) &= \sum_{k=0}^{m} q^{\frac12 k(k+\ell+m+2)}\beta^k \, \frac{1-\beta^2 q^{2k}}{1-\beta^2 q^k} \, \frac{(q^{-\ell},q^{-m},q\beta^2;q)_k}{(q\beta,q\beta,q;q)_k}\\
&\times \frac{\prod_{j=0}^{k-1}\left(4q^{j+\frac12}\beta x^2 - (1+q^{j+\frac12}\beta)^2\right)}{(-q^{\frac12}\beta;q^{\frac12})^2_{2k}} \, R^{q^k\beta;q}_{\ell-k}(x) R^{q^k\beta;q}_{m-k}(x)\\
&\times R_k(q^{-j}+\beta^{-1}q^{j-\ell-m-\frac12};\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12}\mid q). \qquad (5.11)
\end{aligned}
$$

Proof By (5.9) and (5.10)

$$
\begin{aligned}
S^{\beta;q}_{k,\ell,m}(x) &= \frac{(-1)^k q^{\frac12 k(k-\ell-m+1)}}{(-q^{\frac12}\beta;q)^2_k (q^2\beta^2;q^2)^2_k} \, \frac{(q\beta^2;q)_{\ell+k}(q\beta^2;q)_{m+k}(q^{\frac12}\beta;q)_{\ell+m}}{(q^{\frac12}\beta;q)_{\ell+m}(q\beta^2;q)_\ell(q\beta^2;q)_m}\\
&\times h_{0;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q} (\pm q^{\frac14}\beta^{\frac12}z, \pm q^{\frac14}\beta^{\frac12}z^{-1};q^{\frac12})_k R^{q^k\beta,q}_{\ell-k}(x) R^{q^k\beta,q}_{m-k}(x).
\end{aligned}
$$

By Fourier-q-Racah inversion we obtain

$$
\begin{aligned}
R^{\beta,q}_{\ell+m-2j}(x) &= \sum_{k=0}^{m} \frac{(-1)^k q^{\frac12 k(k-\ell-m+1)}}{(-q^{\frac12}\beta;q)^2_k (q^2\beta^2;q^2)^2_k} \, \frac{(q\beta^2;q)_{\ell+k}(q\beta^2;q)_{m+k}(q^{\frac12}\beta;q)_{\ell+m}}{(q^{\frac12}\beta;q)_{\ell+m}(q\beta^2;q)_\ell(q\beta^2;q)_m}\\
&\times \frac{h_{0;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}}{h_{k;\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12};q}} (\pm q^{\frac14}\beta^{\frac12}z, \pm q^{\frac14}\beta^{\frac12}z^{-1};q^{\frac12})_k R^{q^k\beta;q}_{\ell-k}(x) R^{q^k\beta;q}_{m-k}(x)\\
&\times R_k(q^{-j}+\beta^{-1}q^{j-\ell-m-\frac12};\beta q^{-\frac12},\beta q^{-\frac12},q^{-m-1},\beta^{-1}q^{-\ell-\frac12}\mid q).
\end{aligned}
$$

Now use (3.21). □

If we put $\beta = q^\alpha$ in (5.11) and take the limit for $q \uparrow 1$ then we arrive at the dual addition formula (4.7) for ultraspherical polynomials.

5.3 A second proof of the dual addition formula

We will now show that the addition formula (5.3) and the dual additon formula (5.11) coincide when both formulas are suitably restricted in their x or z variable. This will follow from the duality (5.2).

In (5.11) put $\beta = q^{-\frac{1}{2}}a^2$, $x = \frac{1}{2}(z + z^{-1})$ and use (5.1). Then the dual addition formula takes the form

$$\begin{aligned}
&R_{\ell+m-2j}\big[z; a, q^{\frac{1}{2}}a, -a, -q^{\frac{1}{2}}a \mid q\big] \\
&= \sum_{k=0}^{m} (-1)^k q^{\frac{1}{2}k(k+\ell+m+1)} a^{2k} \, \frac{1 - a^4 q^{2k-1}}{1 - a^4 q^{k-1}} \, \frac{(q^{-\ell}, q^{-m}, a^4; q)_k}{(q^{\frac{1}{2}}a^2, q^{\frac{1}{2}}a^2, q; q)_k} \, \frac{(a^2z^2, a^2z^{-2}; q)_k}{(-a^2; q^{\frac{1}{2}})_{2k}^2} \\
&\qquad \times R_{\ell-k}\big[z; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \mid q\big] \\
&\qquad \times R_{m-k}\big[z; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \mid q\big] \\
&\qquad \times R_k\big(q^{-j} + q^{j-\ell-m}a^{-2}; q^{-1}a^2, q^{-1}a^2, q^{-m-1}, q^{-\ell}a^{-2} \mid q\big).
\end{aligned} \tag{5.12}$$

Since both sides of (5.12) are symmetric Laurent polynomials in z, verification of the identity for $z = q^{-\frac{1}{2}n}a^{-1}$ $(n = m, m+1, m+2, \ldots)$ will settle the identity for all z. Thus put $z = q^{-\frac{1}{2}n}a^{-1}$ in (5.12) and use the duality (5.2) in the polynomials $R_{\ell+m-2j}$, $R_{\ell-k}$ and R_{m-k} occurring in (5.12). Furthermore, use (3.18) and (3.6) in order to substitute

$$\begin{aligned}
&R_k(q^{-j} + q^{j-\ell-m}a^{-2}; q^{-1}a^2, q^{-1}a^2, q^{-m-1}, q^{-\ell}a^{-2} \mid q) \\
&\quad = {}_4\phi_3\left(\begin{matrix} q^{-k}, q^{k-1}a^k, q^{-j}, q^{j-\ell-m}a^{-2} \\ a^2, q^{-\ell}, q^{-m} \end{matrix}; q, q\right) \\
&\quad = R_k\big[q^{-\frac{1}{2}(\ell+m-2j)}a^{-1}; q^{-\frac{1}{2}(\ell+m)}a^{-1}, q^{\frac{1}{2}(\ell+m)}a^3, q^{\frac{1}{2}(\ell-m)}a, q^{\frac{1}{2}(m-\ell)}a \mid q\big].
\end{aligned}$$

We obtain

$$\begin{aligned}
&R_n\big[q^{-\frac{1}{2}(\ell+m-2j)}a^{-1}; a, q^{\frac{1}{2}}a, -a, -q^{\frac{1}{2}}a \mid q\big] \\
&\quad = \sum_{k=0}^{n} (-1)^k q^{\frac{1}{2}k(k+\ell+m+1)} a^{2k} \, \frac{1 - a^4 q^{2k-1}}{1 - a^4 q^{k-1}} \, \frac{(q^{-\ell}, q^{-m}, a^4; q)_k}{(q^{\frac{1}{2}}a^2, q^{\frac{1}{2}}a^2, q; q)_k} \, \frac{(q^{-n}, q^n a^4; q)_k}{(-a^2; q^{\frac{1}{2}})_{2k}^2} \\
&\qquad \times R_{n-k}\big[q^{-\frac{1}{2}\ell}a^{-1}; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \mid q\big] \\
&\qquad \times R_{n-k}\big[q^{-\frac{1}{2}m}a^{-1}; q^{\frac{1}{2}k}a, q^{\frac{1}{2}(k+1)}a, -q^{\frac{1}{2}k}a, -q^{\frac{1}{2}(k+1)}a \mid q\big] \\
&\qquad \times R_k\big[q^{-\frac{1}{2}(\ell+m-2j)}a^{-1}; q^{-\frac{1}{2}(\ell+m)}a^{-1}, q^{\frac{1}{2}(\ell+m)}a^3, q^{\frac{1}{2}(\ell-m)}a, q^{\frac{1}{2}(m-\ell)}a \mid q\big].
\end{aligned} \tag{5.13}$$

Because of the factor $(q^{-m}; q)_k$ on the right-hand side and since $n \geq m$ there was no harm to replace m by n as the upper bound of the summation.

On the other hand, for integers j,m,ℓ such that $0\le j\le m\le \ell$ and $m\le n$, substitute $z=q^{-\frac12(\ell+m-2j)}a^{-1}$, $u=q^{-\frac12\ell}a^{-1}$, $v=q^{-\frac12 m}a^{-1}$ in (5.3) in order to obtain

$$\begin{aligned}
&R_n\big[q^{-\frac12(\ell+m-2j)}a^{-1};a,q^{\frac12}a,-a,-q^{\frac12}a\,|\,q\big]\\
&\quad=\sum_{k=0}^n\frac{(-1)^kq^{\frac12k(k+\ell+m+1)}a^{2k}(q^{-n},q^{-\ell},q^{-m},a^2,q^na^4,q^{-1}a^4;q)_k}{(q,q^{\frac12}a^2,-q^{\frac12}a^2,-a^2;q)_k(q^{-1}a^4;q)_{2k}}\\
&\qquad\times R_{n-k}\big[q^{-\frac12\ell}a^{-1};q^{\frac12k}a,q^{\frac12(k+1)}a,-q^{\frac12k}a,-q^{\frac12(k+1)}a\,|\,q\big]\\
&\qquad\times R_{n-k}\big[q^{-\frac12m}a^{-1};q^{\frac12k}a,q^{\frac12(k+1)}a,-q^{\frac12k}a,-q^{\frac12(k+1)}a\,|\,q\big]\\
&\qquad\times R_k\big[q^{-\frac12(\ell+m-2j)}a^{-1};q^{-\frac12(\ell+m)}a^{-1},q^{\frac12(\ell+m)}a^3,q^{\frac12(\ell-m)}a,q^{\frac12(m-\ell)}a\,|\,q\big].
\end{aligned}\tag{5.14}$$

An easy computation shows that (5.13) can be rewritten as (5.14). Thus we have shown that the addition formula (5.3) implies the dual addition formula (5.11).

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Y–BESSEL SAMPLING SERIES OF $L^2(\Omega)$ STOCHASTIC PROCESSES

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Abstract: An irregularly spaced generalization of the Whittaker–Kotel'nikov–Sannon (WKS) sampling theorem in which the deterministic signal (function) represented in the form of a Hankel–transform *via* J_ν, I_ν, Y_ν kernel function is sampled exactly at the at the zeros of Bessel function of the first kind, at the zeros of the modified Bessel function of the first kind or at the zeros of the Bessel function of the second kind Y_ν we call J, I, Y–Bessel sampling, respectively.

The stochastic signals (Piranashvili–type L_2–processes) possessing correlation function representable also in the form of a Hankel–transform integral *via* J_ν, I_ν, Y_ν kernel functions permit mean–square and almost sure P sense Bessel sampling restoration, [14]. These results are presented in this exposure.

Keywords and Phrases: Whittaker–Kotel'nikov–Shannon sampling theorem, irregular sampling, Bessel sampling, Piranashvili L^2–stochastic process, covariance function, spectral representation, Hankel–transform, mean-square sampling restoration, almost sure P restoration

Mathematics subject Classification: 33C10, 41A58, 60G12, 94A20.

1. Brief invitation to sampling series

Let the signal f (either deterministic function or stochastic process) be expressible in a domain D in the form

$$f(x) = \sum_{n\in\mathbb{Z}} f(\lambda_n) S_n(x), \qquad x \in D\,,$$

where $(\lambda_n) \subseteq D$ and $S_n(x) \equiv S_n(x, \lambda_n)$ is some convenient function class. This kind representation is the *sampling series* (of f), since f is restored by its sampled values in the discrete subset $(\lambda_n) \subset D$ by (in general) an infinite linear combination.

The convergence of sampling series is either absolute, pointwise, uniform (deterministic signals) considered in different norms or is in the mean–square, α–mean sense of in almost sure P sense (with probability 1) used.

Te following simple example illustrates the sampling: When $f(x) = kx+n$, chose two x values so, that $\lambda_0 = 0, \lambda_1 = 1$. In turn $kx+n = n(1-x)+(k+n)x$, and the deduced sampling restoration formula reads

$$f(x) = f(0)(1-x) + f(1)x\,,$$

so that [8, p. 2, Eq. (1.1.1)]

$$S_0(0) = 1-x, \quad S_1(1) = x\,.$$

However, the celebrated Cauchy integral formula

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta,$$

where γ is a suitable positively oriented closed integration path enclosing the point z, is another kind of "sampling restoration" formula, but here the sampling set in uncountably infinite.

A short historical development of sampling we begin with the identity

$$\begin{aligned} f(\lambda_0) &+ (x-\lambda_0)f(\lambda_0,\lambda_1) + \cdots + (x-\lambda_0)\cdots(x-\lambda_{N-1})f(\lambda_0,\cdots,\lambda_N) \\ &\equiv \sum_{j=0}^N f(\lambda_j)\, \frac{G_N(x)}{G_N'(\lambda_j)(x-\lambda_j)}\,, \end{aligned}$$

where

$$f(\lambda_0,\lambda_1) = \frac{f(\lambda_1)-f(\lambda_0)}{\lambda_1-\lambda_0}, \quad f(\lambda_0,\lambda_1,\lambda_2) = \frac{f(\lambda_1,\lambda_2)-f(\lambda_0,\lambda_1))}{\lambda_2-\lambda_0}, \dots\,,$$

and

$$G_N(x) = \prod_{j=0}^N \Big(1-\frac{x}{\lambda_j}\Big)\,,$$

we call today *Lagrange interpolation formula* [1].

[1] Lagrange has presented this result on his classes delivered at École Normal Paris during his "Elementary mathematics course" around 1795–1976.

Periodical interoplation formula

$$p(x) = \sum_{|j|\le N} c_j \mathrm{e}^{\mathrm{i}jx}\,,$$

holds for periodical signals using finite sampling sum

$$p(x) = \frac{1}{2N+1}\sum_{j=0}^{2N} p\Big(\frac{2\pi j}{2N+1}\Big)\,\frac{\sin\left(\frac{(2N+1)x}{2} - \pi j\right)}{\sin\left(\frac{x}{2} - \frac{\pi j}{2N+1}\right)};$$

This Cauchy's result was published in [4]. Gauß reported about some similar formula around 1805.

Letting

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} g(t)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t\,,$$

the related Fourier expansion reads

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\,\mathrm{sinc}(x-n), \qquad \mathrm{sinc}(u) = \begin{cases} \dfrac{\sin u}{u} & u\ne 0 \\ 1 & u = 0 \end{cases}, \tag{1}$$

where sinc denotes the *sinus cardinalis*; the notation was introduced by Woodward in 1953, [7, 8]. Here $f(n)$ is the Fourier coefficient of g. For all $m, n \in \mathbb{Z}$ it is $\mathrm{sinc}(m-n) = \delta_{mn}$, which illustrates the interpolation property of sinc kernel.

The scaling of (1) is the famous Whittaker–Kotel'nikov–Shannon (WKS) sampling series expansion theorem:

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(n\,\frac{\pi}{w}\right)\,\mathrm{sinc}\,(wx - n\pi)\,, \qquad w > 0. \tag{2}$$

Here $w > 0$ stands for the so–called *bandwidth*. The *cardinal series*

$$\sum_{n\in\mathbb{Z}} a_n\,\mathrm{sinc}(wx - n\pi)$$

absolutely converges for all $x \in \mathbb{C}$ if and only if [16]

$$\sum_{n\in\mathbb{Z}}{}' \left|\frac{a_n}{n}\right| < \infty\,,$$

we also refer to [12, 17, 18].

2. Preparation and results needed

In the last few years one of my main interests is the so-called Bessel-sampling reconstruction series which belong to the class of generalizations of Whittaker–Kotel'nikov–Shannon sampling series. By these series we sample the input signal (function or stochastic process) exactly at the zeros of Bessel function of the first kind (J–Bessel sampling), at the zeros of the modified Bessel function of the first kind (I–Bessel sampling) or at the zeros of the Bessel function of the second kind Y_ν, which we call *Y–Bessel sampling procedure*.

Here deterministic signals (functions) restoration and stochastic L_2–processes signals sampling *digital* $\to$ *analog* reconstruction, when the correlation function possesses J_ν, Y_ν kernel function Hankel–transform are delivered [14]. The results of [14] belong to the irregular sampling series reconstruction method and consist among others from the stochastic signals generalizations and implementations of certain J–Bessel deterministic findings by Zayed [20] and Knockaert [10], while the Y–Bessel deterministic sampling results by Jankov Maširević *et al.* [9]. The *in medio* and almost sure convergence results are in the first plane. The main derivation tools are the Karhunen–Cramér theorem's generalization by Piranašvili, the integral forms of the Bessel (modified Bessel) functions and certain their appropriate properties.

Now, certain summation of related series will be presented which originate back to the sampling series for Bessel and modified Bessel functions Y_ν, I_ν and K_ν. Related truncation error upper bounds are given in the case of Y–Bessel sampling [2]

The main tools are Kramer's sampling theorem and diverse convenient properties of Bessel functions, when the sampling set (λ_n) coincides with the set of zeros either for Y_ν, I_ν or for K_ν.

First I'll recall Kramer's cornerstone theorem.

[2] I am grateful to Professor T. Koornwinder who draw my attention during the discussion after my talk at the International Conference on Special Functions & Applications (ICSFA-2017) (XVIth Annual Conference of Society for Special Functions and their Applications) held November 2–4 2017 in Bikaner, Rajasthan, India, to the fact that the WKS sampling theorem (2) is actually a kind of J–Bessel sampling series being

$$\operatorname{sinc}(x) = \sqrt{\frac{\pi x}{2}}\, J_{1/2}(x).$$

By this intervention the WKS formula (2) can be re–written into

$$f(x) = \sqrt{\frac{\pi}{2}} \sum_{n\in\mathbb{Z}} f\left(n\,\frac{\pi}{w}\right) \sqrt{wx - n\pi}\; J_{\frac12}(wx - n\pi)\,.$$

Theorem A. [11] *Let $K(x,t) \in L^2(I)$ as the function of x, $t \in \mathbb{R}$, where $I = [a,b]$ and let $E = \{t_k\}_{k\in\mathbb{Z}}$ be a countable real set, for which $\{K(x,t_k)\}_{k\in\mathbb{Z}}$ complete orthogonal in $L^2(I)$. If*

$$f(t) = \int_a^b g(x)K(x,t)\,\mathrm{d}x, \tag{3}$$

for some $g \in L^2[a,b]$ then

$$f(t) = \sum_{k\in\mathbb{Z}} f(t_k)S_k^\star(t),$$

where

$$S_k^\star(t) = \frac{\displaystyle\int_a^b K(x,t)\overline{K(x,t_k)}\,\mathrm{d}x}{\displaystyle\int_a^b |K(x,t_k)|^2\,\mathrm{d}x}\,.$$

In general every f representable in the previous form (3) we call *band–limited on* $[a,b]$.

Theorem B. [19, p. 701] *When for a convenient $g \in L^2(0,a)$ we have*

$$f(t) = \int_0^a g(x)\cos(xt)\,\mathrm{d}x,$$

then

$$f(t) = \sum_{k\in\mathbb{Z}} f\left(\left(k+\frac{1}{2}\right)\frac{\pi}{a}\right)\frac{\sin(at-(k+\frac{1}{2})\pi)}{at-(k+\frac{1}{2})\pi},$$

uniformly on all compact t–subregion of $\mathbb{R}$.

In the same paper using [5, p. 716, Eq. 6.681.1]

$$\int_0^\pi J_{2\nu}\left(2b\cos\frac{x}{2}\right)\cos(tx)\,\mathrm{d}x = \pi J_{\nu+t}(b)J_{\nu-t}(b),$$

it is proved a Bessel–sampling series result [19, p. 703, Eq. (2.8)]

$$J_{\nu+t}(b)J_{\nu-t}(b) = \sum_{k\in\mathbb{Z}} J_{\nu+k+\frac{1}{2}}(b)J_{\nu-k-\frac{1}{2}}(b)\,\mathrm{sinc}(t-k-\tfrac{1}{2})\,, \qquad \Re\{\nu\} > -\tfrac{1}{2}.$$

To ensure that such series expansion makes sense we remind that for real ν, $J_\nu(z)$ and $Y_\nu(z)$ possess countable many simple zeros except the $z=0$, which is the famous von Lommel's theorem, see [1, p. 370].

3. The first set of Bessel–sampling results

By convention we use in the sequel that for $\nu \geq 0$ the notations $j_{\nu,k}$, $y_{\nu,k}$ stand for the kth positive nil of the Bessel functions J_ν, Y_ν of the first and second kind of the order ν, respectively. Now, we are ready to state the first result.

Theorem 1. [9, p. 80, Theorem 1] *For all $\nu > -\frac{1}{4}$ we have*

$$I_{\nu+t}(b)I_{\nu-t}(b) = \sum_{k\in\mathbb{Z}} I_{\nu+k+\frac{1}{2}}(b)I_{\nu-k-\frac{1}{2}}(b)\,\mathrm{sinc}(t-k-\tfrac{1}{2}),$$

uniformly on all compact real t–domains.

Moreover

$$I_\nu^2(b) = \frac{4}{\pi}\sum_{k\geq 0}\frac{(-1)^k}{2k+1}\,I_{\nu+(k+\frac{1}{2})}(b)I_{\nu-(k+\frac{1}{2})}(b),$$

and

$$\sinh^2 b = 2b\sum_{k\geq 0}\frac{(-1)^k}{2k+1}\,I_{k+1}(b)I_{-k}(b)\,.$$

Theorem 2. [9, p. 80, Theorem 2] *For $|\nu| < \frac{1}{2}$ we have*

$$K_\nu(b) = \frac{2}{\pi}\sum_{k\in\mathbb{Z}} K_{2k+1}(b)\frac{\cos\pi(\frac{\nu}{2}-k)}{2k+1-\nu}.$$

Moreover

$$\mathrm{e}^{-b} = \frac{16\sqrt{b}}{\sqrt{\pi^3}}\sum_{k\geq 0}\frac{(-1)^k(2k+1)}{(4k+1)\,(4k+3)}K_{2k+1}(b)\,, \tag{4}$$

where (4) *holds when $b\to\infty$.*

Theorem 3. [9, p. 81, Theorem 3] *There holds true*

$$\begin{aligned} I_\nu(t) = 2\,(2t)^\nu\cosh t &\sum_{k\in\mathbb{Z}}\frac{(-1)^k(\pi+2k\pi)^{1-\nu}}{(\pi+2k\pi)^2+4t^2} \\ &\times J_\nu\left(\pi\left(k+\tfrac{1}{2}\right)\right), \quad t\in\mathbb{R};\ \Re\{\nu\}>0\,. \end{aligned}$$

Also we deduce that

$$\tanh(\pi t) = \frac{8t}{\pi}\sum_{k\geq 1}\frac{1}{(2k-1)^2+4t^2}, \qquad t\in\mathbb{R}\,.$$

It should be mentioned that based on the definition

$$K_\nu(t) = \frac{\pi}{2\sin(\nu\pi)}\left(I_{-\nu}(t) - I_\nu(t)\right)\,,$$

using $\nu = \frac{1}{2}$ we conclude

$$\mathrm{e}^{-t} = \cosh t - \sinh t\,.$$

As the modified Bessel K_ν is expressible *via* I_ν and Y_ν, for integer $\nu \notin \mathbb{Z}$ we have

$$K_\nu(t) = \frac{\pi}{2}\left(\frac{(\mathrm{i}\,t)^{2\nu}\cos(\pi\nu)}{t^{2\nu}} - 1\right)\frac{I_\nu(t)}{\sin(\pi\nu)} - \frac{\pi(\mathrm{i}\,t)^\nu}{2t^\nu}Y_\nu(\mathrm{i}\,t)\,.$$

Now, the summation for Y_ν follows as well.

4. The Y–Bessel sampling series

One of the well–known properties of the Bessel functions J_ν, Y_ν and is that the zeros of two consecutive order functions interlace [2, 15]:

$$j_{\nu,k} < j_{\nu+1,k} < j_{\nu,k+1} < \cdots\,; \qquad y_{\nu,k} < y_{\nu+1,k} < y_{\nu,k+1} < \cdots\,.$$

Therefore $Y_{\nu+1}(y_{\nu,k}) \neq 0, k \in \mathbb{N}_0$ enables the

Theorem 4. [9, p. 80, Theorem 4] *Let $g \in L^2(0,a), a > 0$, and assume that*

$$f(t) = \int_0^a g(x)\sqrt{x}\,Y_\nu(tx)\,\mathrm{d}x\,.$$

Then for all $t \in \mathbb{R}$ and $\nu \in [0,1)$ there holds

$$f(t) = 2\,Y_\nu(at)\sum_{k\geq 1}\frac{y_{\nu,k}\,f(a^{-1}y_{\nu,k})}{(y_{\nu,k}^2 - a^2t^2)\,Y_{\nu+1}(y_{\nu,k})}\,. \tag{5}$$

Corollary 4.1. [9, p. 81, Corollary 4.1] *For $\Re\{\nu\} > 0, t \in \mathbb{R}$ it is*

$$Y_\nu(t)\,J_\nu(t) = 2\sum_{k\geq 1}\frac{(2t)^{\nu+1}\,Y_\nu(2t)\,J_\nu(\frac{y_{\nu,k}}{2})\,Y_\nu(\frac{y_{\nu,k}}{2})}{y_{\nu,k}^{\nu-1}(y_{\nu,k}^2 - 4t^2)\,Y_{\nu+1}(y_{\nu,k})},$$

while for $t \neq k - \frac{1}{2}, k \in \mathbb{N}$ we have

$$\tan(\pi t) = \frac{8t}{\pi}\sum_{k\geq 1}\frac{1}{(2k-1)^2 - 4t^2}\,.$$

5. Truncation error of Y–Bessel sampling series

For the sake of simplicity specify $a = 1$ in looking for the upper bound for the error made in truncating the Y–Bessel series (5) to its Nth partial sum

$$\mathscr{S}_N^Y(f;t) = 2\,Y_\nu(t) \sum_{k=1}^N \frac{y_{\nu,k}\, f(y_{\nu,k})}{(y_{\nu,k}^2 - t^2)\, Y_{\nu+1}(y_{\nu,k})}\,, \qquad t \in \mathbb{R}$$

where f is band–limited with respect to $(0,1)$. The Nth truncation error we write $\mathscr{T}_N^Y(f;t) = \big|f(t) - \mathscr{S}_N^Y(f;t)\big|$, that is

$$\mathscr{T}_N^Y(f;t) = \left|\sum_{k\ge N+1} f(y_{\nu,k})\, \frac{2\, y_{\nu,k}\, Y_\nu(t)}{(y_{\nu,k}^2 - t^2)\, Y_{\nu+1}(y_{\nu,k})}\right|.$$

When f behavior is polynomial for t being large, that is

$$|f(t)| \le A\,|t|^{-(r+1)}, \qquad A > 0, \quad r > \tfrac{1}{2}\,, \tag{6}$$

we are looking for a manageable upper bound for $\mathscr{T}_N^Y(f;t)$. For $\nu \in [0,1)$ this implies

$$\mathscr{T}_N^Y(f;t) \le 2A \sum_{k\ge N+1} \frac{|Y_\nu(t)|}{y_{\nu,k}^r |y_{\nu,k}^2 - t^2|\; |Y_{\nu+1}(y_{\nu,k})|}\,;$$

since $y_{\nu,k} > 0$ for $\nu \ge 0$, see [1, p. 370]. So the following result.

