Comments on the paper *Dunkl shift operators and Bannai–Ito polynomials* by S. Tsujimoto et al., 2012

Notes by Tom Koornwinder, thkmath@xs4all.nl, last modified 13 November 2020

These are comments on the paper

S. Tsujimoto, L. Vinet and A. Zhedanov, *Dunkl shift operators and Bannai–Ito polynomials*, Adv. Math. 229 (2012), 2123–2158.

First a few errata:

p.2130, (3.15):
In the two ${}_4F_3(1)$ expressions the fourth upper parameter should be $-x-r_1+1/2$.

p.2142, unnumbered formulas after (5.19):
The last numerator factor in the expression for $\kappa_n^{(1)}$ should be $(\rho_2-r_2+1/2)_n$.
The last numerator factor in the expression for $\kappa_n^{(2)}$ should be $(\rho_2-r_2+3/2)_n$.

Matching Sections 3 and 5

For low values of n I checked in Mathematica that the expressions (3.15), (3.16) for $P_n(x)$ indeed match with the expression (5.21). In case of $P_{2n}(x)$ we have

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix}-n,n+1+\rho_1+\rho_2-r_1-r_2,x-r_1+\frac12,-x-r_1+\frac12\\ 1-r_1-r_2,\frac12+\rho_1-r_1,\frac12+\rho_2-r_1\end{matrix};1\right)\\
&+\frac{n(x-r_1+\frac12)}{(\frac12+\rho_1-r_1)(\frac12+\rho_2-r_1)}\,{}_4F_3\left(\begin{matrix}-n+1,n+1+\rho_1+\rho_2-r_1-r_2,x-r_1+\frac32,-x-r_1+\frac12\\ 1-r_1-r_2,\frac32+\rho_1-r_1,\frac32+\rho_2-r_1\end{matrix};1\right)\\
&=\frac{(1+\rho_1+\rho_2)_n(\frac12-r_2+\rho_2)_n}{(1-r_1-r_2)_n(\frac12-r_1+\rho_1)_n}\left({}_4F_3\left(\begin{matrix}-n,n+1+\rho_1+\rho_2-r_1-r_2,x+\rho_2,-x+\rho_2\\ 1+\rho_1+\rho_2,\frac12+\rho_2-r_1,\frac12+\rho_2-r_2\end{matrix};1\right)\right.\\
&\qquad+\frac{n(n-r_1-r_2)}{(\frac12+\rho_2-r_1)(\frac12+\rho_2-r_2)(1+\rho_1+\rho_2)}\\
&\qquad\qquad\left.\times(x-\rho_2)\,{}_4F_3\left(\begin{matrix}-n+1,n+1+\rho_1+\rho_2-r_1-r_2,x+\rho_2+1,-x+\rho_2+1\\ 2+\rho_1+\rho_2,\frac32+\rho_2-r_1,\frac32+\rho_2-r_2\end{matrix};1\right)\right).
\end{aligned}
$$

In case of $P_{2n+1}(x)$ we have

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix}-n,n+1+\rho_1+\rho_2-r_1-r_2,x-r_1+\frac12,-x-r_1+\frac12\\ 1-r_1-r_2,\frac12+\rho_1-r_1,\frac12+\rho_2-r_1\end{matrix};1\right)\\
&\quad-\frac{(n+1+\rho_1+\rho_2-r_1-r_2)(x-r_1+\frac12)}{(\frac12+\rho_1-r_1)(\frac12+\rho_2-r_1)}\\
&\qquad\times\,{}_4F_3\left(\begin{matrix}-n,n+2+\rho_1+\rho_2-r_1-r_2,x-r_1+\frac32,-x-r_1+\frac12\\ 1-r_1-r_2,\frac32+\rho_1-r_1,\frac32+\rho_2-r_1\end{matrix};1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{(1+\rho_1+\rho_2)_n]\,(\frac12+\rho_2-r_2)_{n+1}}{(1-r_1-r_2)_n\,(\frac12+\rho_1-r_1)_{n+1}}\\
&\quad\times\Bigg({}_4F_3\left(\begin{matrix}-n,n+1+\rho_1+\rho_2-r_1-r_2,x+\rho_2,-x+\rho_2\\ 1+\rho_1+\rho_2,\frac12+\rho_2-r_1,\frac12+\rho_2-r_2\end{matrix};1\right)\\
&\qquad+\frac{(1+n+\rho_1+\rho_2)(1+n+\rho_1+\rho_2-r_1-r_2)}{(1+\rho_1+\rho_2)(\frac12+\rho_2-r_1)(\frac12+\rho_2-r_2)}\\
&\qquad\quad\times(x-\rho_2)\,{}_4F_3\left(\begin{matrix}-n,n+2+\rho_1+\rho_2-r_1-r_2,x+\rho_2+1,-x+\rho_2+1\\ 2+\rho_1+\rho_2,\frac32+\rho_2-r_1,\frac32+\rho_2-r_2\end{matrix};1\right)\Bigg).
\end{aligned}
$$

