About the paper *On the zeros of a certain class of polynomials and related analytic functions* by A. Aziz and Q. G. Mohammad

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This is a comment to the paper

A. Aziz and Q. G. Mohammad, *On the zeros of a certain class of polynomials and related analytic functions*, J. Math, Anal. Appl. 75 (1980), 495–502.

Their Theorem 5 states:

Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ be analytic in $|z| \le t$. If

$$a_j > 0 \quad \text{and} \quad a_{j-1} - t a_j \ge 0, \quad j = 1, 2, 3, \ldots$$

then $f(z)$ does not vanish in $|z| < t$.

Their proof, using Schwarz's lemma, is extremely short and simple.

Observe that, without loss of generality, we may assume $t = 1$ and $a_0 = 1$. This is just a matter of rescaling. Moreover, we do not need to assume that $f(z)$ is analytic up to the boundary. It suffices to assume that it is analytic on the open disc. This is seen by using the version of Schwarz's lemma which states:

Let $g(z)$ be analytic for $|z| < 1$, let $|g(z)| \le 1$ for $|z| < 1$, and let $g(0) = 0$. Then $|g(z)| \le |z|$ for $|z| < 1$.

Furthermore, Schwarz's lemma becomes trivial in the following case:

If $g(z) := \sum_{j=1}^{\infty} c_j z^j$ with $\sum_{j=1}^{\infty} |c_j| \le 1$ then $|g(z)| \le \sum_{j=1}^{\infty} |c_j|\,|z| \le |z|$ for $|z| < 1$.

An example, where g is continuous on the closed unit disk, but where the convergence of $\sum_{j=1}^{\infty} c_j z^j$ is non-uniform on every arc of $|z| = 1$ (and where thus Schwarz's lemma cannot be proved in the trivial way), is given in Ch. VIII, Theorem (1.17) of the book

A. Zygmund, *Trigonometric Series, Vol. I*, Warszaw, 1935; reprinted by Cambridge University Press.

This example goes back to a paper by Fejér (1917).

Now the following version of Theorem 5 can be formulated.

Theorem Let $1 = a_0 \ge a_1 \ge a_2 \ge \ldots \ge 0$. Then

$$f(z) := \sum_{j=0}^{\infty} a_j z^j \ne 0 \quad \text{for } |z| < 1.$$

Proof We have

$$(1-z)f(z) = 1 - g(z) \quad \text{with} \quad g(z) := \sum_{j=1}^{\infty} (a_{j-1} - a_j) z^j.$$

Then $a_{j-1} - a_j \geq 0$ and $\sum_{j=1}^{\infty} (a_{j-1} - a_j) = 1 - \lim_{j\to\infty} a_j \leq 1$. Hence, by the trivial case of Schwarz's lemma, $|g(z)| \leq |z|$ for $|z| < 1$. So

$$|(1-z)f(z)| \geq 1 - |g(z)| \geq 1 - |z| > 0 \quad \text{for } |z| < 1. \qquad \square$$

Of course, the Theorem remains true if the power series for $f(z)$ is terminating, by which we have also a short proof of the Eneström-Kakeya theorem.

It is surprising that the elegant and potentially useful Theorem 5, with its simple proof, did not make its way to the textbooks.

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