Theorem 5. [9, p. 83, Theorem 5] *Let f behaves according to* (6) *when $\nu \in [0,1)$. Then for all $t \in (0, y_{\nu,N+1})$, $A > 0, r > \frac{1}{2}$ and $N \ge 2$ we have*

$$\mathscr{T}_N^Y(f;t) < U_N^Y(t) := \frac{2AH(t)\, M_N(\nu)\, (y_{\nu,N+1})^{\frac{1}{2}-r}}{\pi^2 \min\limits_{k\ge N+1} \sqrt{y_{\nu,k}}\, |Y_{\nu+1}(y_{\nu,k})|}, \tag{7}$$

for the following Y–Bessel sampling series

$$f(t) = 2\,Y_\nu(t) \sum_{k\ge 1} \frac{y_{\nu,k}\, f(y_{\nu,k})}{(y_{\nu,k}^2 - t^2)\, Y_{\nu+1}(y_{\nu,k})}\,.$$

Here

$$H(t) = 1 + \frac{2}{\pi\nu\, t}, \qquad M_N(\nu) = \exp\left\{\left(N + \frac{1 - \pi + 2(\nu - y_{\nu,2})}{2\pi}\right)^{-1}\right\} - 1\,.$$

Moreover for all fixed $t \in (0, y_{\nu,2})$ and enough large N there holds the estimate

$$\mathscr{T}_N^Y(f;t) = \mathscr{O}\left(N^{-r-\frac{1}{2}}\right).$$

Denote

$$h(t) = \frac{Y_\nu(t)\, J_\nu(t)}{t^\nu\, Y_\nu(2t)}, \qquad \mathscr{S}_N^Y(h;t) = \sum_{k=1}^N \frac{2^{\nu+1}\, J_\nu(\frac{y_{\nu,k}}{2})\, Y_\nu(\frac{y_{\nu,k}}{2})}{y_{\nu,k}^{\nu-1}(y_{\nu,k}^2 - 4t^2)\, Y_{\nu+1}(y_{\nu,k})}\,.$$

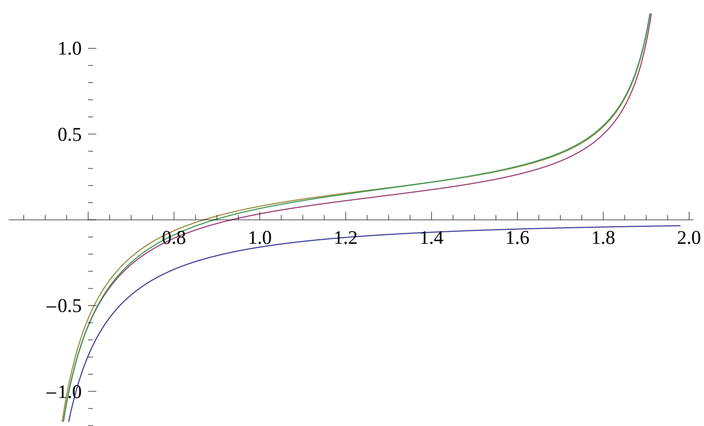

Figure 1: Approximation in Y–Bessel sampling according to Corollary 4.1. $h(t)$ – green, $\mathscr{S}_1^Y(h;t)$ – blue, $\mathscr{S}_{10}^Y(h;t)$ – violet and $\mathscr{S}_{90}^Y(h;t)$ – yellow.

The figure presents $h(t)$ and the truncated Y–Bessel partial sums $\mathscr{S}_N^Y(h;t)$, $N = 1, 10, 90$. We mention that here the t–band becomes $\big[\frac12 y_{0,1}, \frac12 y_{0,2}\big] \approx [0.446788, 1.97884]$, when $\nu = 0$.

Helms and Thomas [6, p. 324, Eq. (7)] have proved certain results for generalized Hankel–transforms. It is worth to mention that the magnitude of their truncation error upper bound behaves like $\mathscr{O}(N^{-1})$ when the input signal is bounded, [6, p. 324, Eq. (7)]; our bound (7) is superior to their for any signal which satisfies (6) with $r > \frac12$.

6. Stochastic counterpart: Piranashvili processes

Let $(\Omega, \mathfrak{A}, \mathsf{P})$ be a standard probability space and $\xi\colon T \times \Omega \mapsto \mathbb{C}$, $T \subseteq \mathbb{R}$, that is $\{\xi(t) \equiv \xi(t,\omega)\colon t \in T,\ \omega \in \Omega\}$ a *stochastic process*. Denote $L^2(\Omega, \mathfrak{A}, \mathsf{P})$ [in short $L^2(\Omega)$] the space of finite second order moment complex valued random variables over $(\Omega, \mathfrak{A}, \mathsf{P})$ with the norm $\|\cdot\|_2 := \sqrt{\mathsf{E}|\cdot|^2}$ endowed. $L^2(\Omega)$ is a Hilbert space with the scalar/inner product $\langle \xi\, \eta\rangle = \mathsf{E}\xi\, \overline{\eta}$.

The mean–square closure of the linear span $\mathscr{H}_t(\xi) := \overline{L^2\{\xi(s)\colon s \le t\}}$ which is generated by the family $\{\xi(s)\colon s \le t\}$, $t \in \mathbb{R}$. This is a subspace of $L^2(\Omega)$. When $\bigcap_{t\in\mathbb{R}} H_t(\xi) = \emptyset$, then ξ is *regular* (purely indeterministic); while $\bigcap_{t\in\mathbb{R}} H_t(\xi) = \mathscr{H}(\xi)$ means that the process is *singular* (purely deterministic) ξ.

We are focused here to a subclass of singular processes.

Consider a zero mean process $m_\xi(t) = \mathsf{E}\xi(t) = 0\,, t \in T$. The correlation function

$$B_\xi(t,s) = \mathsf{E}\xi(t)\,\overline{\xi(s)}, \qquad t,s \in T$$

and $\mathsf{D}\xi(t) := B_\xi(t,t)$ is the variance of ξ for which there holds

$$\mathsf{D}\xi(t) \le \sup_{u\in\mathbb{R}} B_\xi^2(u,u) := \mathfrak{B}_\xi^2 < \infty\,,$$

being the considered process with finite second moment.

Consider non–stationary zero mean $L^2(\Omega)$–process $\xi\colon \mathbb{R}\times\Omega \mapsto \mathbb{R}$ having covariance

$$B(t,s) = \int_\Lambda\int_\Lambda f(t,\lambda) f^*(s,\mu) F_\xi(\mathrm{d}\lambda,\mathrm{d}\mu), \tag{8}$$

in which for $\lambda \in \Lambda$, the function $f(\cdot,\lambda)$ can be analytically continued to a complex, exponentially bounded kernel function and the spectral distribution function F_ξ turns out to be a positive definite measure in the $\mathbb{R}^2$ plane so, that the total variation

$$\|F_\xi\|(\Lambda,\Lambda) = \int_\Lambda\int_\Lambda \big|F_\xi(\mathrm{d}\lambda,\mathrm{d}\mu)\big| = V_{F_\xi} < \infty\,,$$

the constant V_{F_ξ} is also called the *Vitali variation* [22, p. 153]. The type of the trajectory of the process $\xi(t) \equiv \xi(t,\omega_0)$ and $f(t,\lambda)$ coincide by [3, p. 441, Theorem 4], [13, Theorem 3]. Then, by the Karhunen–Cramèr theorem , the spectral representation of the process $\xi(t)$ is the following Lebesgue integral

$$\xi(t) = \int_\Lambda f(t,\lambda) Z_\xi(\mathrm{d}\lambda)\,, \tag{9}$$

where

$$F_\xi(S_1,S_2) = \mathsf{E} Z_\xi(S_1) Z_\xi^*(S_2) \qquad S_1, S_2 \subseteq \sigma(\Lambda).$$

Such processes ξ we call *Piranashvili processes.*

Special cases of Piranashvili processes are [21, 22]:

1. when $F_\xi(x,y) = \delta_{xy} F_\xi(x)$ in (8) then we get the Karhunen – representation

$$B(t,s) = \int_\Lambda f(t,\lambda) f^*(s,\lambda) F_\xi(\mathrm{d}\lambda).$$

2. When $f(t,\lambda) = \mathrm{e}^{\mathrm{i}t\lambda}$ the Loève process is in charge:

$$B(t,s) = \int_\Lambda \int_\Lambda \mathrm{e}^{\mathrm{i}(t\lambda - s\mu)} F_\xi(\mathrm{d}\lambda, \mathrm{d}\mu).$$

3. The Karhunen process with the Fourier kernel $f(t,\lambda) = \mathrm{e}^{\mathrm{i}t\lambda}$ equipped becomes the Hinčin (wide sense stationary) process, for which

$$B(\tau) = \int_\Lambda e^{i\tau\lambda} F_\xi(\mathrm{d}\lambda), \qquad \tau = t - s.$$

4. Finally in the case of $\Lambda = [-w, w], w > 0$, the band–limited process is the result. In that case the WKS theorem reads:

$$\xi(t) = \sum_{k\in\mathbb{Z}} \xi\Big(\frac{\pi}{w}\, k\Big) \,\frac{\sin(wt - k\pi)}{wt - k\pi}\,, \tag{10}$$

where the convergence is uniform in any t–compact in $\mathbb{R}$ in the mean–square sense. Moreover, the last series (10) converges also in almost sure P sense too [3, p. 443].

Now we apply Theorem 4. Denoting

$$\begin{aligned} \mathscr{S}_N^J(\mathfrak{G};t) &:= \frac{2\,J_\nu(bt)}{b^\nu\, t^\nu} \sum_{k=1}^N \frac{j_{\nu,k}^{\nu+1}\, \mathfrak{G}(b^{-1}j_{\nu,k})}{(b^2t^2 - j_{\nu,k}^2)\, J_\nu'(j_{\nu,k})} \\ \mathscr{S}_N^Y(\mathfrak{G};t) &:= 2\,Y_\nu(bt) \sum_{k=1}^N \frac{y_{\nu,k}\, \mathfrak{G}(b^{-1}y_{\nu,k})}{(y_{\nu,k}^2 - b^2t^2)\, Y_{\nu+1}(y_{\nu,k})}\,, \end{aligned}$$

for the $L^2(0,b)$–band-limited signal f or process ξ: $\mathfrak{G} \in \{f, \xi\}$, and

$$\begin{aligned} \mathscr{T}_N^J(\mathfrak{G};t) &:= \mathfrak{G}(t) - \mathscr{S}_N^J(\mathfrak{G};t) = \frac{2\,J_\nu(bt)}{b^\nu\, t^\nu} \sum_{k\ge N+1} \frac{j_{\nu,k}^{\nu+1}\, \xi(b^{-1}j_{\nu,k})}{(b^2t^2 - j_{\nu,k}^2)\, J_\nu'(j_{\nu,k})} \\ \mathscr{T}_N^Y(\mathfrak{G};t) &:= \mathfrak{G}(t) - \mathscr{S}_N^Y(\mathfrak{G};t) = 2\,Y_\nu(bt) \sum_{k\ge N+1} \frac{y_{\nu,k}\, \xi(b^{-1}y_{\nu,k})}{(y_{\nu,k}^2 - b^2t^2)\, Y_{\nu+1}(y_{\nu,k})}\,. \end{aligned}$$

Our main goal is to give an useful upper bounds for the truncation error

$$\begin{aligned} \Delta_N^J(\xi;t) &= \mathsf{E}\big|\xi(t) - \mathscr{S}_N^J(\xi;t)\big|^2 = \mathsf{E}\big|\mathscr{T}_N^J(\xi;t)\big|^2, \\ \Delta_N^Y(\xi;t) &= \mathsf{E}\big|\xi(t) - \mathscr{S}_N^Y(\xi;t)\big|^2 = \mathsf{E}\big|\mathscr{T}_N^Y(\xi;t)\big|^2\,. \end{aligned}$$

To achieve this goal we need the spectral representations for the involved stochastic signals and their series expansions. These constitute the next result.

Theorem 6. [14, p. 13, Theorems 1, 2] *Let $\xi(t), t \in T \subseteq \mathbb{R}$ be a Piranashvili process, that is*

$$\xi(t) = \int_\Lambda f(t,\lambda)\, Z_\xi(\mathrm{d}\lambda)\,,$$

where $f(t,\lambda) \in L^2(0,b)$ with respect to t for all $\lambda \in \Lambda$. Then we have in the means–square

$$\begin{aligned}
\mathscr{S}_N^{\mathscr{B}}(\xi;t) &= \int_\Lambda \mathscr{S}_N^{\mathscr{B}}(f;t)\, Z_\xi(\mathrm{d}\lambda), \qquad \mathscr{B} \in \{J,Y\}\\
\mathscr{T}_N^{\mathscr{B}}(\xi;t) &= \int_\Lambda \mathscr{T}_N^{\mathscr{B}}(f;t)\, Z_\xi(\mathrm{d}\lambda), \qquad \mathscr{B} \in \{J,Y\}\\
\Delta_N^{\mathscr{B}}(\xi;t) &= \int_\Lambda\int_\Lambda \mathscr{T}_N^{\mathscr{B}}(f;t)\, \overline{\mathscr{T}_N^{\mathscr{B}}(f;t)}\, F_\xi(\mathrm{d}\lambda,\mathrm{d}\mu), \qquad \mathscr{B} \in \{J,Y\}\,.
\end{aligned}$$

For the Karhunen process this turns out to be

$$\Delta_N^{\mathscr{B}}(\xi;t) = \int_\Lambda \left|\mathscr{T}_N^{\mathscr{B}}(f;t)\right|^2 F_\xi(\mathrm{d}\lambda), \qquad \mathscr{B} \in \{J,Y\}\,.$$

Observe that the function class

$$L^2(\Lambda;F_\xi) := \left\{\varphi\colon\ \int_\Lambda |\varphi|^2\, F_\xi(\mathrm{d}\lambda) < \infty\right\}$$

is also a H–space and it is isometrically isomorphic to $\mathscr{H}(\xi)$ by the correspondence $\xi(t) \leftrightarrow f(t,\lambda)$.

The special case of the Karhunen process is the Hinčin stationary process. When $\Lambda = [-w,w]$,

$$\begin{aligned}
\Delta_N^{J}(\xi;t) &= \int_{-w}^{w} \left|\mathscr{T}_N^{J}(\mathrm{e}^{\mathrm{i}t\lambda})\right|^2 F_\xi(\mathrm{d}\lambda)\\
\Delta_N^{Y}(\xi;t) &= \int_{-w}^{w} \left|\mathscr{T}_N^{Y}(\mathrm{e}^{\mathrm{i}t\lambda})\right|^2 F_\xi(\mathrm{d}\lambda)\,.
\end{aligned}$$

Theorem 7. [14, p. 14, Theorem 3] *Assume the Piranashvili process $\{\xi(t)\colon t \in \mathbb{T} \subseteq \mathbb{R}\}$ has integral representation* (9) *with kernel $f(t,\lambda) \in L^2(0,b)$ which is a Hankel–transform in J, Y–Bessel form. Then there hold*

$$\begin{aligned}
\xi(t) = \mathscr{S}^J(\xi;t) &= \frac{2\, J_\nu(bt)}{b^\nu\, t^\nu} \sum_{k\ge1} \frac{j_{\nu,k}^{\nu+1}\, \xi(b^{-1}j_{\nu,k})}{(b^2t^2 - j_{\nu,k}^2)\, J_\nu'(j_{\nu,k})}\\
\xi(t) = \mathscr{S}^Y(\xi;t) &= 2\, Y_\nu(bt) \sum_{k\ge1} \frac{y_{\nu,k}\, \xi(b^{-1}y_{\nu,k})}{(y_{\nu,k}^2 - b^2t^2)\, Y_{\nu+1}(y_{\nu,k})}\,,
\end{aligned}$$

in the mean square sense.

Next, we expose the truncation error bound results for Karhunen processes. For the sake of simplicity take $b = 1$.

Theorem 8. [14, p. 16, Theorem 4] *When the kernel function f of the related Karhunen process $\xi(t), t \in \mathbb{R}$ has polynomial decay according to* (6) *then for all $\nu \in [0,1)$, $t \in (\nu, y_{\nu,2})$, $\min\{A, r\} > 0$ and $N \geq 2$ we have*

$$\Delta_N^Y(\xi;t) \leq \frac{A^2\, V_{F_\xi}\, (\pi\nu t + 2)^2\, [(4y_{\nu,N+1} - \nu - 1)^{\frac{3}{2}} + (2\nu+3)(2\nu+5)]}{\pi^5\, \nu^2\, t^2\, y_{\nu,N+1}^{2r}[y_{\nu,N+1}^2 - (2n+3)(2n+7)]} \times \left(\exp\left\{ \left(N + \frac{1 - \pi + 2(\nu - y_{\nu,2})}{2\pi} \right)^{-1} \right\} - 1 \right)^2 ,$$

where V_{F_ξ} it the Vitali (*or total*) *variation of F_ξ. Moreover*

$$\Delta_N^Y(\xi;t) = \mathscr{O}\left(N^{-2r-\frac{5}{2}} \right) .$$

The almost sure convergence of Y–Bessel sampling series for Karhunen processes is established by the following result, which can be proved by the use of Borel–Cantelli Lemma

Theorem 9. [14, p. 16, Theorem 5]

Consider a Karhunen process $\xi(t)$ which kernel f is of polynomial decay (see (6)*). Then for all $\nu \in [0,1)$, $t \in (\nu, y_{\nu,2})$, $\min\{A, r\} > 0$ and $N \geq 2$ we have*

$$\mathsf{P}\left\{ \lim_{N\to\infty} \mathscr{S}_N^Y(\xi;t) = \xi(t) \right\} = 1 .$$

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ON ULTRA GAMMA INTEGRAL

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Abstract

A certain integral in the literature is known by the names: ultra gamma function, generalized gamma, Krätzel integral, inverse Gaussian integral, reaction-rate probability integral, Bessel integral, the unconditional density in a Bayesian structure and the Mellin convolution of a product. Thus, this integral is very important to various people in different disciplines. In this article, this integral is examined for its structural properties and then it is evaluated in computable series form. It is shown that the names, generalized gamma and ultra gamma are not appropriate for this integral. The name a Bessel integral is justifiable. This integral really belongs to the Krätzel family of integrals.

Keywords: Mellin convolution, Krätzel integral, ultra gamma function, generalized gamma function, Bessel integral, reaction-rate probability integral, inverse Gaussian integral, computable series form.

2010 Mathematics Subject Classification Numbers: 26A33, 44A35, 62E15,33C60

1. Introduction

Consider the following integrals which are given to be convergent:

$$A=\int_0^\infty \mathrm{e}^{-ax-b\phi(x)}\mathrm{d}x<\infty, B=\int_0^\infty \mathrm{e}^{-ax}\mathrm{d}x<\infty, C=\int_0^\infty \mathrm{e}^{-b\phi(x)}\mathrm{d}x<\infty. \tag{1.1}$$

For $b = 0$ in A, one has B the exponential integral and for $a = 0$ in A one has the integral in C. Is there any justification in calling A the generalized exponential integral or ultra exponential integral? For $b \neq 0$, A has no connection to an exponential integral as in B. In the same context can we call B or C a contracted form of A? If one has an integral $D = \int_0^\infty \mathrm{e}^{-ax^\delta}\mathrm{d}x < \infty$, for $a > 0, \delta > 0$ then we can say that B is a special case of D or D as a generalization of B, but A is neither a generalization or contraction of D. For $\phi(x) = x^\delta$ or $\phi(x) = x^{-\rho}$ with $\delta > 0, \rho > 0, a > 0, b > 0$ we have the forms

$$A_1 = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-ax-bx^\delta}\mathrm{d}x < \infty, \tag{1.2}$$

$$A_2 = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-ax-bx^{-\rho}}\mathrm{d}x < \infty \text{ for } \alpha > 0. \tag{1.3}$$

These two integrals in A_1 and A_2 are very popular in the literature. These are also the Laplace transforms, with Laplace parameter a, of the functions e^{-bx^δ} and $\mathrm{e}^{-bx^{-\rho}}$ respectively. For $\rho = 1$, A_2 is the basic Krätzel integral and associated with it one has the Krätzel transform. For $\rho = 1$, the integrand in A_2, multiplied with a normalizing constant, is the inverse Gaussian density in stochastic processes. Both A_1 and A_2 are also connected to Bayesian analysis, Mellin convolutions, statistical densities of products and ratios etc. To this end, we will consider a more general integral. Consider the integral

$$B_1 = c_1 \int_0^\infty x^{\gamma-1}\mathrm{e}^{-ax^\delta-bx^{-\rho}}\mathrm{d}x \tag{1.4}$$

for $a > 0, b > 0, \gamma > 0, \delta > 0, \rho > 0$. If the integrand in B_1 is to be a statistical density then we take the constant c_1 as the normalizing constant. In that case the function is defined for $x \geq 0$ and zero otherwise. The integrand in B_1 for $\delta = 1, \rho = 1$ and multiplied by the normalizing constant c_1 is the inverse Gaussian density for appropriate values of a, b, γ, c_1. For $\delta = 1, \rho = \frac{1}{2}$ it is the basic reaction-rate probability integral in nuclear reaction-rate theory. For general values of δ and ρ, Mathai and Haubold (1988) call the integral in (1.4) as the generalized reaction-rate probability integral. For the general parameters case, there is no physical interpretation yet but the theory is worked out in Mathai and Haubold (1988) and in later papers so that one day the corresponding physical interpretations will be found. For $\delta = 1, \rho = 1$ the integral is the basic Krätzel integral in applied analysis, which is also connected to Krätzel transform, see Krätzel (1979), Mathai (2012). Hence one may call (1.4) as the generalized Krätzel integral. If $b = 0$ then it is a generalized gamma integral but when $b = 0$ the special nature of (1.4) is lost. Hence it is not appropriate to call (1.4) as generalized gamma or ultra gamma function because the connection to gamma function is only when $b = 0$ and this is not an admissible value in (1.4). All sorts of studies are done by people working in special functions, treating (1.4) as generalization of gamma function. For two functions $f_1(x_1)$ and $f_2(x_2)$ the Mellin convolution of a product has the basic structure

$$g(u) = \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v) \mathrm{d}v \tag{1.5}$$

where $\frac{1}{v}$ is the Jacobian in the transformation $u = x_1 x_2$ and $v = x_2$. Here the Mellin transform of g, with the Mellin parameter s, denoted by $M_g(s)$, is given by

$$M_g(s) = M_{f_1}(s) M_{f_2}(s). \tag{1.6}$$

This is the Mellin convolution of a product property. How can we identify (1.1) as a Mellin convolution of a product formula? Let

$$f_1(x_1) = \mathrm{e}^{-x_1^\rho}, 0 \leq x_1 < \infty, \rho > 0, u = b^{\frac{1}{\rho}} \tag{1.7}$$

$$f_2(x_2) = x_2^\gamma \mathrm{e}^{-a x_2^\delta}, 0 \leq x_2 < \infty, \delta > 0, \gamma > 0. \tag{1.8}$$

Then

$$\int_0^\infty \frac{1}{v} f_1(\frac{u}{v}) f_2(v) \mathrm{d}v = \int_0^\infty x^{\gamma-1} \mathrm{e}^{-a x^\delta - b x^{-\rho}} \mathrm{d}v \tag{1.9}$$

where $b = u^{\frac{1}{\rho}}$. A constant multiple of (1.9) can also be the statistical density of a product. Let $x_1 > 0, x_2 > 0$ be two real scalar positive random variables with density functions $f_1(x_1)$ and $f_2(x_2)$ respectively, independently distributed or enjoying the the product probability property (PPP) in the sense that their joint density is the product $f_1(x_1)f_2(x_2)$. Then, (1.4), multiplied with the normalizing constant is the density of the product $u = x_1 x_2$. A structure of the type in (1.4) is also of interest for statisticians working on different topics. Connections to inverse Gaussian density and density of a product are already pointed out. It is also of interest for people working in Bayesian analysis. Consider a conditional density of a real scalar positive random variable y, at given value of a parameter or another variable x, which is a generalized gamma density of the following form:

$$f(y|x) = c_1 \mathrm{e}^{-\frac{y}{x^\rho}}, 0 \leq y < \infty, x > 0 \tag{1.10}$$

and zero elsewhere, where c_1 is the normalizing constant. Consider the marginal density of x, a generalized gamma density of the form:

$$f_1(x) = c_2 x^{\gamma-1} \mathrm{e}^{-a x^\delta}, 0 < x < \infty, a > 0, \gamma > 0, \delta > 0 \tag{1.11}$$

and zero elsewhere, where c_2 is the normalizing constant. Then the unconditional density of y, denoted by $g(y)$, is available by integrating over x of the joint density of x and y, namely $f(y|x)f_1(x)$. That is,

$$g(y) = c_1 c_2 \int_0^\infty x^{\gamma-1} \mathrm{e}^{-ax^\delta - yx^{-\rho}} \mathrm{d}x \tag{1.12}$$

which is nothing but (1.4), multiplied by the constant $c_1 c_2$. Hence, from the point of view of Bayesian analysis also the integral in (1.4) is very important. The integral in (1.11) can be interpreted as a continuous mixture in statistical distribution theory. Since it is a very interesting integral in many topics, we will evaluate it explicitly and represent it in computable forms.