The expression (5.21) could have neen obtained immediately in the framework of Section 3 if another ansatz, as suggested by (5.18), (5.19) had been chosen for solving $LP_n(x)=\lambda_nP_n(x)$.

Nonsymmetric Wilson polynomials as limits of nonsymmetric Askey–Wilson polynomials

The polynomials $P_n(x)$ in Sections 3 and 5 are Bannai–Ito polynomials. It is shown in [1] that they can also be considered as nonsymmetric Wilson polynomials. I will obtain the nonsymmetric Wilson polynomials as limits of nonsymmetric Askey–Wilson polynomials. In [3, (97)] the nonsymmetric Askey–Wilson polynomials are given by

$$
\begin{aligned}
E_n(z;a,b,c,d\,|\,q) &:= {}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ ab,ac,ad\end{matrix};q,q\right)-\frac{q^{1-n}(1-q^n)(1-q^{n-1}cd)}{(1-qab)(1-ab)(1-ac)(1-ad)}\\
&\qquad\times az^{-1}(1-az)(1-bz)\,{}_4\phi_3\left(\begin{matrix}q^{-n+1},q^nabcd,qaz,qaz^{-1}\\ q^2ab,qac,qad\end{matrix};q,q\right)\qquad(n\ge0),\\
E_{-n}(z;a,b,c,d\,|\,q) &:= {}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ ab,ac,ad\end{matrix};q,q\right)-\frac{q^{1-n}(1-q^nab)(1-q^{n-1}abcd)}{(1-qab)(1-ab)(1-ac)(1-ad)}\\
&\qquad\times b^{-1}z^{-1}(1-az)(1-bz)\,{}_4\phi_3\left(\begin{matrix}q^{-n+1},q^nabcd,qaz,qaz^{-1}\\ q^2ab,qac,qad\end{matrix};q,q\right)\qquad(n<0),
\end{aligned}
$$

Define the Dunkl–Cherednik operator Y by

$$
\begin{aligned}
(Yf)(z) &:= \frac{z\big(1+ab-(a+b)z\big)\big((c+d)q-(cd+q)z\big)}{q(1-z^2)(q-z^2)}\,f(z)+\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}f(qz)\\
&+\frac{(1-az)(1-bz)\big((c+d)qz-(cd+q)\big)}{q(1-z^2)(1-qz^2)}\,f(z^{-1})+\frac{(c-z)(d-z)\big(1+ab-(a+b)z\big)}{(1-z^2)(q-z^2)}\,f(qz^{-1}).
\end{aligned}
$$

Then

$$
\begin{aligned}
YE_n &= q^{n-1}abcd\,E_n \qquad (n=0,1,2,\ldots),\\
YE_{-n} &= q^{-n}\,E_{-n} \qquad (n=1,2,\ldots).
\end{aligned}
$$

