On two questions by P. J. Larcombe

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In this informal note I answer two questions posed to me by P. J. Larcombe. Sections 1 and 2 give the answers to the two questions, as I mailed them to Larcombe on July 2 and 8, 2000, respectively. I conclude with a short comment in Section 3.

1. The answer to Larcombe's first question (note of July 2, 2000)

In an email of June 20, 2000 P. J. Larcombe conjectured that

$$\lim_{n\to\infty} {}_3F_2\left[\begin{matrix} -n, \frac12, \frac12 \\ \frac12 - n, \frac12 - n \end{matrix}; -1\right] = 2. \tag{1.1}$$

I will give a proof of (1.1). Note that the terms of the (terminating well-poised) ${}_3F_2$-series on the left remain invariant under reversion of the direction of summation. Thus we can write

$${}_3F_2\left[\begin{matrix} -n, \frac12, \frac12 \\ \frac12 - n, \frac12 - n \end{matrix}; -1\right] = \sum_{k=0}^{\infty} c_{n,k},$$

where

$$c_{n,k} = \begin{cases} \dfrac{2\,(-n)_k\,(\frac12)_k\,(\frac12)_k\,(-1)^k}{(\frac12-n)_k\,(\frac12-n)_k\,k!} & \text{if } k < \frac12 n, \\ \dfrac{(-n)_k\,(\frac12)_k\,(\frac12)_k\,(-1)^k}{(\frac12-n)_k\,(\frac12-n)_k\,k!} & \text{if } k = \frac12 n \text{ (occurs only if } n \text{ is even)}, \\ 0 & \text{if } k > \frac12 n. \end{cases}$$

Now I will prove (1.1) by dominated convergence. I use that $\lim_{n\to\infty} c_{n,k} = 2$ if $k = 0$ and $= 0$ otherwise, and that $0 \le c_{n,k} \le 4 \cdot (\frac12)^k$. The last inequality follows because, for $k \le \frac12 n$, we have:

$$\begin{aligned} &\frac{(-n)_k\,(\frac12)_k\,(\frac12)_k\,(-1)^k}{(\frac12-n)_k\,(\frac12-n)_k\,k!} \\ &= \left(\frac{n}{n-\frac12}\,\frac{n-1}{n-\frac32}\cdots\frac{n-k+1}{n-k+\frac12}\right)\left(\frac{\frac12}{1}\,\frac{\frac32}{2}\cdots\frac{k-\frac12}{k}\right)\left(\frac{\frac12}{n-k+\frac12}\,\frac{\frac32}{n-k+\frac32}\cdots\frac{k-\frac12}{n-\frac12}\right) \\ &\qquad \le \frac{n}{n-k+\frac12}\left(\frac{k-\frac12}{n-\frac12}\right)^k \le 2\cdot(\tfrac12)^k. \end{aligned}$$

2. The answer to Larcombe's second question (note of July 8, 2000)

In an email of July 4, 2000 P. J. Larcombe communicated that

$$\lim_{n\to\infty} 2^{n-1}\,{}_3F_2\left[\begin{matrix}-n, -\frac{1}{2}(n-1), -\frac{1}{2}n\\ \frac{1}{2}-n, \frac{1}{2}-n\end{matrix};1\right] = 1. \tag{2.1}$$

He asked for an independent proof.

I will show in this note that for nonnegative integer m we have

$${}_3F_2\left[\begin{matrix}-2m, -m+\frac{1}{2}, -m\\ \frac{1}{2}-2m, \frac{1}{2}-2m\end{matrix};1\right] = \frac{2^{-2m+1}\,\Gamma(m+\frac{1}{2})^4}{\Gamma(m+\frac{1}{4})^2\,\Gamma(m+\frac{3}{4})^2}\,{}_3F_2\left[\begin{matrix}-m, \frac{1}{2}, \frac{1}{2}\\ -m+\frac{1}{2}, m+1\end{matrix};1\right], \tag{2.2}$$