2. Evaluation of the Generalized Krätzel Integral

Comparing (1.12) with B_1 of (1.4) we see that $y = b$. Let us take the Mellin transform with respect to b in (1.4) or with respect to y in (1.12), with Mellin parameter s, denoted by, $M_b(s)$, or take it as the Mellin transform of the function $g(y)$ of (1.12) with $b = y$ or consider $M_y(s)$. Then

$$M_y(s) = \int_0^\infty y^{s-1} [\int_0^\infty x^{\gamma-1} \mathrm{e}^{-ax^\delta - yx^{-\rho}} \mathrm{d}x] \mathrm{d}y. \tag{2.1}$$

Interchange of integrals is valid here and taking the integral over y and then integral over x we have the following:

$$\int_{y=0}^\infty y^{s-1} \mathrm{e}^{-yx^{-\rho}} \mathrm{d}y = \Gamma(s)(x^{-\rho})^{-s} = x^{\rho s}\Gamma(s), \Re(s) > 0. \tag{2.2}$$

Now, the integral over x gives the following:

$$\int_0^\infty x^{\gamma+\rho s-1} \mathrm{e}^{-ax^\delta} \mathrm{d}x = \frac{1}{\delta}\Gamma(\frac{\gamma+\rho s}{\delta}) a^{-(\frac{\gamma+\rho s}{\delta})}. \tag{2.3}$$

Therefore, from (2.2) and (2.3), we have

$$M_y(s) = \frac{1}{\delta a^{\frac{\gamma}{\delta}}} \Gamma(s)\Gamma(\frac{\gamma}{\delta} + \frac{\rho}{\delta}s) a^{-\frac{\rho}{\delta}s}. \tag{2.4}$$

Hence by taking the inverse Mellin transform of (2.4) we get B_1 as the inverse Mellin transform. That is,

$$B_1 = \frac{1}{\delta a^{\frac{\gamma}{\delta}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(\frac{\gamma}{\delta} + \frac{\rho}{\delta}s)(ba^{\frac{\rho}{\delta}})^{-s} \mathrm{d}s, i = \sqrt{-1} \tag{2.5}$$

where the c in the contour is any positive number. The integral in (2.5) can be written as a H-function of the following format: The theory and applications of

H-function may be found from any book on H-function, see for example Mathai et al. (2010). Then it is seen that

$$B_1 = \frac{1}{\delta z^{\frac{\gamma}{\delta}}} H_{0,2}^{2,0}[ba^{\frac{\rho}{\delta}}|_{(0,1),(\frac{\gamma}{\delta},\frac{\rho}{\delta})}]. \tag{2.6}$$

When $\rho = \delta$ one can get an interesting special case in terms of Meijer's G-function. In this special case the coefficient of s in the inverse Mellin transform in (2.5) is 1 and hence one can write it as a G-function. A detailed description of the theory and applications of G-function is available from Mathai (1993). That is, for $\rho = \delta$,

$$B_1 = (\delta a^{\frac{\gamma}{\delta}})^{-1} G_{0,2}^{2,0}[ab|_{0,\frac{\gamma}{\delta}}]. \tag{2.7}$$

3. Explicit Evaluation in Computable Series Forms

For explicit evaluations, the best place to start is the integrand in the Mellin-Barnes representation in (2.5) for the general case when $\delta \neq \rho$ and for the special case $\delta = \rho$. Then (2.5) or (2.7) can be evaluated as the sum of the residues at the poles of the integrand in (2.5). To this end let us examine the poles of the gammas there. The poles of $\Gamma(s)$ are at $s = 0, -1, ...$ and the poles of $\Gamma(\frac{\gamma}{\delta} + \frac{\rho}{\delta}s)$ are $\frac{\gamma}{\delta} + \frac{\rho}{\delta}s = -\nu, \nu = 0, 1, 2, ...$ or $s = -\frac{\gamma}{\rho} - \frac{\delta}{\rho}\nu, \nu = 0, 1, 2, ...$ Hence if $\frac{\gamma}{\rho} + \frac{\delta}{\rho}\nu, \nu = 0, 1, 2, ...$ is not a positive integer then the poles of the integrand in the Mellin-Barnes representation in (2.5) are simple and then the residues will be in simple forms. Then one can sum up these two sets of residues easily. Then evaluating the sums of residues at the poles of $\Gamma(s)$ and $\Gamma(\frac{\gamma}{\delta} + \frac{\rho}{\delta}s)$ we have the explicit series form. Let us look at the sum of the residues at the poles of $\Gamma(s), s = -\nu, \nu = 0, 1, 2, ...$ is

$$(\delta a^{\frac{\gamma}{\delta}})^{-1} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu!} \Gamma(\frac{\gamma}{\delta} - \frac{\rho}{\delta}\nu)(ba^{\frac{\rho}{\delta}})^{\nu}. \tag{i}$$

For computing the sum of residues at the poles of $\Gamma(\frac{\gamma}{\delta} + \frac{\rho}{\delta}s)$ it is convenient to make a transformation $\frac{\gamma}{\delta} + \frac{\rho}{\delta}s = s_1 \Rightarrow s = -\frac{\gamma}{\rho} + \frac{\delta}{\rho}s_1, \mathrm{d}s = \frac{\delta}{\rho}\mathrm{d}s_1$,

$$(ba^{\frac{\rho}{\delta}})^{-s} = (ba^{\frac{\rho}{\delta}})^{\frac{\gamma}{\rho} - \frac{\delta}{\rho}s_1} = b^{\frac{\gamma}{\rho}} a^{\frac{\gamma}{\delta}} (b^{\frac{\delta}{\rho}} a)^{-s_1}.$$

and the sum of the residues is the following:

$$\frac{b^{\frac{\gamma}{\rho}}}{\delta} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu!} \Gamma(-\frac{\gamma}{\rho} - \frac{\delta}{\rho}\nu)(ab^{\frac{\delta}{\rho}})^{\nu}. \tag{ii}$$

Therefore for the simple poles case, B is available as the sum of (i) and (ii). That is,

$$B_1 = (\delta a^{\frac{\gamma}{\delta}})^{-1}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(\frac{\gamma}{\delta}-\frac{\rho}{\delta}\nu)(ba^{\frac{\rho}{\delta}})^{\nu}$$
$$+\frac{b^{\frac{\gamma}{\rho}}}{\delta}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!}\Gamma(-\frac{\gamma}{\rho}-\frac{\delta}{\rho}\nu)(ab^{\frac{\delta}{\rho}})^{\nu}, \tag{3.1}$$

for $\frac{\gamma}{\rho}+\frac{\delta}{\rho}\nu$ is not a positive integer for $\nu=0,1,...,$ $\frac{\gamma}{\delta}-\frac{\rho}{\delta}\nu\neq 0,-1,-2,...$ for $\nu=0,1,....$ For $\rho=\delta$ note that the coefficient of ν is 1 and then both the gammas in the first and second lines of (3.1) with gammas with $+\nu$ in the denominator and then the series will become Bessel series in both the lines. Hence, in this case one can obtain B_1 as linear functions of two Bessel series. Hence it is more appropriate to call B_1 as Bessel integral or extended Bessel integral, not as generalized or ultra gamma integral.

Special case (1): $\rho=\delta$ and $\frac{\gamma}{\delta}$ is not a positive integer

Then

$$\Gamma(\frac{\gamma}{\delta}) = (\frac{\gamma}{\delta}-1)(\frac{\gamma}{\delta}-2)..(\frac{\gamma}{\delta}-\nu)\Gamma(\frac{\gamma}{\delta}-\nu)$$
$$\Gamma(\frac{\gamma}{\delta}-\nu) = \frac{\Gamma(\frac{\gamma}{\delta})}{(-1)^{\nu}(-\frac{\gamma}{\delta}+1)_{\nu}}$$

where, for example, $(a)_n=a(a+1)...(a+n-1), a\neq 0, (a)_0=1$ is the Pochhammer symbol. Also

$$\Gamma(-\frac{\gamma}{\rho}-\nu)=\frac{\Gamma(-\frac{\gamma}{\rho})}{(-1)^{\nu}(\frac{\gamma}{\rho}+1)_{\nu}}.$$

Then for $\rho=\delta$ and $\frac{\gamma}{\delta}$ not a positive integer, we have from (3.1)

$$B_1 = \frac{\Gamma(\frac{\gamma}{\delta})}{\rho a^{\frac{\gamma}{\rho}}}{}_0F_1(\ ;-\frac{\gamma}{\rho}+1;ab)$$
$$+\frac{\Gamma(-\frac{\gamma}{\delta})}{\rho}b^{\frac{\gamma}{\rho}}{}_0F_1(\ ;\frac{\gamma}{\rho}+1;ab) \tag{3.2}$$

Thus, it is the sum of two Bessel series. Hence Bessel integral is an appropriate name to be used for (1.4).

Special case (2): $\rho=\delta, \frac{\gamma}{\delta}=m, m=1,2,...$

In this case the poles at $s = 0, -1, -2, ..., -(m-1)$ are simple and the poles at $s = -m, -m-1, ...$ are of order two each. In this case

$$B_1 = \frac{1}{\rho a^{\frac{\gamma}{\rho}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(m+s)(ab)^{-s} \mathrm{d}s. \tag{3.3}$$

Sum of the residues at the poles $s = 0, -1, .., -(m-1)$ is given by

$$\frac{1}{\delta a^{\frac{\gamma}{\delta}}} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \Gamma(m-\nu)(ab)^\nu. \tag{iii}$$

For $x = -m-\nu, \nu = 0, 1, ...$ the poles are of order two each. Let

$$\phi(s) = \Gamma(s)\Gamma(m+s)(ab)^{-s}.$$

Then the residue at the poles of order two, denoted by R_ν, is given by the following:

$$\begin{aligned}
R_\nu &= \lim_{s \to -\nu} \frac{\mathrm{d}}{\mathrm{d}s} \{(s+\nu)^2 \Gamma(s)\Gamma(m+s)(ab)^{-s}\} \\
&= \lim_{s \to -\nu} \frac{\mathrm{d}}{\mathrm{d}s} \{(s+\nu)^2 \frac{(s+\nu-1)^2 ... (s+m)^2 (s+m-1)...s}{(s+\nu-1)^2 ... (s+m)^2 (s+m-1)...s} \Gamma(s)\Gamma(m+s)(ab)^{-s}\} \\
&= \lim_{s \to -\nu} \frac{\mathrm{d}}{\mathrm{d}s} \{ \frac{\Gamma^2(s+\nu+1)}{(s+\nu-1)^2 ... (s+m)^2 (s+m-1)...s} (ab)^{-s}\}
\end{aligned}$$

Note that $(ab)^{-s} = \mathrm{e}^{-s\ln(ab)}$ and

$$\frac{\mathrm{d}}{\mathrm{d}s} \phi(s) = \phi(s) \frac{d}{\mathrm{d}s} \ln \phi(s).$$

Also

$$\begin{aligned}
\lim_{s \to -\nu} \phi(s) &= \lim_{s \to -\nu} \frac{\Gamma^2(s+\nu+1)}{(s+\nu-1)^2 ... (s+m)^2 (s+m-1)...s} (ab)^{-s} \\
&= \frac{(-1)^m}{\nu!(\nu-m)!} (ab)^\nu, \nu = m, m+1, ...
\end{aligned}$$

$$\begin{aligned}
\lim_{s \to -\nu} \ln \phi(s) &= \lim_{s \to -\nu} \frac{\mathrm{d}}{\mathrm{d}s} \ln \phi(s) \\
&= \lim_{s \to -\nu} [2\psi(s+\nu+1) - \frac{2}{s+\nu-1} - ... - \frac{2}{s+m} \\
&- \frac{1}{s+m-1} - ... - \frac{1}{s} - \ln(ab)] \\
&= 2\psi(1) + 2[1 + \frac{1}{2} + ... + \frac{1}{\nu-m}] + (\frac{1}{\nu-m+1} + ... + \frac{1}{\nu}) - \ln(ab) \\
&= \psi(\nu+1) + \psi(\nu-m+1) - \ln(ab)
\end{aligned}$$

where $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\Gamma(z)$ is the psi function. The above simplification is done by using the properties of the psi function. Hence

$$R_\nu = [\psi(\nu+1) + \psi(\nu - m + 1) - \ln(ab)]\frac{(-1)^m}{\nu!(\nu-m)!}(ab)^\nu, \nu = m, m+1, ... \quad (3.4)$$

Therefore

$$B_1 = \frac{1}{\delta a^{\frac{\gamma}{\delta}}}\{\sum_{\nu=0}^{m-1}\frac{(-1)^\nu}{\nu!}\Gamma(m-\nu)(ab)^\nu$$
$$+ \sum_{\nu=m}^{\infty}[\psi(\nu+1) + \psi(\nu-m+1) - \ln(ab)][\frac{(-1)^m}{\nu!(\nu-m)!}(ab)^\nu]\}. \quad (3.5)$$

By using the same procedure one can write the logarithmic version corresponding to (2.5) when the poles of $\Gamma(s)\Gamma(\frac{\gamma}{\delta} + \frac{\rho}{\delta}s)$ differ by integers. Since the expressions become too lengthy they are not listed here.

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A SHORT NOTE ON SIGN CHANGES OF FOURIER COEFFICIENTS OF CUSP FORM

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Abstract: In this note we have studied the sign changes for the subsequence $\{a(n^j)\}$ for any $j \in \mathbb{N}$ of the Fourier coefficients $a(n)$ of Hecke eigen cusp form for the full modular group $SL_2(\mathbb{Z})$.
Keywords and Phrases: Sign changes, Fourier coefficients, Cusp forms
Mathematics subject Classification: Primary: 11F30 , Secondary: 11M06.

1. Introduction

Sign changes of Fourier coefficients of cusp forms in one or in several variables have been studied in various aspects by many authors. It is known that, if the Fourier coefficients of a cusp form are real then they change signs infinitely often [2]. Further, many quantitative results for the number of sign changes for the sequence of the Fourier coefficients have been established. The sign changes of the subsequence of the Fourier coefficients at prime numbers was first studied by M. Ram Murty [10]. Later, Meher et. al. in [8] studied the problem for the subsequence $\{a(n^j)\}_{n\geq 1}$ $(j = 2, 3, 4)$. W.Kohnen and Y.Martin [4] in 2014 proved that the subsequence $\{a(p^{jn})\}_{n\geq 0}$ has infinitely many sign changes for almost all primes p and $j \in \mathbb{N}$.

Here we investigate the sign changes of the subsequence $\{a(n^j)\}_{n\geq 1}$ for $j = 5, 6, 7, 8$. we also genaralize the result of [8] by showing that, for any $j \in \mathbb{N}$ the subsequence $\{a(n^j)\}_{n\geq 1}$ has infinitely many sign changes under some certain conditions.

2. Preliminaries

Let, $\Gamma = SL_2(\mathbb{Z})$. For k be an even integer and $k \geq 4$, denote $S_k(\Gamma)$ as the space of cusp form of weight k on Γ. Consider, $f(z) = \sum\limits_{n\geq 1} a(n)q^n \in S_k(\Gamma)$ be an Hecke eigenform where, $q = e^{2\pi i z}$.

Denote, normalized Fourier coefficient by $\lambda(n) = \frac{a(n)}{n^{(k-1)/2}}$ where, $\lambda(n)$ is real and satisfies the multiplicative property

$$\lambda(m)\lambda(n) = \sum_{d|(m,n)} \lambda\left(\frac{mn}{d^2}\right)$$

In 1974, P. Deligne [1] proved the Ramanujan-Petersson conjecture

$$|\lambda(n)| \leq d(n)$$

where $d(n)$ is the divisor function.
The j th symmetric power L-function attached to $f \in S_k(\Gamma)$ is defined as,

$$L(sym^j f, s) := \prod_p \prod_{m=0}^{j} (1 - \alpha(p)^{j-m}\beta(p)^m p^{-s})^{-1} \tag{1}$$

where $\alpha(p) + \beta(p) = \lambda(p)$ and $\alpha(p)\beta(p) = 1$.[1]

3. Statement of the results
Here we state our main results.

Theorem 3.1. Let $f \in S_k(\Gamma)$ be a nonzero Hecke eigenform with normalized Fourier coefficients $\lambda(n) \in \mathbb{R}$. Then for $j \in \{5, 6, 7, 8\}$, $\{\lambda(n^j)\}_{n\geq 1}$ has infinitely many sign changes.

Theorem 3.2. Let $f \in S_k(\Gamma)$ be a nonzero Hecke eigenform with normalized Fourier coefficients $\lambda(n) \in \mathbb{R}$. If $L(sym^j f, s)$ is automorphic cuspidal. Then for any $j \in \mathbb{N}$, $\{\lambda(n^j)\}_{n\geq 1}$ has infinitely many sign changes.

4. Proof of Theorem 3.1
We need to use the following two lemmas to prove our result.

Lemma 4.1 [G. Lu (6; Theorem 1.2)] Let $f \in S_k(\Gamma)$ be nonzero Hecke eigenform with normalized Fourier coefficients $\lambda(n) \in \mathbb{R}$. Then for any $j \in \mathbb{N}$ there exist a suitable constant c_1 depending on f and j such that,

$$\sum_{n\leq x} \lambda^2(n^j) d(n-1) = c_1 \; xlogx(1 + o(1)) \tag{2}$$

where d(n) is the divisor function.

lemma 4.2 [G. Lu and H. Tang (7; Theorem 1.1)] Let $f \in S_k(\Gamma)$ be nonzero Hecke eigenform with normalized Fourier coefficients $\lambda(n) \in \mathbb{R}$. Then for $j \in \{5, 6, 7, 8\}$ there exist a suitable constant $c_2 > 0$ such that,

$$\sum_{n\leq x} \lambda(n^j) \ll x \; exp(-c_2\sqrt{logx}) \tag{3}$$

Proof of Theorem 3.1. If possible, let us assume that the sequence $\{\lambda(n^j)\}_{n\geq 1}$ for $j \in \{5, 6, 7, 8\}$ are of constant sign say positive for all $n \in (x, 2x]$.

From lemma 4.1, we get

$$\begin{aligned}\sum_{x<n\leq 2x} \lambda^2(n^j)d(n-1) &= c_1 2x\ log\ 2x - c_1 x\ log\ x + o(x\ log\ x) \\ &= c_1 x\ log\ x \left(\frac{2\ log\ 2x}{log\ x} - 1\right) + o(x\ log\ x) \\ &\gg x\ log\ x \qquad (4)\end{aligned}$$

On the other hand, by using lemma 4.2 and Delign's bound [1] on $\lambda(n)$, we get

$$\begin{aligned}\sum_{x<n\leq 2x} \lambda^2(n^j)d(n-1) &= \sum_{x<n\leq 2x} \lambda(n^j)\lambda(n^j)d(n-1) \\ &\ll \sum_{x<n\leq 2x} \lambda(n^j)d(n^j)d(n-1) \\ &\ll x^{2\epsilon} \sum_{x<n\leq 2x} \lambda(n^j) \quad [\epsilon > 0,\ \text{ sufficiently small}] \\ &\ll x^{2\epsilon}\{2x\ e^{(-c_2\sqrt{log\ 2x})} + x\ e^{(-c_2\sqrt{logx})}\} \\ &\ll x^{2\epsilon} x\ e^{(-c_2\sqrt{logx})} \qquad (5)\end{aligned}$$

Now, by comparing the bounds of $\sum\limits_{x<n\leq 2x} \lambda^2(n^j)d(n-1)$ in equation (4) and (5), we arrive at a contradiction. Therefore, atleast one $\lambda(n^j)$ for $n \in (x, 2x]$ is negetive. Hence the sequence $\{\lambda(n^j)\}_{n\geq 1}$ for $j \in \{5, 6, 7, 8\}$, has infinitely many sign changes.

5. Proof of Theorem 3.2.

We need to use **Lemma 4.1** and the following lemma to prove this result.

Lemma 5.1 [Lau and Lu(5; Theorem 1)] Let $f \in S_k(\Gamma)$ be nonzero Hecke eigenform with normalized Fourier coefficients $\lambda(n) \in \mathbb{R}$. Suppose $L(sym^j f, s)$ is automorphic cuspidal for $j \in \mathbb{N}$. Then for any $j \geq 3$ and $j \in \mathbb{N}$,

$$\sum_{n\leq x} \lambda(n^j) \ll x^{\frac{j}{j+2}} \qquad (6)$$

Proof of Theorem 3.2. If possible, let us assume that the sequence $\{\lambda(n^j)\}_{n\geq 1}$ for $j \geq 3$ and $j \in \mathbb{N}$ are of constant sign say positive for all $n \in (x, 2x]$.

By using lemma 5.1 and Delign's bound [1] on $\lambda(n)$, we get

$$\begin{aligned}\sum_{x<n\leq 2x}\lambda^2(n^j)d(n-1) &= \sum_{x<n\leq 2x}\lambda(n^j)\lambda(n^j)d(n-1)\\ &\ll x^{2\epsilon}\sum_{x<n\leq 2x}\lambda(n^j)\quad [\epsilon>0,\quad \text{sufficiently small}]\\ &\ll x^{2\epsilon}\{(2x)^{\frac{j}{j+2}}+x^{\frac{j}{j+2}}\}\\ &\ll x^{2\epsilon}x^{\frac{j}{j+2}} \qquad (7)\end{aligned}$$

Equation (4) gives the lower bound of $\sum_{x<n\leq 2x}\lambda^2(n^j)d(n-1)$ for any $j\in\mathbb{N}$. Now, by comparing the bounds of $\sum_{x<n\leq 2x}\lambda^2(n^j)d(n-1)$ in equation (4) and (7), we arrive at a contradiction. Hence, we can conclude that our theorem is true for $j\geq 3$. For the case $j=1$ and 2, the theorem holds true by the references [4] and [8].

Remark

The condition of automorphic cuspidality of $L(sym^j f, s)$ is necessary to conclude the result of infinitely many sign changes of the subsequence $\{\lambda(n^j)\}_{n\geq 1}$ in theorem 3.2. By theorem 3.1 and references [4] and [8] the result holds unconditionally for $j\in\{1,2,\cdots,8\}$.

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GENERALIZED RATHIE-SWAMEE DISTRIBUTION AND APPLICATIONS

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Abstract: We present a generalized Rathie-Swamee (GRS) distribution, by introducing one more parameter. We present various properties of GRS distribution, such as Quantiles, moments, order statistics, maximum likelihood estimates, gamma- and beta- generated distributions, Marshall-Olkin-Rathie-Swamee distribution, and corresponding GRS-loglogistic distribution is defined and discussed. The family of GRS distributions is very flexible and fits well asymmetric data. The utility of the GRS distribution is illustrated by analyzing real data sets involving: (a) Old Faithful Geyser waiting times to the next eruption data, (b) Old Faithful Geyser durations of the eruptions data, and (c) Environmental Performance Index (EPI) data.

Keywords: Generalized logistic distribution, GRS distribution, Data analysis.

A.M.S. Subject Classification: 60E05, 33C60

1 Introduction

In this article, we put forward a new multi-modal distribution with four parameters. This family of distribution, called generalized Rathie-Swamee (GRS), is very flexible and fits well asymmetric data. Rathie and Swamee (2006) [see also, Rathie (2011)] introduced the following multimodal distribution for a random variable X, generalizing the logistic distribution:

$$F_0(x) = \frac{1}{\{1 + \exp[-x(a + b|x|^p)]\}}, \tag{1}$$

with the corresponding density function

$$f_0(x) = \frac{[a + b(p+1)|x|^p] \exp[-x(a + b|x|^p)]}{\{1 + \exp[-x(a + b|x|^p)]\}^2}, \tag{2}$$

where $x \in \Re$, a, $b \geq 0$ (not both zeros simultaneously) and $p \geq -1$. The distribution and density functions (1) and (2) reduce to Swamee-Rathie generalized logistic (SRGL) distribution for $a = 0$. The SRGL distribution approximeted very well the standard normal distribution for $b = 1.7255$ and $p = 0.12$ [see, Swamee and Rathie (2007)].

The generalized Rathie-Swamme (GRS) distribution is defined by

$$F(x) = \frac{1}{\{1 + \exp[-x(a + b|x|^p)]\}^\lambda}, \tag{3}$$

with the corresponding density function

$$f(x) = \frac{\lambda[a + b(p+1)|x|^p] \exp[-x(a + b|x|^p)]}{\{1 + \exp[-x(a + b|x|^p)]\}^{\lambda+1}}, \tag{4}$$

where $x \in \Re$, a, $b \geq 0$ (not both zeros simultaneously), $\lambda > 0$ and $p \geq -1$. For $\lambda = 1$, (3) and (4) reduce to (1) and (2) respectively. Symbolically, $X \sim GRS(a, b, p, \lambda)$. The density functions are plotted for various values of a, b, p and λ in Figures 1 and 2. Using a location parameter $\mu \in \Re$, (4) may be changed to $g(x)$ as

$$g(x) = \frac{\lambda[a + b(p+1)|x-\mu|^p] \exp[-(x-\mu)(a + b|x-\mu|^p)]}{\{1 + \exp[-(x-\mu)(a + b|x-\mu|^p)]\}^{\lambda+1}}. \tag{5}$$

In the article, we study (3) and (4) as follows:Section 2 deals with quantiles while moments are given in Section 3. In Section 4, we show that the density function $f(x)$ is an linear combination of densities. Order statistics are indicated in Section 5. Section 6 deals with two generalized gamma-generated GRS distributions while in Section 7 a beta-generated GRS distribution is given. Marshall-Olkin GRS distribution is mentioned in Section 8. Section 9 deals with estimation of parameters by MLE method and three data applications: (a) Old Faithful Geyser waiting times to the next eruption data, (b) Old Faithful Geyser durations of the eruptions data, and (c) Environmental Performance Index (EPI) data. In Section 10, GRS-loglogistic distribution is defined and discussed, while Section 11 concludes the article.

2 Quantiles

Using (3) we have

$$F^{\frac{1}{\lambda}}(x) = \{1 + \exp[-x(a + b|x|^p)]\}^{-1}.$$

Replacing F by $F^{\frac{1}{\lambda}}$ in Rathie et. al (Rathie et al., 2013, eq. (1.5)), we have the following expression for the quantiles:

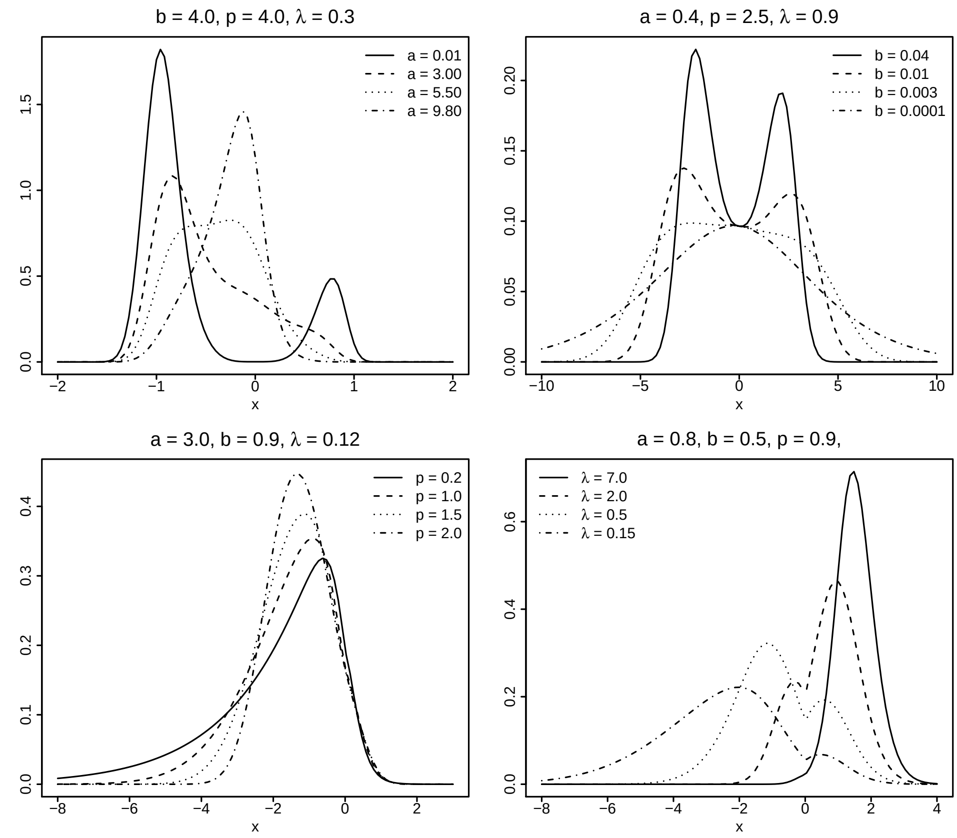

Figure 1: Some shapes for the GRS density.

$$x = \begin{cases} -\displaystyle\sum_{r=0}^{\infty} \frac{(\frac{-b}{a})^r}{r!} \frac{\Gamma(rp+r+1)}{\Gamma(rp+2)} \left[\frac{1}{a}\ln\left(\frac{1-F^{\frac{1}{\lambda}}}{F^{\frac{1}{\lambda}}}\right)\right]^{rp+1}, & F \leq 0.5^{\lambda} \\ \displaystyle\sum_{r=0}^{\infty} \frac{(\frac{-b}{a})^r}{r!} \frac{\Gamma(rp+r+1)}{\Gamma(rp+2)} \left[\frac{1}{a}\ln\left(\frac{F^{\frac{1}{\lambda}}}{1-F^{\frac{1}{\lambda}}}\right)\right]^{rp+1}, & F > 0.5^{\lambda}. \end{cases} \tag{6}$$

3 Moments

Using

$$(1+x)^{-\delta} = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!}(\delta)_r, \ |x| < 1, \tag{7}$$

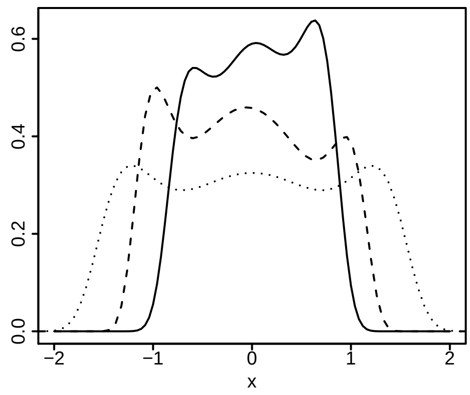

Figure 2: Some trimodal shapes for the GRS density: solid line: $a = 2.3,\ b = 3.0,\ p = 4.0,\ \lambda = 1.1$; dashed line: $a = 1.9,\ b = 0.6,\ p = 6.0,\ \lambda = 0.9$; dotted line: $a = 1.3,\ b = 0.12,\ p = 4.9,\ \lambda = 1.0$.

we have for $x > 0$,

$$f(x) = \lambda[a + b(1+p)x^p] \sum_{r=0}^{\infty} \frac{(-1)^r(\lambda+1)_r}{r!} \exp[-(r+1)x(a+bx^p)] \tag{8}$$

and for $x < 0$,

$$f(x) = \lambda[a + b(1+p)|x|^p] \sum_{r=0}^{\infty} \frac{(-1)^r(\lambda+1)_r}{r!} \exp[-(r+\lambda)|x|(a+b|x|^p)]. \tag{9}$$

Hence,

$$E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx = \lambda \sum_{r=0}^{\infty} \frac{(-1)^r(\lambda+1)_r}{r!}[I_1 + (-1)^n I_\lambda],$$

where

$$I_\eta = \int_0^\infty x^n[a + b(1+p)x^p] \exp[-(r+\eta)x(a+bx^p)]dx. \tag{10}$$

Using (see Mathai et al. (2010, p. 41))

$$\int_0^\infty t^\alpha \exp[-t(a_1 + b_1 t^{p_1})]dt = a_1^{-\alpha-1} H_{1,1}^{1,1}\left[a_1^{p_1+1} b_1^{-1} \middle| \begin{matrix} (1,\ 1) \\ (\alpha+1,\ p_1+1) \end{matrix}\right],$$

we obtain

$$I_\eta = a[a(r+\eta)]^{-n-1} H_{1,1}^{1,1}\left[\frac{[a(r+\eta)]^{p+1}}{b(r+\eta)}\middle|\begin{matrix}(1,\ 1)\\(n{+}1,\ p{+}1)\end{matrix}\right]$$
$$+\, b(1+p)[a(r+\eta)]^{-n-p-1} H_{1,1}^{1,1}\left[\frac{a^{p+1}(r+\eta)^p}{b}\middle|\begin{matrix}(1,\ 1)\\(n{+}p{+}1,\ p{+}1)\end{matrix}\right]. \tag{11}$$

The H-function in (11) is defined by

$$H_{p,q}^{m,n}\left[x\middle|\begin{matrix}(a_1,A_1),\dots,(a_n,A_n),(a_{n+1},A_{n+1}),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_m,B_m),(b_{m+1},B_{m+1}),\dots,(b_q,B_q)\end{matrix}\right]$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - B_j s)\prod_{j=1}^n \Gamma(1-a_j+A_j s)}{\prod_{j=m+1}^q \Gamma(1-b_j+B_j s)\prod_{j=n+1}^p \Gamma(a_j - A_j s)} x^s ds. \tag{12}$$

See Mathai et al. (2010) for more details about (12).