Now replace in the above formulas a, b, c, d, x by q^a, q^b, q^c, q^d, q^x. In the limit for $q \to 1$ the $E_{\pm n}$ yield nonsymmetric Wilson polynomials, which we also denote by $E_{\pm n}$:

$$
\begin{aligned}
E_n(x;a,b,c,d) = {}& {}_4F_3\left(\begin{matrix} -n, n-1+a+b+c+d, a+x, a-x \\ a+b, a+c, a+d \end{matrix}; 1\right) \\
& - \frac{n(n-1+c+d)}{(a+b+1)(a+b)(a+c)(a+d)} \\
& \quad \times (a+x)(b+x)\, {}_4F_3\left(\begin{matrix} -n+1, n+a+b+c+d, a+x+1, a-x+1 \\ a+b+2, a+c+1, a+d+1 \end{matrix}; 1\right), \quad (n \ge 0), \\
E_{-n}(x;a,b,c,d) = {}& {}_4F_3\left(\begin{matrix} -n, n-1+a+b+c+d, a+x, a-x \\ a+b, a+c, a+d \end{matrix}; 1\right) \\
& - \frac{(n+a+b)(n-1+a+b+c+d)}{(a+b+1)(a+b)(a+c)(a+d)} \\
& \quad \times (a+x)(b+x)\, {}_4F_3\left(\begin{matrix} -n+1, n+a+b+c+d, a+x+1, a-x+1 \\ a+b+2, a+c+1, a+d+1 \end{matrix}; 1\right), \quad (n < 0).
\end{aligned}
$$

Let Y_q be the operaotr Y with a, b, c, d, z rescaled as before, acting on f_q depending on rescaled a, b, c, d, z. Let $f_q(q^x)$ tend to $f(x)$. Then

$$
\begin{aligned}
(Lf)(x) := \lim_{q \to 1} \frac{f_q(q^x) - (Y_q f_q)(q^x)}{1-q} &= -\frac{(a+x)(b+x)}{2x}\big(f(x) - f(-x)\big) \\
&+ \frac{(c-x)(d-x)}{2x-1}\big(f(x) - f(1-x)\big) + (a+b+c+d-1) f(x).
\end{aligned}
$$

Note that the term of $f(qx)$ in Y vanishes in the limit. So the eigenvalue equations of L are three-term equations, no longer four-term. The $E_{\pm n}(x)$ are eigenfunctions of L:

$$
\begin{aligned}
LE_n &= (n-1+a+b+c+d) E_n \qquad (n = 0, 1, 2, \ldots), \\
LE_{-n} &= -n\, E_{-n} \qquad (n = 1, 2, \ldots).
\end{aligned}
$$

Define constant multiples of the usual Wilson polynomials by

$$
\mathcal{W}_n(x;a,b,c,d) := {}_4F_3\left(\begin{matrix} -n, n-1+a+b+c+d, a+\mathrm{i}x, a-\mathrm{i}x \\ a+b, a+c, a+d \end{matrix}; 1\right).
$$

Then

$$
\begin{aligned}
E_n(\mathrm{i}x;a,b,c,d) = \mathcal{W}_n(x;a,b,c,d) &- \frac{n(n-1+c+d)}{(a+b+1)(a+b)(a+c)(a+d)} \\
&\times (a+\mathrm{i}x)(b+\mathrm{i}x) W_{n-1}(x;a+1,b+1,c,d), \\
E_{-n}(\mathrm{i}x;a,b,c,d) = \mathcal{W}_n(x;a,b,c,d) &- \frac{(n+a+b)(n-1+a+b+c+d)}{(a+b+1)(a+b)(a+c)(a+d)} \\
&\times (a+\mathrm{i}x)(b+\mathrm{i}x) W_{n-1}(x;a+1,b+1,c,d).
\end{aligned}
$$

The nonsymmetric Wilson polynomials $p_{2n}(x)$ and $p_{2n-1}(x)$ as given in [2, Lemma 3.6] have explicit expressions with the same structure as our $E_{\pm n}$ in view of (3.1) and Theorem 3.10 in [3]. (I did not yet check if they exactly coincide.) The nonsymmetric polynomials are decomposed as a sum of a symmetric and an antisymmetric polynomial, i.e., according to eigenvalues of the element T_1 in the degenerate DAHA, see [2, Section 3.2]. This is different from the ordinary notion of symmetric and antisymmetric polynomials.

