q-SPECIAL FUNCTIONS AND THEIR OCCURRENCE IN QUANTUM GROUPS

T. H. Koornwinder

Summary

In classical Lie group theory irreducible representation spaces are often decomposed into subspaces belonging to irreducible representations of a subgroup (or a chain of subgroups). Matrix elements of the representation w.r.t. a basis obtained in this way can often be expressed in terms of special functions. In particular, *spherical functions* are the matrix elements being left and right invariant w.r.t. a subgroup forming a Gelfand pair with the big group. An elementary example is the group $SU(2)$ with diagonal subgroup isomorphic to $U(1)$. Then the matrix elements of the irreducible representations of $SU(2)$ w.r.t. a $U(1)$-basis can be expressed in terms of Jacobi polynomials and the spherical functions are Legendre polynomials. Nothing essential changes when we take the basis w.r.t. a subgroup conjugate to $U(1)$, for instance $SO(2)$.

For quantum groups the situation is different. Only few quantum subgroups are available, in general, and we cannot take conjugates of quantum subgroups. For instance, quantum $SU(2)$ has a quantum subgroup $U(1)$ (an ordinary group), but no other nontrivial quantum subgroups are known and no conjugates of $U(1)$ can be taken. The matrix elements of the irreducible representations of quantum $SU(2)$ have been computed (cf. [5], [4], [1]) as little q-Jacobi polynomials and the spherical matrix elements as little q-Legendre polynomials. (See Koelink & Koornwinder's q-special functions tutorial in these Proceedings.)

In order to say something about the missing conjugates of a quantum subgroup, we use the quantized universal enveloping algebra. Let $\mathcal{U} = \mathcal{U}_q(sl(2))$ be the algebra generated by A, A^{-1}, B, C with relations $AA^{-1} = 1 = A^{-1}A$, $AB = qBA$, $AC = qCA$, $[B, C] = (A^2 - A^{-2})/(q - q^{-1})$. $\mathcal{U}$ becomes a Hopf algebra with comultiplication $\Delta(A) = A \otimes A$, $\Delta(B) = A \otimes B + B \otimes A^{-1}$, $\Delta(C) = A \otimes C + C \otimes A^{-1}$. Call $X \in \mathcal{U}$ *twisted primitive* if $\Delta(X) = A \otimes X + X \otimes A^{-1}$. Then X has this property iff X ia a linear combination of $A - A^{-1}$, B and C. This three-dimensional space is a kind of quantum analogue of the Lie algebra $sl(2)$.

The Hopf $*$-algebra $\mathcal{A}$ of "polynomial functions" on quantum $SU(2)$ is in doubly non-degenerate Hopf algebra duality with $\mathcal{U}$. Call $a \in \mathcal{A}$ *left (right) invariant* w.r.t. a twisted primitive element $X \in \mathcal{U}$ if $(X \otimes \mathrm{id})(\Delta(a)) = 0$ (respectively $(\mathrm{id} \otimes X)(\Delta(a)) = 0$).

1991 *Mathematics Subject Classification*. Primary 33D45, 33D80, 17B37.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem ([2], [3]). *Let X_1 and X_2 be twisted primitive elements in $\mathcal{U}$. Then the elements of $\mathcal{A}$ being left invariant w.r.t. X_1 and right invariant w.r.t. X_2 form a subalgebra generated by a single element ρ of $\mathcal{A}$ which is quadratic in the generators of $\mathcal{A}$. The intersection of this subalgebra with the subspace of $\mathcal{A}$ belonging to an irreducible representation of quantum $SU(2)$ of odd dimension $2l+1$ is one-dimensional and spanned by a certain Askey-Wilson polynomial of degree l and argument ρ.*

References

1. T. H. Koornwinder, *Representations of the twisted $SU(2)$ quantum group and some q-hypergeometric orthogonal polynomials*, Nederl. Akad. Wetensch. Proc. Ser. A **92** (1989), 97–117.
2. ———, *Orthogonal polynomials in connection with quantum groups*, Orthogonal polynomials: Theory and Practice, P. Nevai (ed.), NATO ASI Series C, Vol. 294, Kluwer Academic Publishers, 1990, pp. 257–292.
3. ———, *Askey-Wilson polynomials as zonal spherical functions on the $SU(2)$ quantum group*, preprint, CWI Rep. AM-R9013, Amsterdam.
4. T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi and K. Ueno, *Representations of the quantum group $SU_q(2)$ and the little q-Jacobi polynomials*, J. Funct. Anal. **99** (1991), 127–151.
5. L. L. Vaksman and Ya. S. Soĭbel'man, *Algebra of functions on the quantum group $SU(2)$*, Functional Anal. Appl. **22** (1988), 170–181.

CWI, P.O. Box 4079, 1009 AB Amsterdam, The Netherlands