Thus, the n-th moments are given by

$$E(X^n) = \lambda \sum_{r=0}^{\infty} \frac{(-1)^r(\lambda+1)_r}{r!}[I_1 + (-1)^n I_\lambda], \tag{13}$$

where I_η is given by (11), and $p \neq 0$.

4 Linear combinations

We have

$$f(x) = \sum_{r=0}^{\infty} \frac{\lambda(-1)^r(\lambda+1)_r}{(r+1)!} g_{1r} \mathrm{I}_{x>0} + \sum_{r=0}^{\infty} \frac{\lambda(-1)^r(\lambda+1)_r}{(r+\lambda)r!} g_{2r} \mathrm{I}_{x<0}, \tag{14}$$

where the density functions g_{1r} and g_{2r} are given by

$$g_{1r}(x) = (r+1)[a + b(1+p)x^p]\exp[-(r+1)x(a+bx^p)], \quad x>0 \tag{15}$$

and

$$g_{2r}(x) = (r+\lambda)[a + b(1+p)|x|^p]\exp[-(r+\lambda)|x|(a+b|x|^p)], \quad x<0. \tag{16}$$

Hence, $f(x)$ is an infinite linear combination of density functions.

5 Order Statistics

The n-th and 1-st order statistics are given below:

$$F_n(x) = F^n(x) = \{1 + \exp[-x(a + b|x|^p)]\}^{-n\lambda}, \tag{17}$$

$$f_n(x) = nF^{n-1}(x)f(x), \tag{18}$$

$$F_1(x) = 1 - [1 - F(x)]^n, \tag{19}$$

$$f_1(x) = n[1 - F(x)]^{n-1}f(x). \tag{20}$$

where $F(x)$ and $f(x)$ are given respectively in (3) and (4).

6 Generalized gamma-generated GRS distributions

Using the distribution function

$$H_1(x) = \frac{\gamma\beta^{\frac{\alpha}{\gamma}}}{\Gamma(\frac{\alpha}{\gamma})}\int_0^{-\ln(1-F(x))} w^{\alpha-1}\exp(-\beta w^{\gamma})dw, \quad \alpha, \beta, \gamma > 0, \tag{21}$$

and Rathie and Silva (2017), we have

$$H_1(x) = \frac{\beta^{\frac{\alpha}{\gamma}}}{\Gamma(1+\frac{\alpha}{\gamma})}\{-\ln[1-F(x)]\}^{\alpha-1}{}_1F_1\left(\frac{\alpha}{\gamma}; 1+\frac{\alpha}{\gamma}; -\beta\{-\ln[1-F(x)]\}^{\gamma}\right), \tag{22}$$

where $F(x)$ is given in (3), and ${}_1F_1$ is confluent hypergeometric function (see, Luke (1969, p.115)).

Using the series expansion for ${}_1F_1$, we get

$$\begin{aligned} H_1(x) &= \frac{\beta^{\frac{\alpha}{\gamma}}}{\Gamma(1+\frac{\alpha}{\gamma})}\sum_{s=0}^{\infty}\frac{\left(\frac{\alpha}{\gamma}\right)_s(-\beta)^s}{s!\left(1+\frac{\alpha}{\gamma}\right)_s}\{-\ln[1-F(x)]\}^{\alpha+\gamma s-1} \\ &= \frac{\beta^{\frac{\alpha}{\gamma}}}{\Gamma(1+\frac{\alpha}{\gamma})}\sum_{s=0}^{\infty}\frac{\left(\frac{\alpha}{\gamma}\right)_s(-\beta)^s}{s!\left(1+\frac{\alpha}{\gamma}\right)_s}[F(x)]^{\alpha+\gamma s-1}\left\{\frac{-\ln[1-F(x)]}{F(x)}\right\}^{\alpha+\gamma s-1}. \end{aligned} \tag{23}$$

Using the Theorem 2.2 of Rathie and Silva (2017), $H_1(x)$ is written as

$$\begin{aligned} H_1(x) = \frac{\beta^{\frac{\alpha}{\gamma}}}{\Gamma(1+\frac{\alpha}{\gamma})}\sum_{s=0}^{\infty}\frac{\left(\frac{\alpha}{\gamma}\right)_s(-\beta)^s}{s!\left(1+\frac{\alpha}{\gamma}\right)_s} \\ \sum_{r=0}^{\infty}\left[\sum_{k=0}^{r}\frac{(-1)^{r-k}(1-\alpha-\gamma s)_{r-k}}{(r-k)!}C_{(k)(r-k)}\right][F(x)]^{r+\alpha+\gamma s-1}, \end{aligned} \tag{24}$$

where

$$C_{(0)(t)} = \left(\frac{1}{2}\right)^t \tag{25}$$

and

$$C_{(m)(t)} = \frac{2}{m}\sum_{l=1}^{m}\frac{[l(t+1)-m]}{l+2}C_{(m-l)(t)}, \quad m \geq 1. \tag{26}$$

Hence $H_1(x)$ is an infinite linear combination of the distribution functions

$$\{1+\exp[-x(a+b|x|^p)]\}^{-\lambda(r+\alpha+\gamma s-1)}. \tag{27}$$

The corresponding result for logistic distribution is obtained by putting $b=0$, $\lambda = 1$ in (22) and is given by

$$\begin{aligned} H_1(x) =& \frac{\beta^{\frac{\alpha}{\gamma}}}{\Gamma(1+\frac{\alpha}{\gamma})}\{-\ln[1+\exp(ax)]\}^{\alpha-1} \\ & {}_1F_1\left(\frac{\alpha}{\gamma};1+\frac{\alpha}{\gamma};-\beta\{-\ln[1+\exp(ax)]\}^{\gamma}\right). \end{aligned} \tag{28}$$

Using another form of generalized gamma distribution function,

$$H_2(x) = 1 - \frac{\gamma\beta^{\frac{\alpha}{\gamma}}}{\Gamma(\frac{\alpha}{\gamma})}\int_0^{-\ln F(x)} w^{\alpha-1}\exp(-\beta w^{\gamma})dw, \quad \alpha,\beta,\gamma>0, \tag{29}$$

and Rathie and Silva (2017), we have

$$H_2(x) = 1 - \frac{\beta^{\frac{\alpha}{\gamma}}}{\Gamma(1+\frac{\alpha}{\gamma})}[-\ln F(x)]^{\alpha}{}_1F_1\left(\frac{\alpha}{\gamma};1+\frac{\alpha}{\gamma};-\beta[-\ln F(x)]^{\gamma}\right), \tag{30}$$

where $F(x)$ is given in (3).

Alternatively,

$$\begin{aligned} H_2(x) &= 1 - \frac{\beta^{\frac{\alpha}{\gamma}}}{\Gamma(1+\frac{\alpha}{\gamma})}\sum_{s=0}^{\infty}\frac{\left(\frac{\alpha}{\gamma}\right)_s(-\beta)^s}{s!\left(1+\frac{\alpha}{\gamma}\right)_s}[-\ln\{1-[1-F(x)]\}]^{s\gamma+\alpha} \\ &= 1 - \frac{\beta^{\frac{\alpha}{\gamma}}}{\Gamma(1+\frac{\alpha}{\gamma})}\sum_{s=0}^{\infty}\frac{\left(\frac{\alpha}{\gamma}\right)_s(-\beta)^s}{s!\left(1+\frac{\alpha}{\gamma}\right)_s} \\ &\qquad [1-F(x)]^{s\gamma+\alpha}\left[\frac{-\ln\{1-[1-F(x)]\}}{1-F(x)}\right]^{s\gamma+\alpha}. \end{aligned} \tag{31}$$

Using the Theorm 2.2 of Rathie and Silva (2017) and (7), $H_2(x)$ takes the following form:

$$\begin{aligned} H_2(x) = 1 - \frac{\beta^{\frac{\alpha}{\gamma}}}{\Gamma(1+\frac{\alpha}{\gamma})} \sum_{s=0}^{\infty} \frac{\left(\frac{\alpha}{\gamma}\right)_s (-\beta)^s}{s! \left(1+\frac{\alpha}{\gamma}\right)_s} \\ \sum_{r=0}^{\infty} \left[\sum_{k=0}^{r} \frac{(-1)^{r-k}(-\alpha-\gamma s)_{r-k}}{(r-k)!} C_{(k)(r-k)} \right] \\ \sum_{q=0}^{\infty} \frac{(-r-\alpha-\gamma s)_q}{q!} [F(x)]^q. \end{aligned} \tag{32}$$

Thus, $H_2(x)$ is an infinite linear combination of the distribution functions

$$\{1+\exp[-x(a+b|x|^p)]\}^{-\lambda q}.$$

For $b=0$, and $\lambda=1$ the corresponding result for logistic distribution is

$$\begin{aligned} H_2(x) = 1 - \frac{\beta^{\frac{\alpha}{\gamma}}}{\Gamma(1+\frac{\alpha}{\gamma})} \{\ln[1+\exp(-ax)]\}^{\alpha} \\ {}_1F_1\left(\frac{\alpha}{\gamma}; 1+\frac{\alpha}{\gamma}; -\beta\{\ln[1+\exp(-ax)]\}^{\gamma}\right). \end{aligned} \tag{33}$$

7 Beta-generated GRS distribution

From Andrade and Rathie (2016), we have the beta-generated distribution given by

$$\begin{aligned} G(x) &= \frac{1}{B(\alpha,\beta)} \int_0^{F(x)} u^{\alpha-1}(1-u)^{\beta-1} du \\ &= \frac{F^{\alpha}(x)}{\alpha B(\alpha,\beta)} \, {}_2F_1(\alpha, 1-\beta; 1+\alpha; F(x)), \end{aligned} \tag{34}$$

for $\alpha, \beta > 0$, with its density function

$$g(x) = \frac{1}{B(\alpha,\beta)} F^{\alpha-1}(x)[1-F(x)]^{\beta-1} f(x), \tag{35}$$

where $F(x)$ and $f(x)$ are given respectively in (3) and (4). In (34), ${}_2F_1$ is Gauss hypergeometric function (see, Luke (1969, p.41)).

Using the series for ${}_2F_1$, we have

$$G(x) = \frac{1}{\alpha B(\alpha,\beta)} \sum_{r=0}^{\infty} \frac{(\alpha)_r (1-\beta)_r}{r!(1+\alpha)_r} [F(x)]^{\alpha+r}. \tag{36}$$

Thus, $G(x)$ is an infinite linear combination of the distribution functions

$$\{1 + \exp[-x(a + b|x|^p)]\}^{-\lambda(\alpha+r)}.$$

The beta-generated density function for logistic base distribution is obtained from (35) by putting $b = 0$, $\lambda = 1$ and is

$$g(x) = \frac{1}{B(\alpha,\beta)} \frac{a\exp(-a\beta x)}{[1+\exp(-ax)]^{\alpha+\beta}}. \tag{37}$$

8 Marshall-Olkin generalized Rathie-Swamee distribution

Using the Marshall-Olkin distribution function (Marshall and Olkin, 1997), the distribution function of Marshall-Olkin Generalized Rathie-Swamee (MOGRS) distribution is defined by

$$H(x) = \frac{F(x)}{F(x) + \gamma \bar{F}(x)}, \quad \gamma > 0, x \in \Re, \tag{38}$$

where $F(x)$ is given in (3). Hence

$$H(x) = \frac{F(x)}{\gamma + (1-\gamma)F(x)} = \frac{1}{1 - \gamma + \gamma\{1 + \exp[-x(a + b|x|^p)]\}^{\lambda}}. \tag{39}$$

For logistic distribution function ($b = 0, \lambda = 1$), (39) reduces to

$$H(x) = \frac{1}{1 + \gamma \exp[-ax]},$$

which is a generalization (asymmetric) of logistic distribution.

9 Estimation and applications

The log-likelihood function for a sample $\underset{\sim}{x} = (x_1, \ldots, x_n)$ from $GRS(a, b, p, \lambda, \mu)$, with density function (5) and $\underset{\sim}{\theta} = (a, b, p, \lambda, \mu)$ is given by

$$\begin{aligned} \ln g(\underset{\sim}{\theta} | \underset{\sim}{x}) = n \ln \lambda + \sum_{i=1}^{n} \ln[a + b(p+1)|x_i - \mu|^p] - \sum_{i=1}^{n} (x_i - \mu)(a + b|x_i - \mu|^p) \\ - (\lambda + 1) \sum_{i=1}^{n} \ln\{1 + \exp[-(x_i - \mu)(a + b|x_i - \mu|^p)]\}. \end{aligned} \tag{40}$$

The parameters a, b, p, λ and μ are estimated by solving simultaneously the equations

$$\frac{\partial \ln g(\underset{\sim}{\theta}|\underset{\sim}{x})}{\partial a}=0,\ \frac{\partial \ln g(\underset{\sim}{\theta}|\underset{\sim}{x})}{\partial b}=0,\ \frac{\partial \ln g(\underset{\sim}{\theta}|\underset{\sim}{x})}{\partial p}=0,\ \frac{\partial \ln g(\underset{\sim}{\theta}|\underset{\sim}{x})}{\partial \lambda}=0,\ \frac{\partial \ln g(\underset{\sim}{\theta}|\underset{\sim}{x})}{\partial \mu}=0.$$

These equations cannot be solved analytically and an iterative method can be used to solve them numerically.

The goodness of fit of the GRS model will be assessed by Kolmogorov-Smirnov (KS), Anderson-Darling (AD) and Cramér-von-Mises (CvM) tests, smaller values indicate better fit to the data. The tests are described in Thas (2010).

The Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Corrected Akaike Information Criterion (AICc) are defined by

$$\begin{aligned}
\text{AIC} &= -2\ln g(\underset{\sim}{\theta}|\underset{\sim}{x})+2p,\\
\text{BIC} &= -2\ln g(\underset{\sim}{\theta}|\underset{\sim}{x})+p\log(n),\\
\text{AICc} &= -2\ln g(\underset{\sim}{\theta}|\underset{\sim}{x})+2p+\left[\frac{2p(p+1)}{n-p-1}\right],
\end{aligned}$$

where $\ln g(\underset{\sim}{\theta}|\underset{\sim}{x})$ is the log-likelihood function, n the sample size and p the number of parameters of the model. The best model has the least value of the criterion used.

The measures of accuracy: Mean Square Error (MSE), Mean Absolute Deviation (MAD) and Maximum Absolute Deviation (MaxAD), which are given by

$$\begin{aligned}
\text{MSE} &= \frac{\sum_{i=1}^{n}(F_e(x_i)-\hat{F}(x_i))^2}{n},\\
\text{MAD} &= \frac{\sum_{i=1}^{n}|F_e(x_i)-\hat{F}(x_i)|}{n},\\
\text{MaxAD} &= \max(|F_e(x_i)-\hat{F}(x_i)|),
\end{aligned}$$

where $F_e(x_i)$ and $\hat{F}(x_i)$, $i=1,\ldots,n$, are the empirical and fitted cumulative distributions of the data. The best model has the least value of the measure used.

Three applications of the generalized Rathie-Swamee (GRS) model defined in (5) are given for the following data sets:

(a) Old Faithfull Geyser of the Yellowstone National Park in Wyoming state, USA, duration and waiting time eruptions.

(b) The Environmental Performance Index (EPI) of 132 countries based on 22 performance indicators and 10 policy categories, given in 2012. The EPI measures how close countries are to established environmental goals (`http://epi.yale.edu/`).

Maximum likelihood method is used to obtain the estimates of the parameters given in Table 1 in all the three data sets. We used the BFGS method available at the `constrOptim` function in R program (R Core Team (2017)).

Table 1: Maximum likelihood estimates for GRS model.

Data set (sample size)	λ	a	b	p	μ
Waiting time (299)	1.5774	0.2820	0.8416	1.2078	-0.4224
Eruption duration (272)	1.4602	0.1944	0.7427	2.8771	-0.3027
EPI (132)	1.9340	0.1246	3×10^{-6}	3.0234	46.5798

The density and distribution functions for histogram and empirical cumulative distribution functions for each data set are given in Figure 3. The data sets for Old Faithful Geyser are standardized by using the observed mean and standard deviation. Thus, we can compare our results for 299 data points about eruption waiting times with the ones given by Abdulah and Elsalloukh (2014) using Epsilon Skew Inverted Gamma (ESIG) distribution. Our results for 272 data points about eruption duration time is compared with the corresponding results given by Ali et al. (2010) where they used Skewed Inverse Reflected Pareto (SIRP) distribution. The Figure 3 indicates that GRS distribution fits all data sets adequately. Table 2 gives the p-values for the goodness of fit tests of the GRS model. The results also indicate that we do not reject the hypothesis to fit the data using GRS model.

The measures of accuracy and criteria for model selection are given in Table 3. Comparing the results of the GRS model with those available for the ESIG and SIRP models, the proposed GRS model is best in all measures.

Table 2: p-values for KS, AD and CvM test.

	Data set		
Test	Waiting time	Eruption duration	EPI
KS	0.4745	0.3250	0.9339
AD	0.6172	0.2200	0.9625
CvM	0.6932	0.3102	0.9423

10 GRS-loglogistic distribution

The GRS-loglogistic distribution density and distribution functions corresponding to (4) and (3) are defined as follows:

$$G(x) = \frac{1}{\{1+\exp[-\ln x(a+b|\ln x|^p)]\}^\lambda}, \tag{41}$$

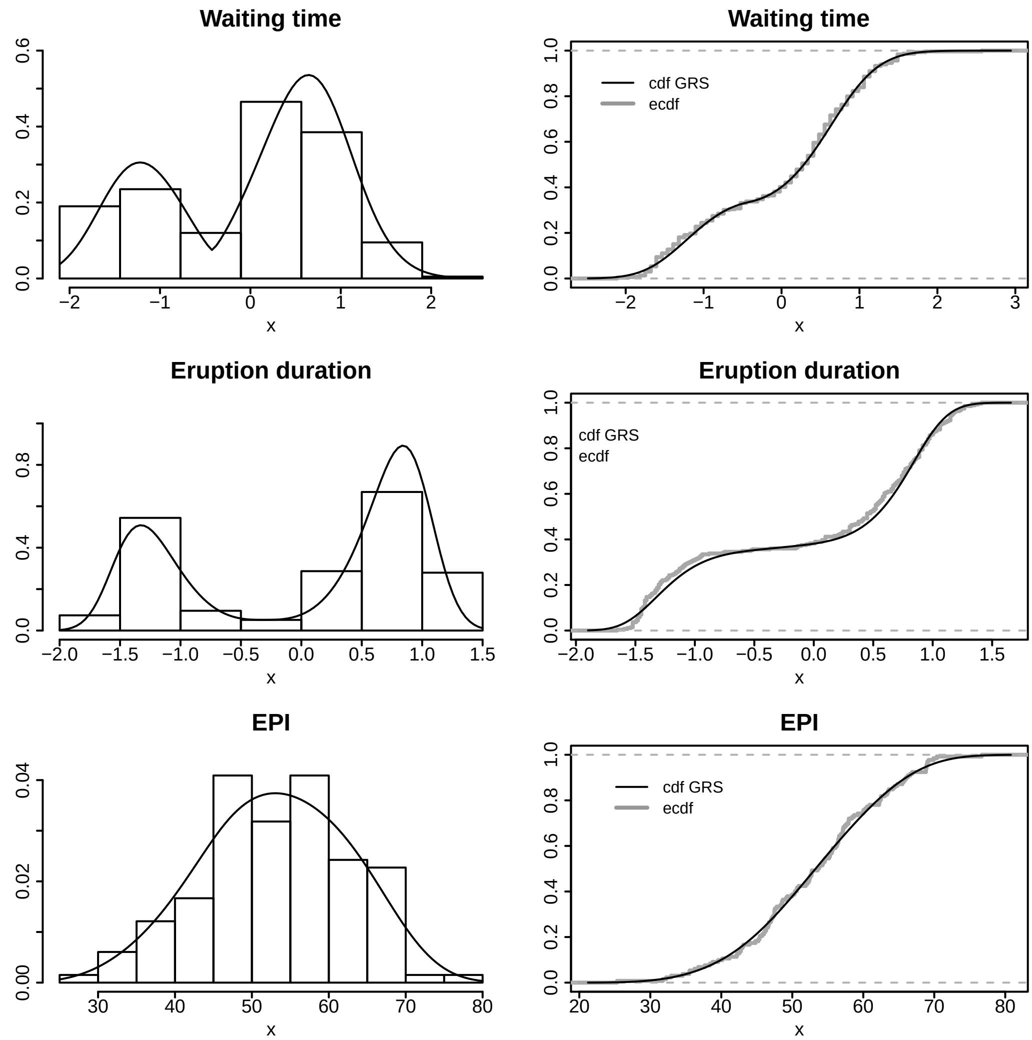

Figure 3: Ajusted GRS density and distribution functions.

with the corresponding density function

$$g(x) = \frac{\lambda[a + b(p+1)|\ln x|^p]\exp[-\ln x(a + b|\ln x|^p)]}{\{1 + \exp[-\ln x(a + b|\ln x|^p)]\}^{\lambda+1}}. \tag{42}$$

For $\lambda = 1$, (41) and (42) yield the RS-loglogistic distribution defined, studied and applied to analyse data earlier by Swamee and Rathie (2007), Ben-Zvi (2009) and Rathie et al. (2011).

Table 3: Log-likelihood, criteria for model selection and measures of accuracy.

	Waiting time		Eruption duration		EPI
	GRS	ESIG	GRS	SIRP	GRS
LogL	-372.59	-566.9	-239.38	-385.5	-487.27
AIC	755.17	1139.8	488.76		984.53
BIC	773.68	1150.9	506.79		998.94
AICc	755.38		488.98		985.01
MSE ($\times 10^3$)	0.63		0.83		0.29
MAD ($\times 10^2$)	1.30		2.26		1.38
MaxAD ($\times 10^2$)	4.88		5.77		4.69

11 Concluding remarks

A generalized Rathie-Swamee (GRS) distribution is introduced and several of its properties studied. The new generalized gamma- and beta-generated density and distribution functions derived in Section 6 and 7, MOGRS distribution of Section 8 and GRS-loglogistic distribution of Section 10 are very flexible and may be used to model data sets. The importance of the GRS distribution is shown by analyzing the Old Faithful Geyser waiting times and duration of the eruption data as well as the Environmental Performance Index data.

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SINGULAR INTEGRAL EQUATIONS CONTAINING EXTENDED MITTAG-LEFFLER FUNCTION

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Abstract: The purpose of this paper is to present the fractional integral operator $\mathcal{G}(\alpha, \beta, \gamma, \delta)$ associated with integral equation involving extended Mittag-Leffler function $\mathcal{E}_{\alpha,\beta}^{\gamma,\delta}(z)$. The operator is discussed for its existence and with regard to its composition with Riemann-Liouville fractional integral operator.

Keywords and Phrases: Singular integral equations; Generalized Mittag-Leffler functions; Fractional calculus

Mathematics subject Classification: Primary 33E12, 45D05; Secondary 34A12, 26A33

1. Introduction

Integral equations of several types have been studied by means of fractional integration with special functions as kernel. Several authors (Ta Li [14], Buschman [1] and Higgins [4]) have applied the Laplace transform to solve convolution equations which are special cases of

$$\int_0^x (x-t)^{b-1} {}_1F_1(a; b; c(x-t)) f(t) dt = g(x); \quad \mathrm{Re}(b) > 0,$$

as discussed by Prabhakar [9].

Erdèlyi [3] investigated the solutions of integral equations whose kernel contain Legendre functions,

$$\int_\alpha^x \left(x^2 - t^2\right)^{-\frac{1}{2}\mu} P_\nu^\mu \left(\frac{x}{t}\right) f(t) dt = g(x),$$

where $\alpha \leq x \leq \beta$, $\alpha, \beta > 0$ and the values of μ and ν are unrestricted, except for $\mathrm{Re}(\mu) < 1$. Higgins [5] and Wimp [13] studied some integral equations involving the hypergeometric function $F(a, b; c; z)$. One of them is

$$\int_x^1 \frac{1}{\Gamma(c)}(t-x)^{c-1} F\left(a, b; c; 1-\frac{t}{x}\right) f(t)dt = g(x),$$

for $\alpha \leq x \leq 1$.
Love [6] considered the following integral equations involving hypergeometric functions in the kernel.

$$\int_0^x \frac{1}{\Gamma(c)}(x-t)^{c-1} F\left(a, b; c; 1-\frac{x}{t}\right) f(t)dt = g(x),$$

for $0 < x < d$ where $0 < d \leq \infty$. Further, he discussed the integral equation

$$\int_0^x \frac{1}{\Gamma(c)}(x-t)^{c-1} F\left(a, b; c; 1-\frac{t}{x}\right) f(t)dt = g(x),$$

for $0 < x < d$ in the same paper.