Orthogonality of the nonsymmetric Wilson polynomials

The polynomials $\mathcal{W}_n(x;a,b,c,d)$ are orthogonal:

$$\frac{1}{2\pi}\int_0^\infty \mathcal{W}_m(x;a,b,c,d)\,\mathcal{W}_n(x;a,b,c,d)\left|\frac{\Gamma(a+\mathrm{i}x)\Gamma(b+\mathrm{i}x)\Gamma(c+\mathrm{i}x)\Gamma(d+\mathrm{i}x)}{\Gamma(2\mathrm{i}x)}\right|^2 \mathrm{d}x$$
$$=\frac{\Gamma(a+b)\cdots\Gamma(c+d)}{\Gamma(a+b+c+d)}\,\frac{(b+c)_n(b+d)_n(c+d)_n}{(a+b)_n(a+c)_n(a+d)_n}\,\frac{(n+a+b+c+d-1)_n\,n!}{(a+b+c+d)_{2n}}\,\delta_{m,n}\,,$$

where $\mathrm{Re}(a,b,c,d)>0$ and non-real paramwters occur in complex conjugate pairs. We abbreviate this orthogonality relation as

$$\frac{1}{2\pi}\int_0^\infty \mathcal{W}_m(x;a,b,c,d)\,\mathcal{W}_n(x;a,b,c,d)\,w(x;a,b,c,d)\,\mathrm{d}x=h_n(a,b,c,d)\,\delta_{m,n}\,.$$

Then

$$\frac{1}{2\pi}\int_0^\infty (a+\mathrm{i}x)(b+\mathrm{i}x)W_{m-1}(x;a+1,b+1,c,d)\,\overline{(a+\mathrm{i}x)(b+\mathrm{i}x)W_{n-1}(x;a+1,b+1,c,d)}$$
$$\times\, w(x;a,b,c,d)\,\mathrm{d}x=h_{n-1}(a+1,b+1,c,d)\,\delta_{m,n}$$

Groenevelt [2, Theorem 3.2] gives an orthogonality for the nonsymmetric Wilson polynomials $E_n(\mathrm{i}x)$ $(n\in\mathbb{Z})$. We formulate it in terms of a hermitian inner product

$$\langle f,g\rangle:=\frac{1}{4\pi}\int_{-\infty}^\infty f(x)\,\overline{g(-x)}\,\frac{(a-\mathrm{i}x)(b-\mathrm{i}x)}{-\mathrm{i}x}\,w(x;a,b,c,d)\,\mathrm{d}x.$$

Keep the assumptions on a,b,c,d, and moreover assume that, if a is nonreal, then $\overline{a}=b$. In [2] the orthogonality was proved by using that the operator L is symmtetric with respect to this inner product. I will show that the orthogonality also follows from the explicit expressions of the $E_{\pm n}(\mathrm{i}x)$ in terms of Wilson polynomials.

Observe the following simple facts. If $f(x)=f_0(x^2)$ and $g(x)=g_0(x^2)$ then

$$\langle f,g\rangle=\frac{a+b}{2\pi}\int_0^\infty f_0(x^2)\,\overline{g_0(x^2)}\,w(x;a,b,c,d)\,\mathrm{d}x.$$

If $f(x)=(a+\mathrm{i}x)(b+\mathrm{i}x)f_0(x^2)$ and $g(x)=(a+\mathrm{i}x)(b+\mathrm{i}x)g_0(x^2)$ then

$$\langle f,g\rangle=-\frac{a+b}{2\pi}\int_0^\infty f_0(x^2)\,\overline{g_0(x^2)}\,w(x;a+1,b+1,c,d)\,\mathrm{d}x.$$