$${}_3F_2\left[\begin{matrix}-2m-1, -m-\frac{1}{2}, -m\\ -\frac{1}{2}-2m, -\frac{1}{2}-2m\end{matrix};1\right] = 2^{-2m}\,\frac{(m+\frac{1}{2})\,\Gamma(m+\frac{1}{2})^2\,\Gamma(m+\frac{3}{2})^2}{(m+1)\,\Gamma(m+\frac{3}{4})^2\,\Gamma(m+\frac{5}{4})^2} \times {}_3F_2\left[\begin{matrix}-m, \frac{1}{2}, \frac{1}{2}\\ -m+\frac{1}{2}, m+2\end{matrix};1\right]. \tag{2.3}$$

If we now use that

$$\lim_{m\to\infty} m^{b-a}\,\frac{\Gamma(m+a)}{\Gamma(m+b)} = 1,$$

see for instance [1, (1.4.3)], then (2.1) will follow from (2.2) and (2.3) if we can show that the two ${}_3F_2(1)$ expressions on the right-hand side of (2.2) and (2.3) tend to 1 as $m\to\infty$. This last result can be shown by writing these ${}_3F_2(1)$ expressions as $\sum_{k=0}^{\infty} c_{m,k}$ where $c_{m,k}=0$ if $k>m$ and where $c_{m,k}$ for $k\le m$ is given by

$$c_{m,k} = \frac{\frac{1}{2}\ldots(k-\frac{1}{2})}{(m-k+\frac{1}{2})\ldots(m-\frac{1}{2})}\,\frac{(m-k+1)\ldots m}{(m+j+1)\ldots(m+j+k)}\,\frac{(\frac{1}{2})_k}{k!}\,.$$

Here $j=0$ for (2.2) and $j=1$ for (2.3). So for $k\le m<2k$ we have

$$0\le c_{m,k}\le \frac{(m-k+1)\ldots m}{(m+1)\ldots(m+k)} \le \frac{k\ldots(2k-1)}{(2k)\ldots(3k-1)} \le (\tfrac{2}{3})^k,$$

and for $m\ge 2k$ we have

$$0\le c_{m,k}\le \frac{\frac{1}{2}\ldots(k-\frac{1}{2})}{(m-k+\frac{1}{2})\ldots(m-\frac{1}{2})} \le \frac{\frac{1}{2}\ldots(k-\frac{1}{2})}{(k+\frac{1}{2})\ldots(2k-\frac{1}{2})} \le (\tfrac{1}{2})^k.$$

Hence for all k we have $0\le c_{m,k}\le(\frac{2}{3})^k$, independently of m. Since $\lim_{m\to\infty} c_{m,k}=\delta_{k,0}$, the desired result follows by dominated convergence.

It remains to prove (2.2) and (2.3). For the proof of (2.2) first revert the order of summation on the left-hand side of (2.1) and next apply the transformation formula

$${}_3F_2\left[\begin{matrix}a, b, c\\ d, e\end{matrix};1\right] = \frac{\Gamma(e)\,\Gamma(d+e-a-b-c)}{\Gamma(e-a)\,\Gamma(d+e-b-c)}\,{}_3F_2\left[\begin{matrix}a, d-b, d-c\\ d, d+e-b-c\end{matrix};1\right] \tag{2.4}$$

(see for instance [1, Corollary 3.3.5]), and use the duplication formula $\Gamma(2z)\,\Gamma(\frac{1}{2}) = 2^{2z-1}\,\Gamma(z)\,\Gamma(z+\frac{1}{2})$ (see for instance [1, (1.5.1)]). This yields the identities