Recently, Desai et al. [2] studied the integral equation containing generalized Mittag-Leffler function as the kernel. The classical Mittag-Leffler function [8] is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1}$$

where z is a complex variable and $\alpha \geq 0$ that occurs as the solution of fractional order differential equation or fractional order integral equations. The Mittag-Leffler function is a direct generalization of exponential function to which it reduces for $\alpha = 1$. For $0 < \alpha < 1$ and $|z| < 1$, it interpolates between the exponential function e^z and a geometric function $\frac{1}{(1-z)} = \sum\limits_{k=0}^{\infty} z^k$.

Wiman [12] suggested the generalization of $E_\alpha(z)$ as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{2}$$

for $\alpha, \beta \in C$, $\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\beta) > 0$, which is known as Wiman's function or the generalized Mittag-Leffler function.

Prabhakar [10] further extended the Mittag-Leffler function as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \tag{3}$$

for $\alpha, \beta, \gamma \in C$, $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma) > 0$, $(\gamma)_n$ is a Pochhammer symbol, $(\gamma)_n = \gamma(\gamma+1)\ldots(\gamma+n-1) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}, n \geq 1, (\gamma)_0 = 1, \gamma \neq 0$.

A new generalization of Mittag-Leffler function was defined by Salim [11] as

$$\mathcal{E}^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_n}, \tag{4}$$

where $\alpha, \beta, \gamma, \delta \in C; Re(\alpha), Re(\beta), Re(\gamma), Re(\delta) > 0$.

Equation (4) is the generalization of exponential function. Equations (1)–(3) can reduce to $\mathcal{E}^{1,1}_{1,1}(z) = \exp(z), \mathcal{E}^{1,1}_{\alpha,1}(z) = \mathcal{E}_{\alpha}(z), \mathcal{E}^{1,1}_{\alpha,\beta}(z) = \mathcal{E}_{\alpha,\beta}(z)$ and $\mathcal{E}^{\gamma,1}_{\alpha,\beta}(z) = \mathcal{E}^{\gamma}_{\alpha,\beta}(z)$. Further, on setting $\gamma = \delta$, we get

$$\mathcal{E}^{\delta,\delta}_{\alpha,\beta}(z) = \mathcal{E}_{\alpha,\beta}(z). \tag{5}$$

In recent times the attention of mathematicians towards the Mittag-Leffler function has increased from both the analytical and numerical point of view. Motivated with the same, in this paper, we discuss the integral equation with $\mathcal{E}^{\gamma,\delta}_{\alpha,\beta}(z)$ in the kernel. For $0 < a < x < b < \infty$,

$$\frac{1}{\Gamma(\delta)} \int_a^x (x-t)^{\delta-1} \mathcal{E}^{\gamma,\delta}_{\alpha,\beta}(t-x) f(t) dt = g(x). \tag{6}$$

2. Preliminary Notes

Definition 1 *L denote the linear space of real (or complex) valued functions $f(x)$ which are L - integrable on a finite $[a, b]$, i.e.*

$$L(a,b) = \left\{ f : \|f\|_1 \equiv \int_a^b |f(t)|\, dt < \infty \right\}. \tag{7}$$

Definition 2 *Riemann-Liouville fractional integrals of order μ (Miller and Ross [7]): Let $f(x) \in L(a,b)$, $\mu \in C; \mathrm{Re}(\mu) > 0$. Then $I^\mu : L \to L$ is a linear operator defined by the fractional integral*

$$I^\mu f(x) = {}_aI_x^\mu f(x) = I_{a+}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}} dt, \tag{8}$$

for almost all $x \in (a,b)$.
It is well known that if I^μ is bounded and f is locally integrable. Then

$$I^\mu f = 0 \Rightarrow f = 0. \tag{9}$$

Hence, inverse operator exists on subspace L_μ of L. If $0 < \mathrm{Re}(\mu) < \mathrm{Re}(\nu)$, then it can be proved that $L_\nu \subset L_\mu \subset L$ and the inclusion is proper. For $\mathrm{Re}(\mu) < 0$, I^μ is defined as the inverse of $I^{-\mu}$. If $\mathrm{Re}(\mu) \neq 0, \mathrm{Re}(\nu) \neq 0$, then $I^\mu I^\nu f = I^{\mu+\nu} f$ for locally integrable functions f. Similarly, for $x < b < \infty$,

$$J^\mu f(x) = {}_xI_b^\mu f(x) = I_{b-}^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \frac{f(t)}{(t-x)^{1-\mu}} dt. \tag{10}$$

3. The fractional operator $\mathcal{G}$ and its properties

We define the operator $\mathcal{G}(\alpha, \beta, \gamma, \delta)$ associated with $\mathcal{E}_{\alpha,\beta}^{\gamma,\delta}$, on space L by

$$\mathcal{G} f(x) = \mathcal{G}(\alpha, \beta, \gamma, \delta) f(x) = \int_a^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)} \mathcal{E}_{\alpha,\beta}^{\gamma,\delta}(t-x) f(t) dt, \tag{11}$$

for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma), \mathrm{Re}(\delta) > 0$. First we need to justify the existence of the integral operator (11).

Theorem 3 *(Existence of the operator) If* $\mathrm{Re}(\mu) > 0, f \in L(a,b), \alpha, \beta, \gamma \in C$ *and* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma), \mathrm{Re}(\delta) > 0$, *then*

$$\int_a^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)} \mathcal{E}_{\alpha,\beta}^{\gamma,\delta}(t-x) f(t) dt$$

defines a function in L.

Proof: To prove $\mathcal{G}$ is a function, it is sufficient to prove $\|\mathcal{G}\|$ is finite, i.e., $\|\mathcal{G}\| < \infty$. By definition of the operator (11), we have

$$\|\mathcal{G}\| = \int_a^b \left| \int_a^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)} \mathcal{E}_{\alpha,\beta}^{\gamma,\delta}(t-x) f(t) dt \right| dx.$$

By change of order of integration, permitted under prescribed condition, and then substituting $t-x=v$, we get

$$\|\mathcal{G}\| = \int_a^b \int_t^b \left| f(t) \frac{(x-t)^{\delta-1}}{\Gamma(\delta)} \mathcal{E}_{\alpha,\beta}^{\gamma,\delta}(t-x) \right| dxdt,$$

$$\leq \int_a^b |f(t)|\, dt \int_0^{b-t} \left| \frac{(-1)^{\delta-1}}{\Gamma(\delta)} v^{\delta-1} \mathcal{E}_{\alpha,\beta}^{\gamma,q}(v) \right| dv.$$

Since $\mathcal{E}_{\alpha,\beta}^{\gamma,\delta}(v)$ is an entire function, it is bounded in (a,b). Hence, the double integral in above equation is finite.

Theorem 4 *If* $\alpha, \beta, \gamma, \delta, \lambda \in C$ *and* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma), \mathrm{Re}(\delta) > 0$, *then*

$$\frac{1}{\Gamma(\lambda)\Gamma(\delta)} \int_t^x (x-s)^{\lambda-1}(s-t)^{\delta-1} \mathcal{E}_{\alpha,\beta}^{\gamma,\delta}(t-s)ds = \frac{(x-t)^{\delta+\lambda-1}}{\Gamma(\lambda+\delta)} \mathcal{E}_{\alpha,\beta}^{\gamma,\lambda+\delta}(t-x) \quad (12)$$

Proof: In accordance with (4), changing order of integration and summation that is permitted under prescribed condition and substituting $s = t + (x-t)u$, we obtain

$$\frac{1}{\Gamma(\lambda)\Gamma(\delta)} \int_t^x (x-s)^{\lambda-1}(s-t)^{\delta-1} \mathcal{E}_{\alpha,\beta}^{\gamma,\delta}(t-s)ds$$

$$= \frac{1}{\Gamma(\delta)\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_n}{\Gamma(\alpha n+\beta)(\delta)_n} \int_0^1 (x-t)^{\lambda-1}(1-u)^{\lambda-1}(x-t)^{\delta+n-1}u^{\delta+n-1}(x-t)du.$$

This yields

$$\frac{1}{\Gamma(\delta)\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_n}{\Gamma(\alpha n+\beta)(\delta)_n} \int_0^1 (x-t)^{\lambda-1}(1-u)^{\lambda-1}(x-t)^{\delta+n-1}u^{\delta+n-1}(x-t)du.$$

The integral immediately leads to

$$\frac{1}{\Gamma(\delta)\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_n}{\Gamma(\alpha n+\beta)(\delta)_n} (x-t)^{\lambda+\delta+n-1} B(\lambda, \delta+n),$$

and (12) is proved.

Since the operator $\mathcal{G}(\alpha,\beta,\gamma,\delta)$ is an integral operator, its composition with Riemann-Liouville fractional integral operator I^{μ}, given by (8), is given in the form of following theorem.

Theorem 5 *(Shifting Property): Let* $\alpha, \beta, \gamma, \delta, \mu \in C$ *and* $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)$, $\mathrm{Re}(\gamma)$, $\mathrm{Re}(\delta)$, $\mathrm{Re}(\mu) > 0$. *Then*

$$I^{\mu}\mathcal{G}\left(\alpha, \beta, \gamma, \delta\right) f(x) = \mathcal{G}\left(\alpha, \beta, \gamma, \mu + \delta\right) f(x). \tag{13}$$

Proof: For $\mathrm{Re}(\mu) > 0$ and any f, which is L-integrable, by Theorem 3, $\mathcal{G}f$ is also L-integrable. Hence, the composition also belongs to L. i.e., $I^{\mu}\mathcal{G}f(x) \in L$. Therefore,

$$I^{\mu}\mathcal{G}f(x) = I^{\mu}\mathcal{G}\left(\alpha, \beta, \gamma, \delta\right) f(x).$$

From (11), we have

$$I^{\mu}\mathcal{G}f(x) = I^{\mu}\left[\int_a^s \frac{(s-t)^{\delta-1}}{\Gamma(\delta)}\mathcal{E}_{\alpha,\beta}^{\gamma,q}(t-s)f(t)dt\right],$$

$$= \frac{1}{\Gamma(\mu)}\int_a^x (x-s)^{\mu-1}ds \int_a^s \frac{(s-t)^{\delta-1}}{\Gamma(\delta)}\mathcal{E}_{\alpha,\beta}^{\gamma,q}(t-s)f(t)dt.$$

Since $\mathcal{E}_{\alpha,\beta}^{\gamma,\delta}$ is bounded in the region of integration, order of integration can be changed by Fubini's theorem.
On using Theorem 4, above equation reduces to

$$I^{\mu}\mathcal{G}f(x) = \int_a^x \frac{(x-t)^{\delta+\mu-1}}{\Gamma(\delta+\mu)}\mathcal{E}_{\alpha,\beta}^{\gamma,\delta+\mu}(t-x)f(t)dt,$$

$$= \mathcal{G}\left(\alpha, \beta, \gamma, \delta + \mu\right).$$

It can be observed that the composition of the fractional operator I^{μ} and the operator $\mathcal{G}$ result into shifting of the forth parameter δ of $\mathcal{G}$ by the order of fractional integral operator μ, whereas, all other parameters remain unaltered. Hence, this property is called the shifting property.

4. Solution of the integral equation (6)
From (6) and (11), we get

$$\mathcal{G}\left(\alpha, \beta, \gamma, \delta\right) f(x) = \int_a^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)}\mathcal{E}_{\alpha,\beta}^{\gamma,\delta}(t-x)f(t)dt = g(x). \tag{14}$$

Let us consider

$$\mathcal{G}\left(\alpha, \beta, \gamma, \delta\right) f(x) = g(x).$$

On invoking the fractional integral operator, given by (8) and using (13), we obtain

$$I^{\delta}\mathcal{G}\left(\alpha,\beta,\gamma,\gamma\right)f(x)=I^{\gamma}g(x).$$

Since inverse operator $I^{-\delta}$ exists, this can be written as

$$\mathcal{G}\left(\alpha,\beta,\gamma,\gamma\right)f(x)=I^{-\delta}I^{\gamma}g(x).$$

Therefore,

$$f(x)=\mathcal{G}^{-1}\left(\alpha,\beta,\gamma,\gamma\right)I^{-\delta}I^{\gamma}g(x). \tag{15}$$

Hence $f(x)$, given by (15), if exists, is a solution of the integral equation (6).

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ON CERTAIN TRANSFORMATION FORMULAS FOR ORDINARY HYPERGEOMETRIC SERIES

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Abstract: In this paper, using Bailey transform and summation formulas for ordinary hypergeometric series due to Verma and Jain, certain interesting transformation formulas for ordinary hypergeometric series have been established.
Keywords and Phrases: Bailey transform, ordinary hypergeometric series, summation formula, transformation formula.
Mathematics subject Classification: Primary 33D15, 33D90, 11A55; Secondary 11F20, 33F05.

1. Introduction, Notations and Definitions

Throughout this paper, we shall employ the Pochhammer symbol defined by,

$$(\lambda)_n = \begin{cases} 1, & \text{if } \; n=0 \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+n-1), & \text{if } \; n=1,2,3,\dots . \end{cases} \tag{1.1}$$

In term of Gamma function, we have

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \tag{1.2}$$

and

$$(\lambda)_{-n} = \frac{\Gamma(\lambda-n)}{\Gamma(\lambda)} = \frac{(-1)^n}{(1-\lambda)_n}, \quad n=1,2,...; \; \lambda \neq 0, \pm 1, \pm 2, \dots . \tag{1.3}$$

The generalized hypergeometric series is defined as,

$${}_rF_s\left[\begin{array}{l} a_1, a_2, ..., a_r; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_r)_n z^n}{(b_1)_n (b_2)_n ... (b_s)_n n!} \tag{1.4}$$

The ${}_rF_s$ series in (1.4) converges for $|z| < \infty$ if $r \leq s$; for $|z| < 1$ if $r = s + 1$ and diverges for all $z \neq 0$ if $r > s + 1$. Also, the denominator parameters $b_i \neq 0, -1, -2, ...$ for $i = 1, 2, ..., s$.

In 1947, Bailey [1] established the following simple but very useful transformation; If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.5}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}, \tag{1.6}$$

where u_r, v_r, α_r and δ_r are any functions of r alone and the series γ_n exists, then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{1.7}$$

provided both series of (1.7) converge.

We shall make use of following results.

$${}_3F_2\left[\begin{array}{l} -n, 1+n+2x+2y, x; 1 \\ 1+x+y, 1+2x \end{array}\right] = \frac{(1)_n (1+x)_m (1+y)_m}{(1+2x)_n (1+x+y)_m (1)_m}, \tag{1.8}$$

[Verma and Jain 3; (2.26)p. 1028]

where m is the greatest integer $\leq \frac{n}{2}$.

$${}_3F_2\left[\begin{array}{l} -n, 1+n+2x+2y, 1+x; 1 \\ 1+x+y, 1+2x \end{array}\right] = \frac{(-1)^n (1)_n (1+x)_m (1+y)_m}{(1+2x)_n (1+x+y)_m (1)_m}, \tag{1.9}$$

[Verma and Jain 3; (2.27)p. 1028]

where m is the greatest integer $\leq \frac{n}{2}$.

$$\begin{aligned} &{}_3F_2\left[\begin{array}{l} -n, 2+n+b+2x, x; 1 \\ 1+\frac{1}{2}b+x, 2+2x \end{array}\right] \\ &= \frac{(1)_n (2+b+x)_n \left(\frac{3}{2}+\frac{1}{2}b+x\right)_m \left(1+\frac{1}{2}b\right)_m \left(1+\frac{1}{2}x\right)_{2m}}{(1+x)_n (2+b+2x)_n (1)_m \left(\frac{3}{2}+x\right)_m \left(1+\frac{1}{2}b+\frac{1}{2}x\right)_{2m}}, \end{aligned} \tag{1.10}$$

[Verma and Jain 3; (3.2)p. 1033]

where m is the greatest integer $\leq \frac{n}{2}$.

$${}_3F_2\left[\begin{array}{l} -n, 2+n+b+2x, 1+x; 1 \\ 1+\frac{1}{2}b+x, 2+2x \end{array}\right] = \frac{(-1)^n (1)_n \left(\frac{3}{2}+\frac{1}{2}b+x\right)_m \left(1+\frac{1}{2}b\right)_m}{(2+b+2x)_n (1)_m (1+x)_m}, \tag{1.11}$$

[Verma and Jain 3; (3.4)p. 1033]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_4F_3\left[\begin{array}{l} x, \frac{1}{2}x-\frac{1}{2}, 1+a+n, -n; 1 \\ x-1, x+1, \frac{1}{2}+\frac{1}{2}a \end{array}\right] = \frac{(1)_n(1+a-x)_n\left(1+\frac{1}{2}a\right)_m\left(\frac{1}{2}+\frac{1}{2}a\right)_m}{(1+a)_n(1+x)_n(1)_m\left(\frac{1}{2}+\frac{1}{2}a-\frac{1}{2}x\right)_m}, \tag{1.12} $$

[Verma and Jain 3; (3.6)p. 1033]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_3F_2\left[\begin{array}{l} \frac{a}{3}, 1+a+n, -n; \frac{3}{4} \\ \frac{1}{2}+\frac{1}{2}a, 1+\frac{1}{2}a \end{array}\right] = \frac{(1)_n\left(1+\frac{a}{3}\right)_m}{(1+a)_n(1)_m}, \tag{1.13} $$

[Verma and Jain 3; (4.6)p. 1036]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_3F_2\left[\begin{array}{l} \frac{a}{3}, 1+a+n, -n; 1 \\ \frac{a}{2}, \frac{1}{2}+\frac{1}{2}a \end{array}\right] = \frac{(-1)^{n-m}(1)_m\left(1+\frac{a}{3}\right)_m}{(1+a)_n(1)_m}, \tag{1.14} $$

[Verma and Jain 3; (4.7)p. 1036]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_4F_3\left[\begin{array}{l} \frac{a}{3}, 1+\frac{a}{2}, 1+a+n, -n; 1 \\ \frac{a}{2}, \frac{1}{2}+\frac{1}{2}a, 2+\frac{a}{2} \end{array}\right] = \frac{(1)_n\left(\frac{a}{2}\right)_n\left(1+\frac{a}{3}\right)_m\left(2+\frac{a}{6}\right)_m}{(1+a)_n\left(2+\frac{a}{2}\right)_n(1)_m\left(\frac{a}{6}\right)_m}, \tag{1.15} $$

[Verma and Jain 3; (4.9)p. 1037]

where m is the greatest integer $\leq \frac{n}{2}$.

$$ {}_1F_0[a; -; z] = (1-z)^{-a}, \quad |z| < 1. \tag{1.16} $$

[Slater 2; (2.2.2.2)p. 46]

2. Main Results

In this section we have established following results

(i)

$$ (1-z)^{-2x-2y-1}\, {}_2F_1\left[\begin{array}{l} x, \frac{1}{2}+x+y; -\frac{4z}{(1-z)^2} \\ 1+2x \end{array}\right] = {}_2F_1\left[\begin{array}{l} 1+y, \frac{1}{2}+x+y; z^2 \\ \frac{1}{2}+x \end{array}\right] $$

$$+\frac{z(1+2x+2y)}{(1+2x)}\,{}_2F_1\left[\begin{array}{l}1+y,\frac{3}{2}+x+y;z^2\\ \frac{3}{2}+x\end{array}\right], \tag{2.1}$$

provided max. $\left(|z|,\left|\frac{4z}{(1-z)^2}\right|\right)<1$.

(ii)

$$(1-z)^{-2x-2y-1}\,{}_2F_1\left[\begin{array}{l}1+x,\frac{1}{2}+x+y;-\frac{4z}{(1-z)^2}\\ 1+2x\end{array}\right]={}_2F_1\left[\begin{array}{l}1+y,\frac{1}{2}+x+y;z^2\\ \frac{1}{2}+x\end{array}\right]$$

$$-\frac{z(1+2x+2y)}{(1+2x)}\,{}_2F_1\left[\begin{array}{l}1+y,\frac{3}{2}+x+y;z^2\\ \frac{3}{2}+x\end{array}\right], \tag{2.2}$$

provided max. $\left(|z|,\left|\frac{4z}{(1-z)^2}\right|\right)<1$.

(iii)

$$(1-z)^{-2-2x-b}\,{}_2F_1\left[\begin{array}{l}\frac{3}{2}+\frac{1}{2}b+x,x;-\frac{4z}{(1-z)^2}\\ 2x+2\end{array}\right]$$

$$={}_7F_6\left[\begin{array}{l}1+x+\frac{b}{2},\frac{3}{2}+x+\frac{b}{2},1+\frac{b}{2}+\frac{x}{2},\frac{3}{2}+\frac{b}{2}+\frac{x}{2},1+\frac{b}{2},\frac{1}{2}+\frac{x}{4},1+\frac{x}{4};z^2\\ \frac{1}{2}+\frac{x}{2},1+\frac{x}{2},1+\frac{b}{2}+x,\frac{3}{2}+x,\frac{1}{2}+\frac{b}{4}+\frac{x}{4},1+\frac{b}{4}+\frac{x}{4}\end{array}\right]$$

$$+\frac{(2+b+x)z}{(1+x)}\,{}_7F_6\left[\begin{array}{l}\frac{3}{2}+\frac{b}{2}+x,2+x+\frac{b}{2},\frac{3}{2}+\frac{b}{2}+\frac{x}{2},2+\frac{b}{2}+\frac{x}{2},\frac{1}{2}+\frac{x}{4},1+\frac{x}{4},1+\frac{b}{2};z^2\\ 1+\frac{x}{2},\frac{3}{2}+x,2+\frac{b}{2}+x,\frac{3}{2}+x,\frac{1}{2}+\frac{b}{4}+\frac{x}{4},1+\frac{b}{4}+\frac{x}{4}\end{array}\right] \tag{2.3}$$

provided max. $\left(|z|,\left|\frac{4z}{(1-z)^2}\right|\right)<1$.

(iv)

$$(1-z)^{-3-2x-b}\,{}_2F_1\left[\begin{array}{l}\frac{3}{2}+\frac{b}{2}+x,1+x;-\frac{4z}{(1-z)^2}\\ 2+2x\end{array}\right]={}_2F_1\left[\begin{array}{l}\frac{3}{2}+\frac{b}{2}+x,1+\frac{b}{2};z^2\\ \frac{3}{2}+x\end{array}\right], \tag{2.4}$$

provided max. $\left(|z|,\left|\frac{4z}{(1-z)^2}\right|\right)<1$.

(v)

$$(1-z)^{-1-a}\,{}_3F_2\left[\begin{array}{l}x,\frac{x}{2}-\frac{1}{2},1+\frac{a}{2};-\frac{4z}{(1-z)^2}\\ x-1,x+1\end{array}\right]={}_3F_2\left[\begin{array}{l}1+\frac{a}{2}-\frac{x}{2},1+\frac{a}{2},\frac{1}{2}+\frac{a}{2};z^2\\ \frac{1}{2}+\frac{x}{2},1+\frac{x}{2}\end{array}\right]$$

$$+\frac{(1+a-x)}{(1+x)}\,{}_4F_3\left[\begin{array}{l}1+\frac{a}{2}-\frac{x}{2},\frac{3}{2}+\frac{a}{2}-\frac{x}{2},1+\frac{a}{2},\frac{1}{2}+\frac{a}{2};z^2\\ 1+\frac{x}{2},\frac{3}{2}+\frac{x}{2},\frac{1}{2}+\frac{a}{2}-\frac{x}{2}\end{array}\right], \tag{2.5}$$

provided max. $\left(|z|, \left|\frac{4z}{(1-z)^2}\right|\right) < 1$.

(vi)

$$(1-z)^{-1-a}\ {}_1F_0\left[\frac{a}{3};-;-\frac{3z}{(1-z)^2}\right] = (1+z+z^2)\ {}_1F_0\left[1+\frac{a}{3};-;z^3\right] \tag{2.6}$$

provided max. $\left(|z|, \left|\frac{3z}{(1-z)^2}\right|\right) < 1$.

(vii)

$$(1-z)^{-1-a}\ {}_2F_1\left[\begin{array}{l}\frac{1}{2}+\frac{a}{2},\frac{a}{3};-\frac{3z}{(1-z)^2}\\ \frac{a}{2}\end{array}\right] = (1-z+z^2)\ {}_1F_0\left[1+\frac{a}{3};-;z^3\right] \tag{2.7}$$

provided max. $\left(|z|, \left|\frac{3z}{(1-z)^2}\right|\right) < 1$.

(viii)

$$\begin{aligned}(1-z)^{-1-a}\ {}_3F_2\left[\begin{array}{l}\frac{a}{3},1+\frac{a}{2},1+\frac{a}{2};-\frac{3z}{(1-z)^2}\\ \frac{a}{2},2+\frac{a}{2}\end{array}\right] &= {}_3F_2\left[\begin{array}{l}\frac{1}{3}+\frac{a}{6},2+\frac{a}{6},1+\frac{a}{3};z^3\\ 1+\frac{a}{6},\frac{4}{3}+\frac{a}{6}\end{array}\right]\\ &+\frac{az}{a+4}\ {}_4F_3\left[\begin{array}{l}\frac{1}{3}+\frac{a}{6},2+\frac{a}{6},\frac{2}{3}+\frac{a}{6},1+\frac{a}{3};z^3\\ \frac{a}{6},\frac{4}{3}+\frac{a}{6},\frac{5}{3}+\frac{a}{6}\end{array}\right]\\ &+\frac{a(a+2)z^2}{(a+4)(a+6)}\ {}_3F_2\left[\begin{array}{l}\frac{2}{3}+\frac{a}{6},1+\frac{a}{6},1+\frac{a}{3};z^3\\ \frac{a}{6},\frac{5}{3}+\frac{a}{6}\end{array}\right],\end{aligned} \tag{2.8}$$

provided max. $\left(|z|, \left|\frac{3z}{(1-z)^2}\right|\right) < 1$.

Proof of (2.1)-(2.8)

As an illustration, we give the proof of (2.1).
Choosing $u_r = \frac{1}{(1)_r}$, $v_r = (1+2x+2y)_r$, $\alpha_r = \frac{(x)_r(-1)^r}{r!(1+x+y)_r(1+2x)_r}$ and $\delta_r = z^r$ in (1.5) and (1.6) and applying the summation formula (1.8) and binomial theorem (1.16) respectively we find the values of β_n and γ_n. Putting these values of α_n, β_n, γ_n and δ_n in (1.7) we get (2.1) after some simplifications.
Proceeding as above by choosing properly u_r, v_r, α_r and δ_r in (1.5) and (1.6) and then applying the summation formulas (1.9)-(1.15) and binomial theorem (1.16) one can easily calculate β_n and γ_n. Putting these values in (1.7) we find (2.2)-(2.8).

3. Special Cases

In this section we have deduced certain results as special cases of the results of section 2.