If $f(x)=(a+\mathrm{i}x)(b+\mathrm{i}x)f_0(x^2)$ and $g(x)=g_0(x^2)$ or if $f(x)=f_0(x^2)$ and $g(x)=(a+\mathrm{i}x)(b+\mathrm{i}x)g_0(x^2)$ then $\langle f,g\rangle=0$. It follows that $E_m(\mathrm{i}x)$ and $E_n(\mathrm{i}x)$ are orthogonal if $m,n\in\mathbb{Z}$ and $m\ne\pm n$. We have yet to check the orthogonality of $E_n(\mathrm{i}x)$ and $E_{-n}(\mathrm{i}x)$. Indeed,

$$\frac{1}{a+b}\langle E_n(\mathrm{i}x),E_{-n}(\mathrm{i}x)\rangle=h_n(a,b,c,d)\\ -\frac{n(n-1+c+d)(n+a+b)(n-1+a+b+c+d)}{(a+b+1)^2(a+b)^2(a+c)^2(a+d)^2}\,h_{n-1}(a+1,b+1,c,d)=0.$$

However, the orthogonality is not positive definite. We haver

$$\frac{1}{a+b}\langle E_n(\mathrm{i}x),E_n(\mathrm{i}x)\rangle=\left(1-\frac{n(n-1+c+d)}{(n+a+b)(n-1+a+b+c+d)}\right)h_n(a,b,c,d)>0,$$
$$\frac{1}{a+b}\langle E_{-n}(\mathrm{i}x),E_{-n}(\mathrm{i}x)\rangle=\left(1-\frac{(n+a+b)(n-1+a+b+c+d)}{n(n-1+c+d)}\right)h_n(a,b,c,d)<0.$$

Matching the nonsymmetric Wilson polynomials with the Bannai–Ito polynomials

For low values of n it can be checked in Mathematica that

$$\begin{aligned}
&{}_4F_3\left(\begin{matrix}-n,n+1+\rho_1+\rho_2-r_1-r_2,\rho_2+x,\rho_2-x\\ 1+\rho_1+\rho_2,\frac12+\rho_2-r_1,\frac12+\rho_2-r_2\end{matrix};1\right)\\
&\quad+\frac{n(n-r_1-r_2)}{(1+\rho_1+\rho_2)(\frac12+\rho_2-r_1)(\frac12+\rho_2-r_2)}\\
&\qquad\times(x-\rho_2)\,{}_4F_3\left(\begin{matrix}-n+1,n+1+\rho_1+\rho_2-r_1-r_2,\rho_2+x+1,\rho_2-x+1\\ 2+\rho_1+\rho_2,\frac32+\rho_2-r_1,\frac32+\rho_2-r_2\end{matrix};1\right)\\
&={}_4F_3\left(\begin{matrix}-n,n+\rho_1+\rho_2-r_1-r_2,\rho_2+x,\rho_2-x\\ \rho_1+\rho_2,\frac12+\rho_2-r_1,\frac12+\rho_2-r_2\end{matrix};1\right)\\
&\quad-\frac{n(n-r_1-r_2)}{(1+\rho_1+\rho_2)(\rho_1+\rho_2)(\frac12+\rho_2-r_1)(\frac12+\rho_2-r_w)}\\
&\qquad\times(x-\rho_1)(x-\rho_2)\,{}_4F_3\left(\begin{matrix}-n+1,n+1+\rho_1+\rho_2-r_1-r_2,\rho_2+x+1,\rho_2-x+1\\ 2+\rho_1+\rho_2,\frac32+\rho_2-r_1,\frac32+\rho_2-r_2\end{matrix};1\right)
\end{aligned}$$