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}-2m,-m+\frac{1}{2},-m\\ \frac{1}{2}-2m,\frac{1}{2}-2m\end{matrix};1\right]\\
&=\frac{(-1)^m\,(m+1)_m\,(\frac{1}{2})_m}{\big((m+\frac{1}{2})_m\big)^2}\,{}_3F_2\left[\begin{matrix}-m,m+\frac{1}{2},m+\frac{1}{2}\\ m+1,\frac{1}{2}\end{matrix};1\right]\\
&=\frac{(m+1)_m\,(\frac{1}{2})_m}{\big((m+\frac{1}{2})_m\big)^2}\,{}_3F_2\left[\begin{matrix}-m,\frac{1}{2},\frac{1}{2}\\ -m+\frac{1}{2},m+1\end{matrix};1\right]\\
&=\frac{2^{-2m+1}\,\Gamma(m+\frac{1}{2})^4}{\Gamma(m+\frac{1}{4})^2\,\Gamma(m+\frac{3}{4})^2}\,{}_3F_2\left[\begin{matrix}-m,\frac{1}{2},\frac{1}{2}\\ -m+\frac{1}{2},m+1\end{matrix};1\right].
\end{aligned}
$$

For the derivation of (2.3) we have a similar string of identities:

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}-2m-1,-m-\frac{1}{2},-m\\ -\frac{1}{2}-2m,-\frac{1}{2}-2m\end{matrix};1\right]\\
&=\frac{(-1)^m\,(m+2)_m\,(\frac{3}{2})_m}{\big((m+\frac{3}{2})_m\big)^2}\,{}_3F_2\left[\begin{matrix}-m,m+\frac{3}{2},m+\frac{3}{2}\\ m+2,\frac{3}{2}\end{matrix};1\right]\\
&=\frac{(m+2)_m\,(\frac{3}{2})_m}{\big((m+\frac{3}{2})_m\big)^2}\,{}_3F_2\left[\begin{matrix}-m,\frac{1}{2},\frac{1}{2}\\ -m+\frac{1}{2},m+2\end{matrix};1\right]\\
&=2^{-2m}\,\frac{(m+\frac{1}{2})\,\Gamma(m+\frac{1}{2})^2\,\Gamma(m+\frac{3}{2})^2}{(m+1)\,\Gamma(m+\frac{3}{4})^2\,\Gamma(m+\frac{5}{4})^2}\,{}_3F_2\left[\begin{matrix}-m,\frac{1}{2},\frac{1}{2}\\ -m+\frac{1}{2},m+2\end{matrix};1\right].
\end{aligned}
$$

3. Concluding remarks

Formulas (1.1) and (2.1) were earlier obtained in a quite different way by Larcombe *et al.* in [2]. It was pointed out by Larcombe and French in [3] that the two results are related by the identity

$$
2^n\,{}_3F_2\left[\begin{matrix}-\frac{1}{2}n,-n,-\frac{1}{2}(n-1)\\ \frac{1}{2}-n,\frac{1}{2}-n\end{matrix},1\right]={}_3F_2\left[\begin{matrix}-n,\frac{1}{2},\frac{1}{2}\\ \frac{1}{2}-n,\frac{1}{2}-n\end{matrix},-1\right], \tag{3.1}
$$

which is a special case of an identity of Whipple (see formula (7.3) in [5] and formula (9.5) in [4]).

Larcombe and French are preparing a paper, where the above sketchy proofs will be given in more detail and where a precise reference will be given for the dominated convergence theorem in the context of infinite series (equivalent to Tannery's theorem).

References

[1] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Cambridge University Press, 1999.

[2] P. J. Larcombe, D. R. French and E. J. Fennessey, *A note on the asymptotic behaviour of the Catalan-Larcombe-French sequence* $\{1, 8, 80, 896, 10816, \ldots\}$, submitted, 2000.

[3] P. J. Larcombe and D. R. French, *On the 'other' Catalan numbers: a historical formulation re-examined*, Cong. Num., to appear.

[4] F. J. W. Whipple, *On well-poised series, generalized hypergeometric series having parameters in pairs, each pair with the same sum*, Proc. London Math. Soc. (2) 24 (1925), 247–263.

[5] F. J. W. Whipple, *Some translations of generalized hypergeometric series*, Proc. London Math. Soc. (2) 26 (1927), 257–272.