(i) Taking $y = -1$ in (2.1) we get,

$$ {}_2F_1\left[\begin{array}{l} x, x-\frac{1}{2}; -\frac{4z}{(1-z)^2} \\ 1+2x \end{array}\right] = \left\{1 + \frac{z(2x-1)}{(2x+1)}\right\}(1-z)^{2x-1}. \tag{3.1} $$

(ii) For $y = 0$ (2.1) yields,

$$ {}_2F_1\left[\begin{array}{l} x, x+\frac{1}{2}; -\frac{4z}{(1-z)^2} \\ 1+2x \end{array}\right] = (1-z)^{2x}. \tag{3.2} $$

(iii) Taking $y = -1$ in (2.2) we get,

$$ {}_2F_1\left[\begin{array}{l} 1+x, x-\frac{1}{2}; -\frac{4z}{(1-z)^2} \\ 2x+1 \end{array}\right] = \left\{1 - \frac{z(2x-1)}{(2x+1)}\right\}(1-z)^{2x-1}. \tag{3.3} $$

(iv) Putting $y = 0$ in (2.2) we find,

$$ {}_2F_1\left[\begin{array}{l} 1+x, \frac{1}{2}+x; -\frac{4z}{(1-z)^2} \\ 2x+1 \end{array}\right] = \frac{(1-z)^{2x+1}}{(1+z)}. \tag{3.4} $$

(v) Adding (3.1) and (3.3) we have,

$$ {}_2F_1\left[\begin{array}{l} x, x-\frac{1}{2}; -\frac{4z}{(1-z)^2} \\ 2x+1 \end{array}\right] + {}_2F_1\left[\begin{array}{l} 1+x, x-\frac{1}{2}; -\frac{4z}{(1-z)^2} \\ 2x+1 \end{array}\right] $$
$$ = 2(1-z)^{2x-1}. $$

(vi) Substracting (3.3) from (3.1) we obtain,

$$ {}_2F_1\left[\begin{array}{l} x, x-\frac{1}{2}; -\frac{4z}{(1-z)^2} \\ 2x+1 \end{array}\right] - {}_2F_1\left[\begin{array}{l} 1+x, x-\frac{1}{2}; -\frac{4z}{(1-z)^2} \\ 2x+1 \end{array}\right] $$
$$ = \frac{2z(2x-1)}{(2x+1)}(1-z)^{2x-1}. \tag{3.6} $$

(vii) Taking $b = -2$ in (2.3) we get,

$$ {}_2F_1\left[\begin{array}{l} x, x+\frac{1}{2}; -\frac{4z}{(1-z)^2} \\ 2x+2 \end{array}\right] = \left(1 + \frac{x}{(1+x)}\right)(1-z)^{2x}. \tag{3.7} $$

(viii) Putting $b = -2$ in (2.4) we get,

$$ {}_2F_1\left[\begin{array}{l} 1+x, x+\frac{1}{2}; -\frac{4z}{(1-z)^2} \\ 2x+2 \end{array}\right] = (1-z)^{2x+1}. \tag{3.8} $$

(ix) For $b = 0$, (2.4) yields,

$$ {}_2F_1\left[\begin{array}{l} \frac{3}{2}+x, 1+x; -\frac{4z}{(1-z)^2} \\ 2x+2 \end{array}\right] = \frac{(1-z)^{2x+2}}{(1+z)}. \tag{3.9}$$

(x) From (3.8) and (3.9) we get

$$ (1-z)\ {}_2F_1\left[\begin{array}{l} 1+x, x+\frac{1}{2}; -\frac{4z}{(1-z)^2} \\ 2x+2 \end{array}\right] = (1+z)\ {}_2F_1\left[\begin{array}{l} \frac{3}{2}+x, 1+x; -\frac{4z}{(1-z)^2} \\ 2x+2 \end{array}\right] \tag{3.10}$$

(xi) Taking $a = -1$ in (2.5) we have

$$ {}_3F_2\left[\begin{array}{l} x, \frac{x}{2}-\frac{1}{2}, \frac{1}{2}; -\frac{4z}{(1-z)^2} \\ x-1, x+1 \end{array}\right] = \left(1 - \frac{xz}{1+x}\right). \tag{3.11}$$

(xii) For $a = -3$, (2.7) yields

$$ {}_2F_1\left[\begin{array}{l} -\frac{1}{2}, -1; -\frac{3z}{(1-z)^2} \\ -\frac{3}{2} \end{array}\right] = \frac{(1-z+z^2)}{(1-z)^2}. \tag{3.12}$$

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SET–THEORETICAL ENTROPIES OF EULER'S TOTIENT FUNCTION AND OTHER NUMBER THEORETICAL SPECIAL FUNCTIONS

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Abstract: In the following text we show set–theoretical entropy of Euler's totient function and contravariant set–theoretical entropy of Dedekind psi function are zero. Also contravariant set–theoretical entropy of Euler's totient function and set–theoretical entropy of Dedekind psi function are $+\infty$. We pay attention to some of the other number theoretical special functions too. We continue our studies on Alexandroff topologies induced by Euler's totient function and Dedekind psi function.
Keywords and Phrases: Alexandroff topology, Dedekind psi function, Euler's totient function, Infinite orbit number, Infinite anti–orbit number, Set–theoretical entropy
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1. Introduction

Various types of entropies have been studied in different branches of mathematics. In category Set one may consider set–theoretical and contravariant set–theoretical entropies of self–maps. Our main aim in this text is to study set–theoretical behaviour of some well–known number theoretical maps like Euler's totient function, Dedekind psi function and their generalizations. However set–theoretical and contravariant set–theoretical entropies of a "*nice*" self–maps have interactions with infinite orbit number's concept and infinite anti–orbit number's concept so we pay attention to these concepts too. We continue our studies in topological arising concepts in this regard, our main emphasis in topological point of view deals

with Alexandroff topological spaces' approach.
Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers and $\mathbb{P} = \{2, 3, 5, 7, \ldots\}$ the set of prime numbers. For finite set A by $\sharp A$ we mean the number of elements of A. Also we say $\lambda : X \to X$ is finite fibre if $\lambda^{-1}(x)$ is finite for all $x \in X$.
Background on infinite orbit number and infinite anti–orbit number of a self–map. For self–map $\lambda : X \to X$ we say the one–to–one sequence $\{a_n\}_{n\geq 1}$ is:
• an infinite $\lambda-$orbit if for all $n \geq 1$, $a_{n+1} = \lambda(a_n)$,
• an infinite $\lambda-$anti–orbit if for all $n \geq 1$, $a_n = \lambda(a_{n+1})$ (see e.g., [6, Definition 1.2] and [7, Definition 1.1]).
Moreover we set [4]:

$$\begin{aligned}\mathfrak{o}(\lambda) &:= \sup(\{n \geq 1 : \text{there exists } n \text{ disjoint infinite } \lambda-\text{orbit sequences}\} \cup \{0\}),\\ \mathfrak{a}(\lambda) &:= \sup(\{n \geq 1 : \text{there exists } n \text{ disjoint infinite } \lambda-\text{anti–orbit}\\ &\qquad \text{sequences}\} \cup \{0\}),\end{aligned}$$

we call $\mathfrak{o}(\lambda)$ infinite orbit number of λ and $\mathfrak{a}(\lambda)$ infinite anti–orbit number of λ.
Background on set–theoretical and contravariant set–theoretical entropies. For $\lambda : X \to X$ and finite subset A of X the following limit exists

$$\mathrm{ent}_{set}(\lambda, A) = \lim_{n\to\infty} \frac{\sharp(A \cup \lambda(A) \cup \cdots \cup \lambda^{n-1}(A))}{n}$$

and we call $\mathrm{ent}_{set}(\lambda) := \sup\{\mathrm{ent}_{set}(\lambda, B) : B \text{ is a finite subset of } X\}$ set–theoretical entropy of λ [2]. Moreover for finite fibre onto map $\mu : X \to X$ and finite sunset A of X the following limit exists

$$\mathrm{ent}_{cset}(\mu, A) = \lim_{n\to\infty} \frac{\sharp(A \cup \mu^{-1}(A) \cup \cdots \cup \mu^{-(n-1)}(A))}{n}$$

and we call $\mathrm{ent}_{cset}(\mu) := \sup\{\mathrm{ent}_{cset}(\mu, B) : B \text{ is a finite subset of } X\}$ set–theoretical entropy of μ. On the other hand if $\lambda : X \to X$ is finite fibre and $sc(\lambda) := \bigcap\limits_{n\geq 1} \lambda^n(X)$, then $\lambda \restriction_{sc(\lambda)}: sc(\lambda) \to sc(\lambda)$ is finite fibre and onto, we call $\mathrm{ent}_{cset}(\lambda) := \mathrm{ent}_{cset}(\lambda \restriction_{sc(\lambda)})$ contravariant set–theoretical entropy of λ [5].
Note 1.1. For $\lambda : X \to X$ we have $\mathrm{ent}_{set}(\lambda) = \mathfrak{o}(\lambda)$ [2, Proposition 2.16], also for finite fibre $\lambda : X \to X$ we have $\mathrm{ent}_{cset}(\lambda) = \mathfrak{a}(\lambda)$ [5, Theorems 3.2, 3.9].
Some number theoretical special functions. Let's recall the following functions ($n \geq 1$ and for convenient suppose all of them map 1 to 1):
• Jordan's totient function (for $k \geq 1$): $J_k(n) = \{(s_1, \ldots, s_k) : s_1, \ldots, s_k \in \{1, \ldots, n\}, \gcd(s_1, \ldots, s_k, n) = 1\}(n^k \prod\{1 - \frac{1}{p^k} : p \in \mathbb{P}, p|n\})$ (so well–known Euler's totient function φ is J_1) (see e.g., [13])
• Generalized Dedekind psi function (for $k \geq 1$): $\psi_k(n) = n^k \prod\{1 + \frac{1}{p^k} : p \in$

$\mathbb{P}, p|n\} = \frac{J_{2k}(n)}{J_k(n)}$ (so well–known Dedekind psi function, $\psi(=\psi_1)$ is $\frac{J_2}{J_1}$) [12]
• Unitary totient function: $\varphi^*(n) = \prod\{p^\alpha - 1 : p^\alpha|n, p^{\alpha+1} \not| n, p \in \mathbb{P}, \alpha \geq 1\}$ (see e.g., [8, 11])
• $\Omega(n) = \sum\{\alpha : p^\alpha|n, p^{\alpha+1} \not| n, p \in \mathbb{P}\}$ [9]
• $\omega(n) = \sum\{1 : p^\alpha|n, p^{\alpha+1} \not| n, p \in \mathbb{P}, \alpha \geq 1\}$
• $d_l(n) = \sharp\{(s_1, \cdots, s_l) \in \mathbb{N}^l : s_1 \cdots s_l = n\}$ $(l \geq 2)$ (we denote d_2 with d) [9]
• $\sigma_l(n) = \sum\limits_{d|n, d\leq n} d^l$ $(l \geq 2)$ [9]

2. Infinite orbit number and infinite anti–orbit number of φ

In this section we compute infinite orbit number and infinite anti–orbit number of Euler's totient function, Dedekind psi function and some other well–known maps.

Lemma 2.1. For $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \leq n$ we have $\mathfrak{o}(f) = 0$.
Proof. Suppose $\{x_n\}_{n\geq 1}$ is an infinite $f-$orbit and for all $n \geq 1$ we have $f(n) \leq n$, thus $x_1, x_2 = f(x_1), x_3 = f^2(x_1), \ldots \in \{1, 2, \ldots, x_1\}$, thus $\{x_n\}_{n\geq 1}$ is not infinite and one–to–one sequence.

■

Lemma 2.2. For $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \geq n$ we have $\mathfrak{a}(f) = 0$.
Proof. Suppose $\{x_n\}_{n\geq 1}$ is an infinite $f-$anti–orbit and for all $n \geq 1$ we have $f(n) \geq n$, thus for all $n \geq 1$ we have $x_1 = f^n(x_{n+1}) \geq f^{n-1}(x_{n+1}) \geq \cdots \geq x_{n+1}$, so $x_1, x_2, x_3, \ldots \in \{1, 2, \ldots, x_1\}$ thus $\{x_m\}_{m\geq 1}$ is not infinite and one–to–one.

■

Lemma 2.3. For $f : \mathbb{N} \to \mathbb{N}$ with $f(n) > n$ for all $n > 1$, we have $\mathfrak{o}(f) > 0$.
Proof. Let $x \geq 2$, then $\{f^n(x)\}_{n\geq 1}$ is an infinite $f-$orbit, thus $\mathfrak{o}(f) > 0$.

■

Lemma 2.4. For $k \geq 1$ let $S_k := \{2^k 3^n\}_{n\geq 1}$, then $S_1, S_2, \ldots$ are disjoint infinite $\varphi-$anti–orbit sequences, so $\mathfrak{a}(\varphi) = +\infty$.
In addition for all $n \geq 1$, $\varphi(n) \leq n$ so $\mathfrak{o}(\varphi) = 0$.
Proof. For $n, k, s, t \geq 1$ we have $\varphi(2^k 3^{n+1}) = 2^{k-1}(2-1)3^k(3-1) = 2^k 3^n$ moreover $2^k 3^n = 2^s 3^t$ if and only if $k = s$ and $n = t$.

■

Lemma 2.5. For $p \in \mathbb{P} \setminus \{2\}$ let $S_p := \{x_n^p\}_{n\geq 1}$ with

- $x_1^p = p$,
- $x_{n+1}^p = p^{x_n^p - 1}$ $(n \geq 1)$,

then $S_3, S_5, S_7, S_{11}, \ldots$ are disjoint infinite $d-$anti–orbit sequences, so $\mathfrak{a}(d) = +\infty$.
In addition for all $n \geq 1$, $d(n) \leq n$ so $\mathfrak{o}(d) = 0$.
Proof. For $n, m \geq 1$ and $p, q \in \mathbb{P} \setminus \{2\}$ we have $d(x^p_{n+1}) = d(p^{x^p_n - 1}) = x^p_n - 1 + 1 = x^p_n$ moreover if $x^p_n = x^q_m$ then the unique prime divisor of x^p_n is p and the unique prime divisor of x^q_m is q, so $p = q$ thus $x^p_n = x^p_m$ moreover for all $i \geq 1$ we have $x^p_i < x^p_{i+1}$, so $x^p_n = x^p_m$ leads to $n = m$.

■

Lemma 2.6. For $p \in \mathbb{P}$ let $S_p := \{x^p_n\}_{n \geq 1}$ with

- $x^p_1 = p$,
- $x^p_{n+1} = p^{x^p_n}$ $(n \geq 1)$,

then $S_2, S_3, S_5, S_7, S_{11}, \ldots$ are disjoint infinite $\Omega-$anti–orbit sequences, so $\mathfrak{a}(\Omega) = +\infty$.
In addition for all $n \geq 1$, $\Omega(n) \leq n$ so $\mathfrak{o}(\Omega) = 0$.
Proof. For $n, m \geq 1$ and $p, q \in \mathbb{P}$ we have $\Omega(x^p_{n+1}) = \Omega(p^{x^p_n}) = x^p_n$ moreover if $x^p_n = x^q_m$ then the unique prime divisor of x^p_n is p and the unique prime divisor of x^q_m is q, so $p = q$ thus $x^p_n = x^p_m$ moreover for all $i \geq 1$ we have $x^p_i < x^p_{i+1}$, so $x^p_n = x^p_m$ leads to $n = m$.

■

Lemma 2.7. For $p \in \mathbb{P} \setminus \{2\}$ let $S_p := \{x^p_n\}_{n \geq 1}$ with (suppose q_n is the nth prime number):

- $x^p_1 = p = q_j$,
- $x^p_{n+1} = pq_{j+1}q_2 \cdots q_{j+x^p_n-1}$ $(n \geq 1)$,

then $S_3, S_5, S_7, S_{11}, \ldots$ are disjoint infinite $\omega-$anti–orbit sequences, so $\mathfrak{a}(\omega) = +\infty$.
In addition for all $n \geq 1$, $\omega(n) \leq n$ so $\mathfrak{o}(\omega) = 0$.
Proof. For $n, m \geq 1$ and $p, q \in \mathbb{P} \setminus \{2\}$ we have

$$\omega(x^p_{n+1}) = \omega(pq_{j+1}q_{j+2} \cdots q_{j+x^p_n-1}) = x^p_n$$

moreover if $x^p_n = x^q_m$ then the least prime divisor of x^p_n is p and the least prime divisor of x^q_m is q, so $p = q$ thus $x^p_n = x^p_m$ moreover for all $i \geq 1$ we have $x^p_i < x^p_{i+1}$, so $x^p_n = x^p_m$ leads to $n = m$.

■

Lemma 2.8. For $k \geq 1$ let $S_k := \{3^k 2^n\}_{n\geq 1}$, then $S_1, S_2, \dots$ are disjoint infinite $\psi-$orbit sequences, so $\mathfrak{o}(\psi) = +\infty$.
In addition for all $n \geq 1$, $\psi(n) \geq n$ so $\mathfrak{a}(\psi) = 0$.
Proof. For $n, m, s, t \geq 1$ we have $\psi(3^m 2^n) = 3^{m-1}(3+1)2^{n-1}(2+1) = 3^m 2^{n+1}$ moreover if $3^m 2^n = 3^s 2^t$ if and only if $m = s$ and $n = t$.

■

Lemma 2.9. For $k \geq 1$ let $S_k := \{2^{2^{n+1}k+2^n-1}3\}_{n\geq 1}$, then $S_1, S_2, \dots$ are disjoint infinite J_2-orbit sequences, so $\mathfrak{o}(J_2) = +\infty$.
In addition for all $n \geq 1$, $J_2(n) \geq n$ so $\mathfrak{a}(J_2) = 0$.
Proof. For $n, m, s, t \geq 1$ we have $J_2(2^{2^{n+1}m+2^n-1}3) = 2^{2(2^{n+1}m+2^n-1)-2}(2^2 - 1)3^{2-2}(3^2-1) = 2^{2(2^{n+1}m+2^n-1)+1}3 = 2^{2^{(n+1)}m+2^{n+1}-1}3$ moreover:

$$\begin{aligned} 2^{2^{n+1}m+2^n-1}3 = 2^{2^{s+1}t+2^s-1}3 &\Leftrightarrow 2^{n+1}m + 2^n - 1 = 2^{s+1}t + 2^s - 1 \\ &\Leftrightarrow 2^n(2m+1) = 2^s(2t+1) \\ &\Leftrightarrow n = s \wedge m = t\,. \end{aligned}$$

■

Note 2.10. Suppose $f : \mathbb{N} \to \mathbb{N}$ is a multiplicative function $g : \mathbb{P} \times \mathbb{N} \to \mathbb{N} \cup \{0\}$ and $h : \mathbb{P} \to \mathbb{N} \cup \{0\}$ such that $f(1) = 1$ and $f(p^n) = p^{g(p,n)}h(p)(\geq 1)$ for all $p \in \mathbb{P}$ and $n \geq 1$. Also suppose there exist distinct $p, q \in \mathbb{P}$ and $u, v \geq 1$ with $p^u = h(q)$ and $q^v = h(p)$ define $s, t : \mathbb{N} \to \mathbb{N}$ with $s(x) = g(p, x) + u$, $t(x) = g(q, x) + v$. If there exists $(x_1, y_1), (x_2, y_2), \dots \in \mathbb{N} \times \mathbb{N}$ such that $\underset{(n,m)\mapsto(s^n(x_m),t^n(y_m))}{\mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}}$ is one–to–one, then $\mathfrak{o}(f) = +\infty$ since for $S_n := \{p^{s^i(x_n)}q^{t^i(y_n)}\}_{i\geq 1}$, the sequences $S_1, S_2, \dots$ are disjoint infinite $f-$orbit sequences (use the fact that

$$\begin{aligned} f(p^{s^i(x_n)}q^{t^i(y_n)}) &= p^{g(p,s^i(x_n))}h(p)q^{g(q,t^i(y_n))}h(q) \\ &= p^{g(p,s^i(x_n))+u}q^{g(q,t^i(y_n))+v} = p^{s^{i+1}(x_n)}q^{t^{i+1}(y_n)}\ ). \end{aligned}$$

Lemmas 2.8 and 2.9 are examples of the above construction.
Note 2.11. As a generalization of Note 2.10 suppose $f : \mathbb{N} \to \mathbb{N}$ is a multiplicative function $g : \mathbb{P} \times \mathbb{N} \to \mathbb{N} \cup \{0\}$ and $h : \mathbb{P} \to \mathbb{N} \cup \{0\}$ such that $f(1) = 1$ and $f(p^n) = p^{g(p,n)}h(p)(\geq 1)$ for all $p \in \mathbb{P}$ and $n \geq 1$. Also suppose there exist distinct $p_1, \dots, p_m \in \mathbb{P}$ and $(u_1^i, \dots, u_m^i) \in \mathbb{N}^m$ (for $i = 1, \dots, m$) with $u_j^i + g(p_i, x) \geq 1$ for all i, j, x and $p_1^{u_1^i} \cdots p_m^{u_m^i} = h(p_i)$ define $s_i : \mathbb{N} \to \mathbb{N}$ with $s_i(x) = g(p_i, x) + (u_i^1 + \cdots + u_i^m)$. If there exists $(x_1^1, x_1^2, \dots, x_1^m), (x_2^1, x_2^2, \dots, x_2^m), \dots \in \mathbb{N} \times \mathbb{N}$ such that $\underset{(i,j)\mapsto(s_1^i(x_j^1),s_2^i(x_j^2),\dots,s_m^i(x_j^m))}{\mathbb{N}^l \to \mathbb{N}^l}$ is one–to–one, then $\mathfrak{o}(f) = +\infty$ since for $S_n := \{p_1^{s_1^i(x_n^1)}p_2^{s_2^i(x_n^2)} \cdots p_m^{s_m^i(x_n^m)}\}_{i\geq 1}$, the sequences $S_1, S_2, \dots$ are disjoint infinite

$f-$orbit sequences.

Table 2.12. We have the following table:

	λ	$\mathfrak{o}(\lambda)$	$\mathfrak{a}(\lambda)$
$1st.\ row$	$\varphi = J_1, d(=d_2), \Omega, \omega$	0	$+\infty$
$2nd.\ row$	φ^*	0	
$3rd.\ row$	$J_2, \psi(=\psi_1)$	$+\infty$	0
$4th.\ row$	$\sigma_k, \psi_k, J_{k+2}(k \geq 1)$	> 0	0

Proof. For the 1st. row use Lemmas 2.4, 2.5, 2.6, and 2.7.
For the 2nd. row use Lemma 2.1.
For the 3rd. row use Lemmas 2.8 and 2.9.
For the 4th. row use Lemmas 2.2, 2.3 and the fact that for all $n \geq 2$ we have $\sigma_k(n) > n, \psi_k(n) > n, J_{k+2}(n) > n$ (for $k \geq 1$).

■

Note 2.13. 1. For $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \geq n$ (for all $n \geq 1$) we have $f^{-1}(m) \subseteq \{1, \ldots, m\}$ (for all $m \geq 1$) and $f : \mathbb{N} \to \mathbb{N}$ is finite fibre, thus $\sigma_k, \psi_k, J_{k+1}$ (for $k \geq 1$) are finite fibre.
2. For distinct prime numbers $p_1, \ldots, p_n$ and $\alpha_1, \ldots, \alpha_n \geq 1$ with $\varphi(p_1^{\alpha_1} \cdots p_n^{\alpha_n}) = m$ we have $p_i - 1 \leq m$ and $2^{\alpha_i - 1} \leq p_i^{\alpha_i - 1} \leq m$ for all $i = 1, \ldots, n$, so $p_1, \ldots, p_n \leq m + 1$ and $\alpha_1, \ldots, \alpha_n \leq \dfrac{\log m}{\log 2} + 1$ therefore

$$p_1^{\alpha_1} \cdots p_n^{\alpha_n} \leq \prod \left\{ p^{\left[\frac{\log m}{\log 2} + 1\right]} : p \in \mathbb{P}, p \leq m + 1 \right\},$$

hence for all $m \geq 1$ we have

$$\varphi^{-1}(m) \subseteq \left\{ 1, \ldots, \prod \left\{ p^{\left[\frac{\log m}{\log 2} + 1\right]} : p \in \mathbb{P}, p \leq m + 1 \right\} \right\}.$$

Thus for all $m \geq 1$, $\varphi^{-1}(m)$ is finite and φ is finite fibre.
3. For $k \geq 2$ we have $\mathbb{P} \subseteq \omega^{-1} \cap \Omega^{-1}(1) \cap d_k^{-1}(k)$ thus ω, Ω, d_k are not finite fibre.
Table 2.14. By Table 2.12 and Note 2.13 we have the following table (where "$-$" indicates that for the corresponding case λ is not finite fibre and contravariant set–theoretical entropy of λ is undefined):

λ	$\mathrm{ent_{set}}(\lambda)$	$\mathrm{ent_{cset}}(\lambda)$
$\varphi(=J_1)$	0	$+\infty$
Ω, ω	0	$-$
$J_2, \psi(=\psi_1)$	$+\infty$	0
$\sigma_k, \psi_k, J_{k+2}(k \geq 1)$	> 0	0

Problem 2.15. Consider $k \geq 1$:
• Compute $\mathfrak{a}(\varphi^*), \mathfrak{o}(d_{k+2}), \mathfrak{a}(d_{k+2})$.
• For $\lambda = \sigma_k, \psi_{k+1}, J_{k+2}$ compute $\mathrm{ent}_{set}(\lambda)$.

3. Some notes on Euler's totient function and Alexandroff topologies on $\mathbb{N}$

We call topological space X Alexandroff, if intersection of any nonempty family of open sets is open [1]. In Alexandroff topological space (X, τ) for every $x \in X$ we denote the smallest open neighbourhood of $x \in X$ with $V(x, \tau)$. For $f : X \to X$:

- $\mathcal{B} = \{\bigcup\{f^{-n}(x) : n \geq 0\} : x \in X\}$
- $\overline{\mathcal{B}} = \{\{f^n(x) : n \geq 0\} : x \in X\}$

are basis of Alexandroff topologies on X. We call topology generated by $\mathcal{B}$, functional Alexandroff topology on X (with respect to f) and denote this topology by τ_f [3]. We call topology generated by $\overline{\mathcal{B}}$, Alexandroff topology on X with respect to f and denote this topology by $\overline{\tau}_f$ [10]. For $f : X \to X$ and $x \in X$, we have:

- $V(x, \tau_f) = \bigcup\{f^{-n}(x) : n \geq 0\}$,
- $V(x, \overline{\tau}_f) = \{f^n(x) : n \geq 0\}$.

As it has been mentioned in [4], set–theoretical entropies of $f : X \to X$ interact with cellularities of the above mentioned Alexandroff spaces on X. So we devote this section to arising Alexandroff topologies from some of number theoretical functions.

Lemma 3.1. For $f : \mathbb{N} \to \mathbb{N}$ and $k \in \mathbb{N}$ we have:

1. if for $n \geq 1$ we have $f(n) \geq n$, then $V(k, \tau_f) \subseteq \{1, \ldots, k\}$;
2. if for $n \geq 1$ we have $f(n) \leq n$, then $V(k, \overline{\tau}_f) \subseteq \{1, \ldots, k\}$.

Proof. 1) Suppose for all $n \geq 1$ we have $f(n) \geq n$. For $k \geq 1$ suppose $x \in V(k, \tau_f)$, then there exists $m \geq 0$ with $k = f^m(x) \geq x$.
2) Suppose for all $n \geq 1$ we have $f(n) \leq n$. For $k \geq 1$ suppose $x \in V(k, \overline{\tau}_f)$, then there exists $m \geq 0$ with $x = f^m(k) \leq k$.

■

Lemma 3.2. For $f : \mathbb{N} \to \mathbb{N}$ and $k \in \mathbb{N}$ if for $n > 1$ we have $f(n) < n$ and $f(1) = 1$, then $1 \in V(k, \overline{\tau}_f)$ and $(\mathbb{N}, \overline{\tau}_f)$ is connected. Also $V(1, \tau_f) = \mathbb{N}$ and $(\mathbb{N}, \tau_f)$ is connected too.
Proof. Suppose $m = \min V(k, \overline{\tau}_f)$, then $f(m) \in V(k, \overline{\tau}_f)$, so

$$m \geq f(m) \geq \min V(k, \overline{\tau}_f)$$

and $m = f(m)$ hence $m = 1$ and $1 \in V(k, \overline{\tau}_f)$. Since 1 belongs to every nonempty subset of $(\mathbb{N}, \overline{\tau}_f)$, thus it does not have any disjoint nonempty open subset and it is connected.
For all $n \geq 1$ we have $f^n(n) = 1$, so $n \in V(1, \tau_f)$ and $V(1, \tau_f) = \mathbb{N}$. For nonempty open subsets of U, V of $(\mathbb{N}, \tau_f)$ with $\mathbb{N} = U \cup V$ we may suppose $1 \in U$ so $V(1, \tau_f) \subseteq U$ and $\mathbb{N} = U$ which leads to connectivity of $(\mathbb{N}, \tau_f)$.