and

$$\begin{aligned}
&{}_4F_3\left(\begin{matrix}-n+1,n+\rho_1+\rho_2-r_1-r_2,\rho_2+x,\rho_2-x\\ 1+\rho_1+\rho_2,\frac12+\rho_2-r_1,\frac12+\rho_2-r_2\end{matrix};1\right)\\
&\quad+\frac{(n+\rho_1+\rho_2)(n+\rho_1+\rho_2-r_1-r_2)}{(1+\rho_1+\rho_2)(\frac12+\rho_2-r_1)(\frac12+\rho_2-r_2)}\\
&\qquad\times(x-\rho_2)\,{}_4F_3\left(\begin{matrix}-n+1,n+1+\rho_1+\rho_2-r_1-r_2,\rho_2+x+1,\rho_2-x+1\\ 2+\rho_1+\rho_2,\frac32+\rho_2-r_1,\frac32+\rho_2-r_2\end{matrix};1\right)\\
&={}_4F_3\left(\begin{matrix}-n,n+\rho_1+\rho_2-r_1-r_2,\rho_2+x,\rho_2-x\\ \rho_1+\rho_2,\frac12+\rho_2-r_1,\frac12+\rho_2-r_2\end{matrix};1\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(n+\rho_1+\rho_2)(n+\rho_1+\rho_2-r_1-r_2)}{(1+\rho_1+\rho_2)(\rho_1+\rho_2)(\frac12+\rho_2-r_1)(\frac12+\rho_2-r_w)}\\
&\qquad \times (x-\rho_1)(x-\rho_2)\,{}_4F_3\left(\begin{matrix}-n+1, n+1+\rho_1+\rho_2-r_1-r_2, \rho_2+x+1, \rho_2-x+1\\ 2+\rho_1+\rho_2, \frac32+\rho_2-r_1, \frac32+\rho_2-r_2\end{matrix};1\right)
\end{aligned}
$$

Hence, with $P_n(x)$ given by (5.21), we have

$$
\begin{aligned}
P_{2n}(x) &= \text{const.}\ E_n(-x;\rho_2,\rho_1,\tfrac12-r_1,\tfrac12-r_2),\\
P_{2n-1}(x) &= \text{const.}\ E_{-n}(-x;\rho_2,\rho_1,\tfrac12-r_1,\tfrac12-r_2).
\end{aligned}
$$

This could also have been derived from our eigenvalue equations for $E_{\pm n}$. Indeed, when we rewrite these eigenvalue equations for $E_{\pm n}(-x)$ then they are seen to give (2.11) for $P_{2n}(x)$ and $P_{2n-1}(x)$ in view of (2.5), (2.12) and (2.13). It is essentially this argument which is used in the proof of [1, Proposition 1].

The above identities can be rewrittden in terms of Wilson polynomials:

$$
\begin{aligned}
E_n(\mathrm{i}x;a,b,c,d) &= \mathcal{W}_n(x;a,b+1,c,d) - \frac{n(n-1+c+d)}{(1+a+b)(a+c)(a+d)}\\
&\qquad\qquad \times (a-\mathrm{i}x)W_{n-1}(x;a+1,b+1,c,d),\\
E_{-n}(\mathrm{i}x;a,b,c,d) &= \mathcal{W}_{n-1}(x;a,b+1,c,d) - \frac{(n+a+b)(n-1+a+b+c+d)}{(a+b+1)(a+c)(a+d)}\\
&\qquad\qquad \times (a-\mathrm{i}x)(b+\mathrm{i}x)W_{n-1}(x;a+1,b+1,c,d).
\end{aligned}
$$

The paper [5] considers certain nonsymmetric Wilson polynomials as images of certain nonsymmetric Jacobi polynomials under the Cherednik–Opdam–Jacobi transform [4]. In view of the Plancherel formula for this transform, the nonsymmetric Wilson polynomils in [5] are orthogonal with respect to a rather intricate positive definite hermitian inner product.

References

[1] V. X. Genest, L. Vinet and A. Zhedanov, *The non-symmetric Wilson polynomials are the Bannai–Ito polynomials*, Proc. Amer. Math. Soc. 144 (2016), 5217–5226.

[2] W. Groenevelt, *Fourier transforms related to a root system of rank 1*, Transform. Groups 12 (2007), 77–116.

[3] T. H. Koornwinder and M. Mazzocco, *Dualities in the q-Askey scheme and degenerate DAHA*, Studies Appl. Math. 141 (2018), 424–473.

[4] E. M. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math. 175 (1995), 75–121.

[5] L. Peng and G. Zhang, *Nonsymmetric Jacobi and Wilson-type polynomials*, Int. Math. Res. Not. (2006), Art. ID 21630, 13 pp.