■

Lemma 3.3. For $f : \mathbb{N} \to \mathbb{N}$ suppose for $n > 1$ we have $f(n) \geq n$ and $f(1) = 1$, then $(\mathbb{N}, \overline{\tau}_f)$ and $(\mathbb{N}, \tau_f)$ are disconnected.
Proof. $\{1\}, \mathbb{N} \setminus \{1\}$ is a separation of $(\mathbb{N}, \overline{\tau}_f)$ (and $(\mathbb{N}, \tau_f)$).

■

Example 3.4. For $1 \leq \alpha \leq \aleph_0$ suppose M is a partition of $\mathbb{N}$ to α infinite subsets of $\mathbb{N}$. For $D \in M$ suppose $D = \{n_k^D\}_{k \geq 1}$ with $n_1^D < n_2^D < \cdots$ and define $f_D : D \to D$ with $f_D(n_k^D) = n_{k+1}^D$ for $k \geq 1$, then for $f := \bigcup\limits_{D \in M} f_D : \mathbb{N} \to \mathbb{N}$, we have $f(n) > n$ for all $n \geq 1$ and M is the collection of all connected components of $(\mathbb{N}, \tau_f)$.
Table 3.5. By Lemmas 3.2 and 3.3 we have following table:

λ	$(\mathbb{N}, \tau_\lambda)$ and $(\mathbb{N}, \overline{\tau}_\lambda)$
$\varphi(= J_1), \varphi^*, \omega, \Omega, d(= d_2)$	connected
$\sigma_k, \psi_k, J_{k+1} (k \geq 1)$	disconnected

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LARGE VALUES OF APPROXIMATE SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS DUE TO THE LAPLACE TRANSFORMS AND THEIR COMPUTATIONS

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Abstract: In this paper, we present a method to evaluate the Laplace transformations of any $\{pth\ (p > 1)\}$ power of a function and nonlinear terms of the differential equations and then obtain the large values of approximate solution of nonlinear differential equations to compute them from the problems occurring in the damped oscillatory motions and other branches of physical, chemical and biological sciences.

Keywords and Phrases: The Hölder's inequality, the Laplace transformations, approximate solution of nonlinear differential equations, computations, Volterra integral equation.
Mathematics subject Classification: 34A08, 34A12, 34A18, 34K37, 45D05, 41D05 and 44A08.

1. Introduction

This paper is devoted to the study of nonlinear differential equations which arise in many branches of Physical Sciences, for example in damped oscillatory motions, electrical circuits, electromagnetic fields and other dynamical systems of chemical and biological sciences (see, van der Pol ([32] and [35]), Lalesco [20], Raleigh [33], Shohat [35], Pipes ([29], [30] and [31]), Jacobsen [13], Liu et al. [23]). These differential equations have been solved by many methods of approximations as by equating the like powers of the coefficients of the series, stability analysis, autonomous systems, Kryloff and Bogoliuboff methods (see, Pipes [31, p.688]). Some nonlinear differential equations are solved by decomposition techniques such as; Adomian decompositions, homotopy perturbation methods on applying He's polynomials [12] and other operational methods etcetera. (See, Bougoffa [5], Mohyuddin [25], El-Sayed [7], Gorbani and Nadjfi [8]).
The Riccati equation has frequently used in the engineering field and in branch of optimal control ([4] and [38]). These equations have been solved by various methods (see Abbasbandy ([1] and [2]), Liu et al. [23], Mak and Harko [24], Zeidan [38]).

The Abel differential equations occur in the modeling of real problems in various areas such as big picture in oceanic circulation [3], in problems of magneto-statics ([14] and [22]), control theory [27], cosmology [9], fluid mechanics ([28] and [5]), solid mechanics [26], biology ([11] and [34]), and cancer therapy [10].

The basic motivation of this paper is to develop and explore some of the consequences of the approximation methods of Laplace transformations which need not additional polynomials (He's or Adomian polynomials) as in the methods developed by other researchers (see [1], [2], [5], [7], [8], [12] and [25]).

In our work we claim the Hölder's inequality (for $\frac{1}{p}+\frac{1}{q}=1, p>0, q>0$) given by (see, Steele [36, p.151]) $\int_0^\infty u(t)v(t)dt \leq (\int_0^\infty (|u(t)|)^p dt)^{\frac{1}{p}} (\int_0^\infty |v(t)|^q dt)^{\frac{1}{q}}$ to find the approximation solution due to Laplace transformation.

Remark 1 It is noted that the Hölder's inequality (for $\frac{1}{p}+\frac{1}{q}=1, p>0, q>0$) becomes a Cauchy-Schwarz inequality, when, $p=q=2$. The Cauchy-Schwarz inequality is a very powerful tool, which is given by

$\int_0^\infty u(t)v(t)dt \leq (\int_0^\infty (|u(t)|)^2 dt)^{\frac{1}{2}} (\int_0^\infty |v(t)|^2 dt)^{\frac{1}{2}}$. Recently, Kumar [15] has applied it to analyze the characteristics of Lupas-Kumar-Pathan type integral operators.

2. Theorems on approximation methods of Laplace transformation

Here in this section, we introduce two theorems on approximation methods of Laplace transformation to obtain the approximate solution of various nonlinear differential equations.

Theorem 1

If the Laplace transformation of *pth* power of any function $v(t)$ is denoted by $\mathcal{L}[\{v(t)\}^p]$, and

$$\lim_{s\to 0^+}\int_0^\infty e^{-st}|v(t)|^p dt \ = A_p \ < \infty, 2\leq p<\infty,$$

$$\lim_{s\to 0^+}\int_0^\infty e^{-st}v(t)dt = \tilde{v}(0^+) < \infty,\ v(t)=0,\ \text{when } t<0. \tag{1}$$

Here, the Laplace transform of $v(t)$ is denoted by $\mathcal{L}[v(t)] = \tilde{v}(s) = \int_0^\infty e^{-st}v(t)dt$. Then, there exists an analytic function $\Phi(p,k;s) = (-1)^k \int_0^\infty e^{-st}t^k\{v(t)\}^p dt$, and $\Phi(p,k;s)$ is bounded by

$(-1)^k(\Gamma(2k+1))^{\frac{1}{2}}(s)^{p-(k+1)}\{\tilde{v}(s)\}^p \geq \Phi(p,k;s)$

$$\geq \begin{cases} \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \frac{\Gamma(p)}{\Gamma(p-r)} \frac{d^{k-r}}{ds^{k-r}}\{\tilde{v}(s)\}^p,\ \text{when } r<p-1, \\ \sum_{p=1}^{k+1} \frac{\Gamma(k+1)}{\Gamma(k+2-p)} \frac{d^{k+1-p}}{ds^{k+1-p}}\{\tilde{v}(s)\}^p,\ \text{when } r=p-1, \\ 0,\ \text{when } r>p-1, \quad \forall 1\leq p<\infty \text{ and } k=0,1,2,3,\ldots \end{cases} \tag{2}$$

Again then, the $\mathcal{L}[\{v(t)\}^p]$ exists in discrete form

$$\mathcal{L}[\{v(t)\}^p] = \begin{cases} \cong s^{(p-1)}\{\tilde{v}(s)\}^p, \text{ when } \quad 2 \leq p < \infty, \\ = \tilde{v}(s), \text{ when } \quad p = 1. \end{cases} . \tag{3}$$

Further the inverse $[\{v(t)\}^p]$ is presented as

$$[\{v(t)\}^p] = \begin{cases} \cong \mathcal{L}^{-1}\{s^{(p-1)}\{\tilde{v}(s)\}^p\}, \text{ when } \quad 2 \leq p < \infty, \\ = \mathcal{L}^{-1}\{\tilde{v}(s)\}, \text{ when } p = 1. \end{cases} . \tag{4}$$

Proof
We prove this theorem on applying the $H\ddot{o}lder's$ inequality (for $\frac{1}{p} + \frac{1}{q} = 1, p > 0, q > 0$) to find that

$$\int_0^\infty e^{-st}v(t)dt = \int_0^\infty e^{-\frac{st}{p}}e^{-\frac{st}{q}}v(t)dt \leq (\int_0^\infty (e^{-\frac{st}{p}})^p dt)^{\frac{1}{p}} (\int_0^\infty (e^{-\frac{st}{q}})^q |v(t)|^q dt)^{\frac{1}{q}},$$

so that due to our assumptions given in Eqn. (1) of the Theorem 1, there exists

$$\int_0^\infty e^{-st}|v(t)|^q dt = \mathcal{L}[\{v(t)\}^q] \geq s^{(q-1)}\{\tilde{v}(s)\}^q, \ q > 0.$$

Otherwise, we may write

$$\mathcal{L}[\{v(t)\}^p] \geq s^{(p-1)}\{\tilde{v}(s)\}^p, \ p > 0. \tag{5}$$

Again, we are familiar with that Laplace transformation of a function be analytic, hence, $\mathcal{L}[\{v(t)\}^p]$ is analytic.
Then, with the aid of assumptions of Eqns. (1) and on differentiating Eqn. (5) both of the sides as $k - times$ with respect to s and to get

$$\Phi(p, k; s) \geq \sum_{r=0}^{k} \left(\begin{array}{c} k \\ r \end{array} \right) \frac{\Gamma(p)}{\Gamma(p-r)} \frac{d^{k-r}}{ds^{k-r}}\{\tilde{v}(s)\}^p. \tag{6}$$

Therefore, from Eqn. (6) we may write

$$\Phi(p, k; s) \geq \begin{cases} \sum_{r=0}^{k} \left(\begin{array}{c} k \\ r \end{array} \right) \frac{\Gamma(p)}{\Gamma(p-r)} \frac{d^{k-r}}{ds^{k-r}}\{\tilde{v}(s)\}^p, \text{ when } r < p-1, \\ \sum_{p=1}^{k+1} \frac{\Gamma(k+1)}{\Gamma(k+2-p)} \frac{d^{k+1-p}}{ds^{k+1-p}}\{\tilde{v}(s)\}^p, \text{ when } r = p-1, \\ 0, \text{ when } r > p-1, \quad \forall 1 \leq p < \infty \text{ and } k = 0, 1, 2, 3, \ldots \end{cases} \tag{7}$$

Further use the Cauchy-Schwarz inequality (see Remark 1) in the integral

$$\int_0^\infty e^{-st}(-t)^k |v(t)|^q dt, \forall k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\},$$

and appeal to the assumptions given in Eqns. (1) and the result (5), we find

$$\Phi(p, k; s) \leq (-1)^k (\Gamma(2k+1))^{\frac{1}{2}} (s)^{p-(k+1)} \{\tilde{v}(s)\}^p, \forall k \in \mathbb{N}_0 \\ = \{0, 1, 2, 3, \dots\}, 1 \leq p < \infty. \quad (8)$$

Then, make an appeal to the inequalities (7) and (8), we obtain the function $\Phi(p, k; s)$ is bounded by Eqn. (2) $\forall k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}, 1 \leq p < \infty$.
Again, in Eqn. (2) the function $\Phi(p, k; s)$ is monotonic $\forall k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}, 1 \leq p < \infty$ and hence it may be expressible as Taylor's series

$$\Phi(p, k; s) = \sum_{m=0}^{\infty} \Phi(p, k; a) \frac{(s-a)^m}{m!}, \ (0 < s < 2a)$$

and again, this series has every term positive for $s < a$, thus

$$\lim_{s \to 0^+} \Phi(p, k; s) = \Phi(p, k; 0^+) = \sum_{m=0}^{\infty} \Phi(p, k; a) \frac{(-a)^m}{m!} \forall k \in \mathbb{N}_0 \\ = \{0, 1, 2, 3, \dots\}, 1 \leq p < \infty.$$

The series $\Phi(p, k; 0^+)$ will be convergent if $|\frac{\Phi(p,k;a)}{(a)^m}| < 1, \ \forall m \in \mathbb{N}_0$. Then, we also write $\Phi(p, k; s) = \mathcal{O}(s^{-m}) \forall k, m \in \mathbb{N}_0, 1 \leq p < \infty$. Thus our assumptions given in Eqns. (1) and (2) are valid.
Now, to evaluate the uniqueness of inverse of the approximate formula of Laplace transform given in Eqn. (2), we define a sequence of functions identical to Widder [37, p. 288] operator given by for any real positive numbers t and $\forall k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}, 1 \leq p < \infty$:

$$\mathcal{L}_{k,t}^{-1} \left\{ \Phi \left(p, k; s = \frac{k}{t} \right) \right\} = \frac{1}{k!} \left(\frac{k}{t} \right)^{k+1} \int_0^\infty e^{-\frac{k}{t}u} u^k \{v(u)\}^p du. \quad (9)$$

Here, in Eqn. (9), $\lim_{k \to \infty} \mathcal{L}_{k,t}^{-1} \{\Phi(p, k; s = \frac{k}{t})\} = \{v(t)\}^p$.
So that due to Eqn. (9), we have $\mathcal{L}_{k,t}^{-1} \{\Phi_1(p, k; s = \frac{k}{t})\} = \frac{1}{k!} (\frac{k}{t})^{k+1} \int_0^\infty e^{-\frac{k}{t}u} u^k \{v_1(u)\}^p du$ and $\mathcal{L}_{k,t}^{-1} \{\Phi_2(p, k; s = \frac{k}{t})\} = \frac{1}{k!} (\frac{k}{t})^{k+1} \int_0^\infty e^{-\frac{k}{t}u} u^k \{v_2(u)\}^p du$. Then, for $\{v_1(t)\}^p \neq \{v_2(t)\}^p$, we find $\lim_{k \to \infty} \mathcal{L}_{k,t}^{-1} \{\Phi_1(p, k; s = \frac{k}{t})\} \neq \lim_{k \to \infty} \mathcal{L}_{k,t}^{-1} \{\Phi_2(p, k; s = \frac{k}{t})\}$.
Hence, for a function $\{v(t)\}^p$ the inversion formula $\mathcal{L}_{k,t}^{-1} \{\Phi(p, k; s)\}$ is unique.
Therefore, on applying Eqns. (3), the Eqn. (4) is computed.

Finally, for getting the large values of the results, we now take the limit $k \to 0$ in the Eqns. (7) and (8), it implies that the result (3)

$$s^{(p-1)}\{\tilde{v}(s)\}^p \leq \mathcal{L}[\{v(t)\}^p] \leq s^{(p-1)}\{\tilde{v}(s)\}^p, \forall 1 \leq p < \infty. \tag{10}$$

Hence, the Theorem1 is proved.
In our investigations, we also make an application of the Laplace transformation of nth derivative of a function $v(t)$ which is given by (see Churchill [6])

$$\mathcal{L}\{v(t)^{(n)}\} = s^n\tilde{v}(s) - s^{n-1}v(0) - s^{n-2}v(0)^{(1)} - \cdots - v(0)^{(n-1)},$$

$$\text{where , } v(t)^{(n)} = \frac{d^n}{dt^n}v(t), v(0)^{(n-1)} = \frac{d^{n-1}}{dt^{n-1}}v(t)|_{\;t=0}\;. \tag{11}$$

Now, applying above methods of Theorem 1 and Eqn. (11), we state and prove following theorem:
Theorem 2 If the Laplace transformation of u be$\mathcal{L}\{u\} = \tilde{u}(s)$, then, there exists

$$\mathcal{L}\left\{u^2\frac{du}{dt}\right\} = \frac{1}{3}s^3\{\tilde{u}(s)\}^3 - \frac{1}{3}(u(0))^3. \tag{12}$$

Proof Take Laplace transformation of $u^2\frac{du}{dt}$, and then on integrate it by parts, we have

$$\mathcal{L}\left\{u^2\frac{du}{dt}\right\} = \int_0^\infty e^{-st}u^2\frac{du}{dt}dt = \{e^{-st}u^3\}|_{\;\substack{t=\infty \\ t=0}} .-\int_0^\infty \left\{-se^{-st}u^2 + e^{-st}2u\frac{du}{dt}\right\}udt \tag{13}$$

Again, here we have $\mathcal{L}\{u\} = \tilde{u}(s) = \int_0^\infty e^{-st}u(t)dt$. Thus, the Eqn. (13) gives us

$$\mathcal{L}\left\{u^2\frac{du}{dt}\right\} = \frac{1}{3}s\int_0^\infty e^{-st}u^3dt - \frac{1}{3}(u(0))^3 = \frac{1}{3}s\mathcal{L}\{u^3\} - \frac{1}{3}(u(0))^3 \tag{14}$$

Finally, make an appeal to the formulae (3) and (11) in Eqn. (14), we get the Eqn. (12).

3. Illustrative Examples

Example. Solution of van der Pol's Equation [32], [35].
Analyze the solution of the van der Pol's differential equation that may be found in the damped oscillatory motions as

$$\frac{d^2u}{dt^2} + \mu(u^2-1)\frac{du}{dt} + u = 0, 0 < \mu \leq 1, \tag{15}$$

provided that $u'(0) = 0, u(0) = u_0$.

Solution. Take the Laplace transformation of both sides of Eqn. (15) (as setting $\mathcal{L}\{u\} = \tilde{u}(s)$) and then use the Theorems 1 and 2 and the result (12), to get

$$\frac{\mu}{3}s^3\{\tilde{u}(s)\}^3 + [s^2 - \mu s + 1]\tilde{u}(s) - \frac{\mu}{3}(u(0))^3 + \mu u(0) - u'(0) - su(0) = 0 \quad (16)$$

Now, make some manipulations in Eqn. (16) and put it in the form

$$\left\{\frac{\mu}{3}[\frac{s^2\{\tilde{u}(s)\}^3}{s^2}]\right\} + \left\{\left[\frac{1}{s} - \frac{\mu}{s^2} + \frac{1}{s^3}\right]\tilde{u}(s)\right\} + \left\{\frac{\mu u(0) - \frac{\mu}{3}(u(0))^3 - u'(0)}{s^3}\right\} - \left\{\frac{u(0)}{s^2}\right\} = 0 \quad (17)$$

Then, with the use of the formula (4), the inverse Laplace transformation of Eqn. (17) gives us the equation

$$\frac{\mu}{3}\int_0^t (t-\tau)u^3(\tau)d\tau + \int_0^t u(\tau)d\tau - \mu\int_0^t (t-\tau)u(\tau)d\tau + \frac{1}{2}\int_0^t (t-\tau)^2 u(\tau)d\tau$$
$$+ \left[\mu u(0) - \frac{\mu}{3}(u(0))^3 - u'(0)\right]\frac{t^2}{2} - [u(0)]t = 0 \quad (18)$$

If we differentiate the Eqn. (18), we get the required solution

$$u(t) = \mu\int_0^t u(\tau)d\tau - \frac{\mu}{3}\int_0^t u^3(\tau)d\tau - \int_0^t (t-\tau)u(\tau)d\tau - [\mu u(0) - \frac{\mu}{3}(u(0))^3 - u'(0)]t + [u(0)] \quad (19)$$

Sequential solutions and analysis: To find approximate solution of problem (15), first we evaluate the sequential solutions to suppose that $u = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} u_k$ and now consider the Eqn. (19) in the form

$$u_k(t) = \mu\int_0^t u_{k-1}(\tau)d\tau - \frac{\mu}{3}\int_0^t {u_{k-1}}^3(\tau)d\tau - \int_0^t (t-\tau)u_{k-1}(\tau)d\tau$$
$$+ \left[\frac{\mu}{3}(u(0))^3 + u'(0) - \mu u(0)\right]t + [u(0)], \forall k = 1, 2, 3, \ldots \quad (20)$$

Put $k = 1$, $u'(0) = 0, u(0) = u_0$ in Eqn. (20), we find that

$$u_1(t) = u_0\left\{\frac{t^2}{2} + 1\right\} \quad (21)$$

Again put $k = 2$ in Eqn. (21), we find that

$$u_2(t) = u_0\{1 - \frac{t^2}{2} + \frac{\mu}{6}(1 - {u_0}^2)t^3 - \frac{t^4}{24} - \frac{\mu}{20}{u_0}^2 t^5 - \frac{\mu}{168}{u_0}^2 t^7\} \quad (22)$$

Finally, making an appeal to the Eqns. (21) and (22) in ($u = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} u_k$), we find the approximate solution

$$u(t) = u_0 \left\{ 1 + \frac{\mu}{12}(1 - {u_0}^2)t^3 - \frac{t^4}{48} - \frac{\mu}{40}{u_0}^2 t^5 - \frac{\mu}{336}{u_0}^2 t^7 + \dots \right\} \tag{23}$$

Again, the series (23) may be written by

$$u(t) = u_0 + t - \frac{t^2}{2} + 1 - \left\{ 1 - \frac{t^2}{2} + u_0 \frac{t^4}{48} \right\} - \left\{ t - \frac{\mu}{12}(1 - {u_0}^2)u_0 t^3 + \frac{\mu}{40}{u_0}^3 t^5 - \frac{\mu}{336}(-u_0)^3 t^7 + \dots \right\}. \tag{24}$$

Thus, due to the series (24), we approach the oscillatory solution of **van der Pol's Equation [15]**

$$u(t) \approx u_0 + t - \frac{t^2}{2} + 1 - \cos\omega_1 t - \sin\omega_2 t, \text{ when,}$$
$$\frac{1}{2} = \frac{u_0(\omega_1)^2}{2!}, \frac{u_0}{48} = \frac{(\omega_1)^4}{4!}, \frac{\mu}{12}(1 - {u_0}^2)u_0 = \frac{(\omega_2)^3}{3!}, \frac{\mu}{40}{u_0}^3 = \frac{(\omega_2)^5}{5!} \text{ and } \frac{\mu}{336}(-u_0)^3 = \frac{(\omega_2)^7}{7!}. \tag{25}$$

Computation of the large oscillations of van der Pol's equation due to the result (23)

On substituting $u_0 = 5, \mu = .5$, $t = 0$ to $t = 10$, in result (23), we draw u, (*vertically down ward*), with respect to $t > 0$, (*horizontal*), as:

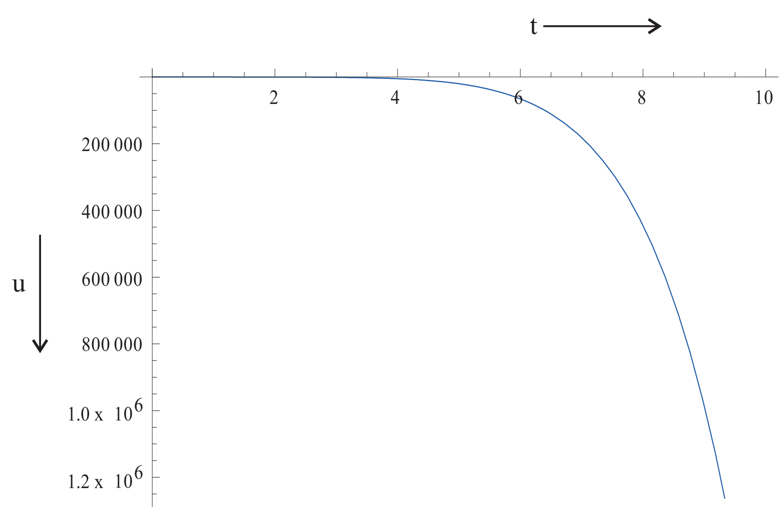

Figure 1: Large Oscillations due to van der Pol's Eqn. (15)

Example. Compute the initial value problem

$$\frac{du}{dt} = u^p + f(t), p \geq 2,\ u(0) = \ u_0. \tag{26}$$

Solution: Take Laplace transforms of $u(t)$ *and* $f(t)$ are $\hat{u}(s)$ *and* $\hat{f}(s)$, respectively, then on making the Laplace transformations of the Eqn. (26) and then use the Eqn. (11) with approximation methods of Theorem 1, we get

$$\{\hat{u}(s)\}^p - \frac{\hat{u}(s)}{s^{p-2}} + \frac{u(0)}{s^{p-1}} + \frac{\hat{f}(s)}{s^{p-1}} = 0,\ 2 \leq p < \infty \tag{27}$$

From, Eqn. (27), we find the discrete equations

$$\begin{cases} \mathcal{L}^{-1}\left\{\frac{s^{p-1}\{\hat{u}(s)\}^p}{s^{p-1}}\right\} - \mathcal{L}^{-1}\left\{\frac{\hat{u}(s)}{s^{p-2}}\right\} + \mathcal{L}^{-1}\left\{\frac{u(0)}{s^{p-1}}\right\} + \mathcal{L}^{-1}\left\{\frac{\hat{f}(s)}{s^{p-1}}\right\} = 0, \forall 3 \leq p < \infty; \\ \mathcal{L}^{-1}\left\{\frac{s\{\hat{u}(s)\}^2}{s}\right\} - \mathcal{L}^{-1}\{\hat{u}(s)\} + \mathcal{L}^{-1}\left\{\frac{u(0)}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{\hat{f}(s)}{s}\right\} = 0,\ \text{when},\ p = 2 \end{cases}. \tag{28}$$

Then, on using inverse Laplace transformations in conjugation with formulae (4) in Eqn. (28), we find that

$$\begin{cases} \frac{1}{\Gamma(p-1)}\int_0^t (t-\tau)^{p-2}(u(\tau))^p d\tau - \frac{1}{\Gamma(p-2)}\int_0^t (t-\tau)^{p-3}u(\tau)d\tau + \frac{1}{\Gamma(p-1)}\int_0^t (t-\tau)^{p-2}f(\tau)d\tau + \\ \quad + u_0\frac{t^{p-2}}{\Gamma(p-1)} = 0,\ \forall 3 \leq p < \infty; \\ u(t) = \int_0^t (u(\tau))^2 d\tau + \int_0^t f(\tau)d\tau + u_0,\ when, p = 2. \end{cases}. \tag{29}$$

Case I. When we put $p = 2$ and $f(t) = t$ in the problem (26), it becomes **Riccati's nonlinear** differential equation [21, p. 1092, Eqn. (20.36)]

$$\frac{du}{dt} = u^2 + t,\ u(0) = \ u_0. \tag{30}$$

Then, making an appeal to the second part of Eqn. (29), we get the integral solution of the Eqn. (30), given by

$$u(t) = \int_0^t (u(\tau))^2 d\tau + \frac{t^2}{2} + u_0 \tag{31}$$

Sequential solutions and analysis:

The Eqn. (31) gives the sequential solutions

$$u_k(t) = \int_0^t (u_{k-1}(\tau))^2 d\tau + \frac{t^2}{2} + u_0, \forall k = 1, 2, 3, \ldots \tag{32}$$

The Eqn. (32) gives us

$$u_1(t) = \frac{t^2}{2} + {u_0}^2 t + u_0, u_2(t) = \frac{t^5}{20} + \frac{{u_0}^2}{4} t^4 + \frac{u_0}{3}({u_0}^3 + 1)t^3 + ({u_0}^3 + \frac{1}{2})t^2 + {u_0}^2 t + u_0 \tag{33}$$

Finally, with the help of Eqn. (33), we find the approximate solution of the **Riccati's nonlinear** differential equation (30) as

$$u(t) = \frac{t^5}{40} + \frac{{u_0}^2}{8} t^4 + \frac{u_0}{6}({u_0}^3 + 1)t^3 + \frac{1}{2}({u_0}^3 + 1)t^2 + {u_0}^2 t + u_0 + \dots \tag{34}$$

Computation of large solutions of Riccati's equation (30) with aid of the result (34):

Substitute $u_0 = 5, t = 0$ to $t = 10$, in result (34), we draw u, (*vertically upward*), with respect to $t > 0$, (*horizontal*), as:

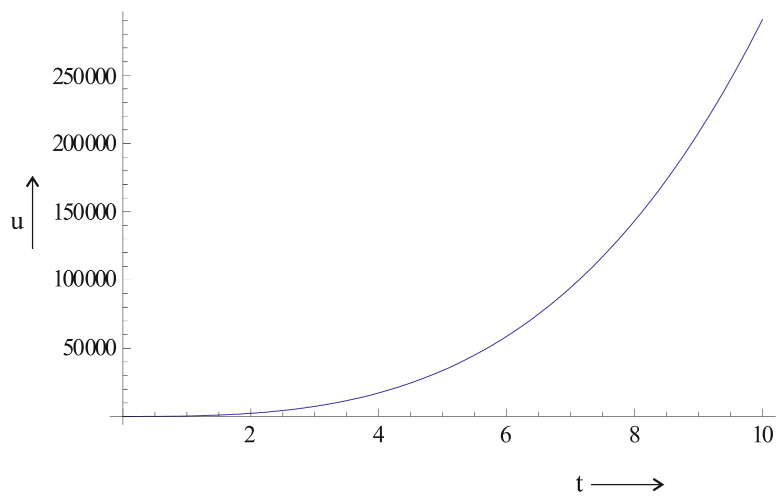

Figure 2: Large values due to Riccati's problem (30)

Case II. When we put $p = 3$ and $f(t) = t$ in the problem (26), it becomes **Abel's nonlinear** differential equation [21, p. 1092, Eqn. (20.37)]

$$\frac{du}{dt} = u^3 + t, u(0) = u_0. \tag{35}$$

Then, making an appeal to the first part of Eqn. (29), and differentiating with respect to t, we get the integral solution of the Eqn. (35), given by

$$u(t) = \int_0^t (u(\tau))^3 d\tau + \frac{t^2}{2} + u_0 \tag{36}$$

Sequential solutions and Analysis:
The Eqn. (36) gives the sequential solutions

$$u_k(t) = \int_0^t (u_{k-1}(\tau))^3 d\tau + \frac{t^2}{2} + u_0, \forall k = 1, 2, 3, \ldots \tag{37}$$

The Eqn. (37) gives us

$$u_1(t) = \frac{t^2}{2} + {u_0}^3 t + u_0,$$
$$u_2(t) = \frac{t^7}{56} + \frac{{u_0}^3}{8} t^6 + \frac{3u_0}{10}({u_0}^5 + \frac{1}{2})t^5 + {u_0}^4(\frac{{u_0}^5}{4} + \frac{3}{4})t^4 + {u_0}^2({u_0}^5 + \frac{1}{2})t^3 + (\frac{3}{2}{u_0}^5 + \frac{1}{2})t^2 + {u_0}^3 t + u_0 \tag{38}$$

Finally, with the help of Eqn. (38), we find the approximate solution of the **Abel's nonlinear** differential equation (35)

$$u(t) = \frac{t^7}{112} + \frac{{u_0}^3}{16} t^6 + \frac{3u_0}{20}\left({u_0}^5 + \frac{1}{2}\right) t^5 + {u_0}^4 \left(\frac{{u_0}^5}{8} + \frac{3}{8}\right) t^4 + \frac{{u_0}^2}{2}\left({u_0}^5 + \frac{1}{2}\right) t^3$$
$$+ \left(\frac{3}{4}{u_0}^5 + \frac{1}{2}\right) t^2 + {u_0}^3 t + u_0 + \ldots \tag{39}$$

Computation of the large solutions of Abel's equation (35) with the aid of result (39):
Substitute $u_0 = 5, t = 0$ to $t = 10$, in the solution (39), we draw u, (*vertically upward*), with respect to $t > 0$, (*horizontal*), as:
Example. (**The Troesch's problem** [25],). Analyze the nonlinear differential equation

$$\frac{d^2}{dt^2} u(t) = \beta \sin h\beta u(t), 0 < t < 1, \tag{40}$$

$$u(0) = 0, u(1) = 1.$$

Solution. In the right hand side of Eqn. (40), we apply the formula $\sin h\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots + \frac{\theta^{2n-1}}{2n-1!} + \ldots$, and then taking Laplace transformation of both of the sides of it and using Theorem 1 together with the result (11), we get

$$\frac{\hat{u}(s)}{s^{2n-4}} = \frac{u'(0)}{s^{2n-2}} + \beta^2 \frac{\hat{u}(s)}{s^{2n-2}} + \frac{\beta^4}{3!}\frac{(\hat{u}(s))^3}{s^{2n-4}} + \frac{\beta^6}{5!}\frac{(\hat{u}(s))^5}{s^{2n-6}} + \cdots + \frac{\beta^{2n+2}}{2n-1!}(\hat{u}(s))^{2n-1} + \ldots \tag{41}$$

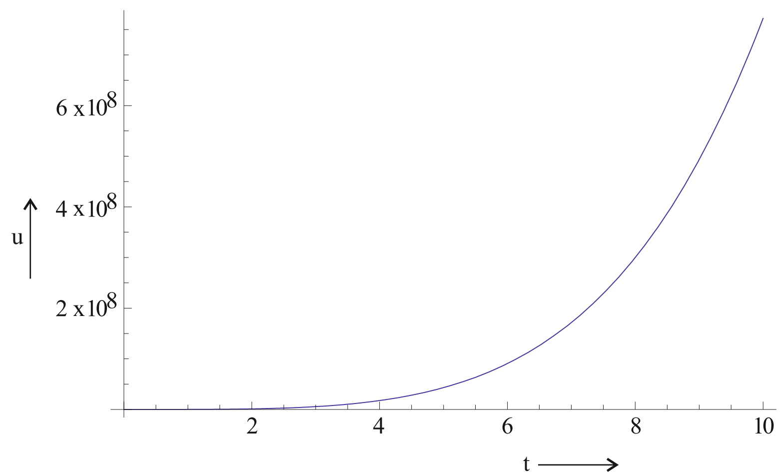

Figure 3: Large values due to Abel's problem (35)

Now for large n, take inverse Laplace transformation of both the sides of Eqn. (41), and use Eqn. (4) of Theorem 1 to find that

$$\frac{1}{2n-5!}\int_0^t (t-\tau)^{2n-5}u(\tau)d\tau = \frac{u'(0)}{2n-3!}t^{2n-3} + \frac{\beta^2}{2n-3!}\int_0^t (t-\tau)^{2n-3}u(\tau)d\tau + \frac{\beta^4}{3!2n-3!}\int_0^t (t-\tau)^{2n-3}(u(\tau))^3 d\tau + \frac{\beta^6}{5!2n-3!}\int_0^t (t-\tau)^{2n-3}(u(\tau))^5 d\tau + \cdots + \frac{\beta^{2n+2}}{2n+1!2n-3!}\int_0^t (t-\tau)^{2n-3}(u(\tau))^{2n+1}d\tau \quad (42)$$

Now, differentiating both of the sides of Eqn. (42) $2n-4$ times with respect to t, we get

$$u(t) = u'(0)t + \frac{\beta^2}{1!}\int_0^t (t-\tau)u(\tau)d\tau + \frac{\beta^4}{3!}\int_0^t (t-\tau)(u(\tau))^3 d\tau + \frac{\beta^6}{5!}\int_0^t (t-\tau)(u(\tau))^5 d\tau + \cdots + \frac{\beta^{2n+2}}{2n+1!}\int_0^t (t-\tau)(u(\tau))^{2n+1}d\tau \quad (43)$$

When, $n \to \infty$, the Eqn. (43) may be written as

$$u(t) = u'(0)t + \beta\int_0^t (t-\tau)sinh\beta u(\tau)d\tau \quad (44)$$

It is noted that the Eqn. (44) satisfies the first condition of Eqn. (40) as

$$u(0) = 0. \quad (45)$$

Further due to another condition of Eqn. (40), the Eqn. (44) gives us

$$u'(0) = 1 - \beta \int_0^1 (1-\tau) sinh\beta u(\tau) d\tau \tag{46}$$

Hence, using the Eqns (44) and (46), finally, we find the approximate solution of the problem (40) as

$$u(t) = t\{1 - \beta \int_0^1 (1-\tau) sinh\beta u(\tau) d\tau\} + \beta \int_0^t (t-\tau) sinh\beta u(\tau) d\tau \tag{47}$$

Sequential solutions and analysis:
From Eqn. (47), we find the sequential solutions

$$u_n(t) = t\{1 - \beta \int_0^1 (1-\tau) sinh\beta u_{n-1}(\tau) d\tau\} + \beta \int_0^t (t-\tau) sinh\beta u_{n-1}(\tau) d\tau, \tag{48}$$

provided that $u_0(\tau) = 0, \forall 0 \leq \tau \leq t < 1$.
Therefore, the Eqn. (48) for $n = 1$, gives us

$$u_1(t) = t \tag{49}$$

Again, from Eqn. (48), for $n = 2$, we find that

$$u_2(t) = t(1 - \frac{sinh\beta}{\beta}) + \frac{sinh\beta t}{\beta} \tag{50}$$

Since, both the solutions $u_1(t)$ and $u_2(t)$, independently, satisfies the boundary conditions, hence the function $u(t)$ will also satisfy the Eqn. (40), given by

$$u(t) = t - \frac{t}{2}\frac{sinh\beta}{\beta} + \frac{1}{2}\frac{sinh\beta t}{\beta} \tag{51}$$

Computation of approximate solution of the Troesch's problem due to Laplace Transformation Methods
Numerical solution for Troesch's problem (40) due to the result (51), when $\beta = .5$ and $t = 0.1\ to\ 1.0$:
Now, to find the validity of our results, we claim the work of Mohyud-din [25] who has presented following VIM formula
$u(t) = t + p\int_0^t \int_0^t \beta^2 \sin h\beta u(\tau) d\tau d\tau,\ u(t) = u_0(t) + pu_1(t) + p^2u_2(t) + \dots$, which may be written as

$$u(t) = t + p\beta^2 \int_0^t (t-\tau) \sin h\beta u(\tau) d\tau,\ u(t) = u_0(t) + pu_1(t) + p^2u_2(t) + \dots \tag{52}$$

t	Exact Solution	Approximate solution	% Error
0.1	0.0951769	0.0979113	2.87
0.2	0.1906338	0.195948	2.78
0.3	0.2866534	0.294235	2.64
0.4	0.3835229	0.392898	2.44
0.5	0.4815373	0.492065	2.18
0.6	0.5810019	0.591863	1.87
0.7	0.6822351	0.692423	1.49
0.8	0.7855717	0.793876	1.06
0.9	0.8913669	0.896356	0.56
1.0	0.9999999	1.000000	0.1

$$\%\text{Error} = \frac{\text{Approximate solution - Exact value}}{\text{Exact value}} \times 100$$

Table 1: Percentage error in approximate solutions with respect to exact values

Computation of series solutions of the Troesch's problem due to VIM formula

Numerical solution for Troesch's problem (40) due to the result (52), when $\beta = .5$ and $t = 0.1$ to 1.0

4. Conclusion

The fact that the proposed technique of our paper solves nonlinear problems without using He's or Adomian's polynomials is a clear advantage over the approximation method of Laplace transformations studied extensively by others which needs an additional polynomials (see [1], [2], [5], [7], [8], [12], [25]). This is a simple method and by this method, we may compute large required values in a small screen (see Figures 1, 2, and 3). In the same vein , in solving the Troesch's problem, by our method we find very near values to the numerical exact solutions and the percentage error due to VIM formula is more than that of Laplace transformation methods (see Tables 1 and 2). That is the unifying theme of this work.

Innovation and Future Directions

The Abel differential equations occur in the modeling of real problems in various areas such as big picture in oceanic circulation [3], in problems of magneto-statics ([14] and [22]), control theory [27], cosmology [9], fluid mechanics ([28] and [5]), solid mechanics [26], biology ([11] and [34]), and cancer therapy [10]. Kumar, Pathan and Srivastava ([16], [17], [18]),and Kumar, Pathan and Yadav [19] have

t	Exact Solution	Series Solution	% Error
0.1	0.0951769	0.100042	5.11
0.2	0.1906338	0.200334	5.09
0.3	0.2866534	0.301128	5.05
0.4	0.3835229	0.402677	4.99
0.5	0.4815373	0.505241	4.92
0.6	0.5810019	0.609082	4.83
0.7	0.6822351	0.71447	4.72
0.8	0.7855717	0.821682	4.59
0.9	0.8913669	0.931008	4.45
1.0	0.9999999	1.04274	4.27

$$\%\text{Error} = \frac{\text{Series solution - Exact value}}{\text{Exact value}} \times 100$$

Table 2: Percentage error in series solutions with respect to exact values

computed the space-and-time fractional initial value problems, anomalous diffusion problem, advection-dispersion problem with the help of sequential solutions, hence, in further extensions in the researches to these problems may be computed by the approximation method of Laplace transformations in small screen.

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COMPOSITIONS OF PATHWAY INTEGRAL OPERATOR ON MITTAG-LEFFLER TYPE HYPERGEOMETRIC FUNCTIONS

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Abstract: The aim of the present paper is to obtain new relations using Pathway integral operator on Mittag-Leffler type Hypergeometric functions. The formulas established here are basic in nature and are likely to have useful applications in the field of science and engineering. Pathway integral operator generalizes the classical Riemann - Liouville fractional integration operator, and when $\alpha \rightarrow 1$ it reduces to the Laplace integral transform.

Keywords and Phrases: Mittag-Leffler type Hypergeometric Functions, Pathway Model, Generalized Wright function, Fractional derivative.

Mathematics subject Classification: 26A33, 33E20, 33E12

1. Introduction and Preliminaries:

(a) Definitions

The Wright function: The Wright function plays an important role in the solution of a linear partial differential equation. The Wright function, which we denote by W (z; α, β), is so named in honor of Wright, who introduced and investigated this function in a series of notes starting from 1993 in the framework of the asymptotic theory of partitions. This function is related to Mittag$-$Leffler function [19, 20, 21, 23]. We obtain a number of useful relationships between the Mittag$-$Leffler functions and the Wright functions.

Definition 1.1:The Wright function is defined by the series representation, convergent in the whole z-complex plane [2]

$$W(z; \alpha, \beta) = \sum_{K=0}^{\infty} \frac{z^k}{k!\Gamma(ak + \beta)} \tag{1.1.1}$$

Definition 1.2: The Pathway fractional integration operator

Let us recall the definition of left sided Riemann−Liouville fractional integral operator. Let $f(x) \in L(a, b)$, then

$$(I^{\alpha}_{a+}\Psi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\Psi(t)}{(x-t)^{1-\alpha}} dt, \;\; (\alpha \in C, Re(\alpha) > 0) \tag{1.2.1}$$

For more details, see [4], [5], [15] and other books on fractional calculus.

If $f(t)$ is replaced by $t^{\gamma} f(t)$ in (1.2.1), the above operator turns out to be the Erdélyi – Kober fractional integral; if it is replaced by${}_2F_1(\eta+\beta, -\gamma; \eta; 1-\frac{t}{x})f(t)$, then (1.2.1) takes the form of the Saigo hypergeometric fractional integral, see e.g. [12]:

$$\frac{\Gamma(\eta)}{x^{-\eta-\beta}} I^{\eta,\beta,\gamma}_{0+} f(x) = \int_0^x (x-t)^{\eta-1} \, {}_2F_1(\eta+\beta, -\gamma; \eta; 1-\frac{t}{x}) f(t) dt \tag{1.2.2}$$

Many other operators of generalized fractional calculus can be obtained if on the place of $f(t)$ one can use $\phi(t)f(t)$ with a suitably chosen special function $\phi(t)$. As it is done in Kriyakova [5] for a well known Fox's H- Function.

In this paper we introduce compositions of Pathway integral operator on Mittag – Leffler type Hypergeometric Functions.

Let $f(x) \in L(a, b), \rho \in C, Re(\rho) > 0, a > 0$ and let us take a pathway parameter $\alpha < 1$. Then the pathway fractional integration operator, as an extension of (1.2.1), is defined and represented as follows (see [11, p. 239]):

$$(P^{\rho,\alpha,a}_{0+} f)(t) = t^{\rho} \int_0^{\frac{t}{a(1-\alpha)}} \left[1 - \frac{a(1-\alpha)\tau}{t}\right]^{\frac{\rho}{1-\alpha}} f(\tau) d\tau \tag{1.2.3}$$

where $L(a, b)$ is the set of Lebesgue measurable functions defined on (a, b).

The pathway model is introduction by Mathai [6] and studied further by Mathai and Haubold [7], [8].

Definition 1.3: The generalized Wright's function is defined as follows (see, e.g. [22]):

$$ {}_p\Psi_q \left[\begin{array}{l} (\alpha_1, A_1), ..., (\alpha_p, A_p); \\ (\beta_1, B_1), ..., (\beta_p, B_p); \end{array} z \right] = \sum_{n=o}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z_n}{n!} \tag{1.3.1} $$

where the coefficients A_1 ,...,A_p and B_1 ,...,B_p are positive real numbers such that

$$ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0 $$

Definition 1.4: The generalized Mittag-Leffler function

In 1971, Prabhakar (1971) introduced the generalized Mittag-Leffler function $E^{\gamma}_{\rho,\mu}(z)$ (see [1], [9]):

$$ E^{\gamma}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\rho k + \mu) k!} \tag{1.4.1} $$

$(\rho, \mu, \gamma \in C, Re(\rho) > 0, Re(\beta) > 0, Re(\gamma) > 0)$

where at$(\gamma = 1, E^{1}_{\rho,\mu}(z)$ coincides with the classical Mittag-Leffler function $E_{\rho,\mu}(z)$ and in particular$E_{1,1}(z) = e^z$ and when $\rho = 1$ it coincides with Kummer's confluent hypergeometric function $\phi(\gamma; \mu; z)$ with the exactness to the constant multiplier $[\Gamma(\mu)]^{-1}$.In 2007, Shukla and Prajapati (2007) (cf. [16]) introduced the function $E^{\gamma,q}_{\rho,\mu}(z)$, which is defined for $\rho, \mu, \gamma \in C, Re(\rho) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0, 1) \cup N$ as

$$ E^{\gamma,q}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{\Gamma(\rho k + \mu) k!} \tag{1.4.2} $$

In 2009, Tariq O. Salim (2009) (cf. [13]) introduced the function$E^{\gamma,\delta}_{\rho,\mu}(z)$, which is defined for $\rho, \mu, \gamma, \delta \in C; Re(\rho) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0$ as

$$ E^{\gamma,\delta}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\rho k + \mu)(\delta)_n} \tag{1.4.3} $$

In 2012, a new generalization of Mittag – Leffler function was defined by Salim (2012) (cf. [14]) as

$$ E^{\gamma,\delta,q}_{\rho,\mu,p}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{\Gamma(\rho k + \mu)(\delta)_{pk}} \tag{1.4.4} $$

where$\rho, \mu, \gamma, \delta \in C; \min(Re(\rho), Re(\beta), Re(\gamma), Re(\delta)) > 0$.
Compositions of Pathway integral operator on (1.4.2) and (1.4.3) are studied by Nagar and Tripathi [10]. In this paper we introduce new composition of Pathway Operator on generalized Mittag – Leffler function defined by Khan and Ahmed [3],

$$E^{\gamma,q}_{\alpha,\beta,\delta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{\Gamma(\alpha k + \beta)(\delta)_k} \tag{1.4.5}$$

where$\alpha, \beta, \gamma, \delta \in C; \min(Re(\alpha), Re(\beta), Re(\gamma), Re(\delta)) > 0 \, and \, q \in (0,1) \cup N$.

Further the generalization of definition (1.4.5) defined and investigated as follows by Khan and Ahmed [3]

$$E^{\tau,\zeta,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}(\tau)_{\zeta k} z^k}{\Gamma(\alpha k + \beta)(\nu)_{\sigma k}(\delta)_{pk}} \tag{1.4.6}$$

(b) Required Result:

The following formula is required (see [11, eq. (12)])

$$P^{(\rho,\alpha,a)}_{0+}\{t^{\beta-1}\} = \frac{t^{\rho+\beta}}{[a(1-\alpha)]^{\beta}} \frac{\Gamma(\beta)\Gamma(1+\frac{\rho}{1-\alpha})}{\Gamma(\frac{\rho}{1-\alpha}+\beta+1)} \tag{1.4.7}$$

where $\alpha < 1; Re(\rho) > 0; Re(\beta) > 0$.

2. Main Results:

Theorem 2.1 : Let $\eta, \mu, \rho, \lambda \in C, a > 0, \rho > 0, Re(\eta) > 0, Re(\mu) > 0, Re(\lambda) > 0, \beta > 0 \, and \, \alpha < 1$.
Then we have the following relation:

$$P^{(\rho,\alpha,a)}_{0+}\{z^{\lambda-1}W(z;\eta,\mu)\}$$

$$= \frac{z^{\rho+\lambda}\Gamma(1+\frac{\rho}{1-\alpha})}{[a(1-\alpha)]^{\lambda}} {}_1\Psi_2 \left[\begin{matrix} (\lambda, 1); \\ \left(1+\lambda+\frac{\rho}{1+\alpha}, 1\right), (\mu, \eta); \end{matrix} \left(\frac{z}{a(1-\alpha)} \right) \right] \tag{2.1.1}$$

Proof of theorem (2.1): To prove the relation in (2.1.1), we denote left – hand side of the relation by Δ_1 i.e.

$$\Delta_1 = P^{(\rho,\alpha,a)}_{0+}\{z^{\lambda-1}W(z;\eta,\mu)\}$$

Now using the definition (1.1.1), we get

$$\Delta_2 = P_{0+}^{(\rho,\alpha,a)}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}\frac{(z)^n}{n!\Gamma(\eta n+\mu)}\right\}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(\eta n+\mu)}P_{0+}^{(\rho,\alpha,a)}\{z^{n+\lambda-1}\}$$

By using the well – known relationship between the Beta function and the Gamma function (ef. [17, pp. 9-11] and [18, pp. 7-10])and using (1.4.7) with β replaced by $n+\lambda$ to the pathway integral and finally after a simplification, we get

$$\Delta_2 = \sum_{n=0}^{\infty}\frac{\Gamma(1+\frac{\rho}{1-\alpha})\Gamma(n+\lambda)}{n!\Gamma(\eta n+\mu)\Gamma(n+\lambda+\frac{\rho}{1-\alpha}+1)}\frac{z^{n+\rho+\lambda}}{[a(1-\alpha)]^{n+\lambda}}$$

$$= \frac{z^{\rho+\lambda}\Gamma(1+\frac{\rho}{1-\alpha})}{[a(1-\alpha)]^{\lambda}}\sum_{n=0}^{\infty}\frac{\Gamma(\lambda+\eta)}{\Gamma(\mu+\eta n)\Gamma(\eta+\lambda+\frac{\rho}{1-\alpha}+1)n!}\frac{(z)^n}{[a(1-\alpha)]^n}$$

Now in view of the result (1.3.1) therein, we at once arrive at the desired result in (2.1.1).

Theorem 2.2: Let $\eta,\mu,\gamma,\delta,\rho,\lambda \in C, a>0, C\in R, \rho>0, Re(\eta)>0, Re(\mu)>0, Re(\gamma)>0, Re(\delta), Re(\lambda)>0, \beta>0\, and\, \alpha<1, q\in(0,1)\cup N$. Then we have the following relation:

$$P_{0+}^{(\rho,\alpha,a)}\{z^{\lambda-1}E_{\eta,\mu,\delta}^{\gamma,q}(cz^{\beta})\}$$

$$= \frac{z^{\rho+\lambda}\Gamma(1+\frac{\rho}{1-\alpha})}{[a(1-\alpha)]^{\lambda}\Gamma(\gamma)}{}_3\Psi_3\left[\begin{array}{c}(\gamma,q),(\lambda,\beta),(1,1);\\(\mu,\eta),(\delta,1)(1+\lambda+\frac{\rho}{1+\alpha},\beta);\end{array} c\left(\frac{z}{a(1-\alpha)}\right)^{\beta}\right] \quad (2.2.1)$$

Proof of theorem (2.2): To prove the relation in (2.2.1), we denote left – hand side of the relation by Δ_2 i.e.

$$\Delta_2 = P_{0+}^{(\rho,\alpha,a)}\{z^{\lambda-1}E_{\eta,\mu,\delta}^{\gamma,q}(cz^{\beta})\}$$

Now using the definition (1.4.5), we get

$$\Delta_2 = P_{0+}^{(\rho,\alpha,a)}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(cz^{\beta})^n}{\Gamma(\eta n+\mu)(\delta)_n}\right\}$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}c^n}{\Gamma(\eta n+\mu)(\delta)_n}P_{0+}^{(\rho,\alpha,a)}\{z^{\beta n+\lambda-1}\}$$

Here, using (1.4.7) with β replaced by $\beta n+\lambda$ to the pathway integral and after a simplification, we get

$$\Delta_2=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}c^n\Gamma(1+\frac{\rho}{1-\alpha})\Gamma(\beta n+\lambda)}{\Gamma(\eta n+\mu)\Gamma(\beta n+\lambda+\frac{\rho}{1-\alpha}+1)(\delta)_n}\frac{z^{\beta n+\rho+\lambda}}{[a(1-\alpha)]^{\beta n+\lambda}}$$

$$=\frac{z^{\rho+\lambda}\Gamma(1+\frac{\rho}{1-\alpha})\Gamma(\delta)}{[a(1-\alpha)]^{\lambda}\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\Gamma(\lambda+\beta n)\Gamma(n+1)}{\Gamma(\mu+\eta n)\Gamma(\beta n+\lambda+\frac{\rho}{1-\alpha}+1)\Gamma(\delta+n)n!}\frac{(cz^{\beta})^n}{[a(1-\alpha)]^{\beta n}}$$

Now in view of the result (1.3.1) therein, we at once arrive at the desired result in (2.2.1).

Theorem 2.3: Let $\eta,\mu,\gamma,\delta,\rho,\lambda\in C, a>0, C\in R, \rho>0, Re(\eta)>0, Re(\mu)>0, Re(\gamma)>0, Re(\delta), Re(\lambda)>0, \beta>0\, and\, \alpha<1, q\in(0,1)\cup N$. Then we have the following relation:

$$P_{0+}^{(\rho,\alpha,a)}\{z^{\lambda-1}E_{\eta,\mu,\nu,\sigma,\delta p}^{\tau,\zeta,\gamma,q}(cz^{\beta})\}$$

$$=\frac{z^{\rho+\lambda}\Gamma(1+\frac{\rho}{1-\alpha})\Gamma(\nu)\Gamma(\delta)}{[a(1-\alpha)]^{\lambda}\Gamma(\gamma)\Gamma(\tau)}{}_4\Psi_4\left[\begin{array}{c}(\tau,\zeta)(\gamma,q),(\lambda,\beta),(1,1);\\(\mu,\eta),(\nu,\sigma)(\delta,p)\big(1+\lambda+\frac{\rho}{1+\alpha},\beta\big);\end{array}c\left(\frac{z}{a(1-\alpha)}\right)^{\beta}\right] \tag{2.3.1}$$

Proof of theorem (2.3) To prove the relation in (2.3.1), we follow the same technique, which we used to prove the relation (2.2.1) in view of the definition in (1.4.6).

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