Review of "Jack, Hall-Littlewood and Macdonald polynomials", V. B. Kuznetsov and S. Sahi (eds.)

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Jack, Hall-Littlewood and Macdonald polynomials, by Vadim B. Kuznetsov and Siddharta Sahi (eds.), Contemporary Mathematics 417, American Mathematical Society, 2006; ISBN: 978-0-8218-3683-5, xix+360 pp., $99.00.

The book under review contains the Proceedings of a Workshop held in 2003 in Edinburgh, UK. This workshop paid attention to (in fact, celebrated) the pioneering work by the mathematicians Henry Jack, Philip Hall and D. E. Littlewood on two new families of symmetric polynomials, and to the magnificent job done by Ian Macdonald to bring these two classes of polynomials together in a more general family of symmetric polynomials. But most lectures described new developments involving these polynomials. Jack, Hall and Littlewood all lived and worked in England, Scotland or Wales, as does Macdonald, so it was very appropriate to hold this workshop in the UK. The remarkable story about the life and work of these four people fills the first 125 pages of the volume. The remaining two third of the volume is filled with 13 papers describing recent research. All authors of the research papers have earlier made important contributions to the development of the subject.

I will pay most attention in this review to the first, historical part of the Volume. It contains short biographies of the four main characters, reprints of the original papers introducing the various polynomials, including an unpublished manuscript by Jack followed by comments by Macdonald. Some relevant letters are also included.

It is striking to read in the biographies about the isolation of Jack, Hall and Littlewood. Jack was a Reader in Dundee, Scotland. He never ventured far from Scotland and he did not attend conferences (except for ICM, Amstrdam, 1954). Hall was the only algebrist in Cambridge. He was reticent and cared little for large gatherings. Littlewood was a full professor in Bangor, Wales. He also kept away from mathematical meetings. He never met Hall. Eventually he stopped publishing because "there is no point in writing papers of which nobody takes any notice".

The Hall algebra and Hall-Littlewood symmetric functions

Let λ be a *partition*, i.e., a finite sequence of integers $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. Then $|\lambda| := \lambda_1 + \cdots + \lambda_r$ is the *weight* of the partition λ. Let p be a prime number. By a *finite abelian p-group of type* λ we mean a group G which is isomorphic to $\mathbb{Z}_{p^{\lambda_1}} \times \mathbb{Z}_{p^{\lambda_2}} \times \cdots \times \mathbb{Z}_{p^{\lambda_r}}$ (a direct produt of specific cyclic groups). If H is a subgroup of G then H and G/H are again finite abelian p-groups. For G of type λ let $g^{\lambda}_{\mu\nu}(p)$ denote the number of distinct subgroups H of G such that H has type μ and G/H has type ν. For instance, $G = \mathbb{Z}_p \times \mathbb{Z}_p$ has $p + 1$ distinct subgroups H which are isomorphic to $\mathbb{Z}_p$, and for all such H we have that G/H is isomorphic to $\mathbb{Z}_p$. Hence $g^{(1,1)}_{(1)(1)} = p + 1$. Philip Hall showed in 1955 (lectures in St. Andrews at a Colloquium of the Edinburgh Mathermatical Society, short account published in 1957) that $g^{\lambda}_{\mu\nu}(p)$ is a polynomial function of p with integer coefficients. These polynomials $g^{\lambda}_{\mu\nu}$ are called the *Hall polynomials*. If $g^{\lambda}_{\mu\nu}(p) \neq 0$ then $|\lambda| = |\mu| + |\nu|$. Next, as observed by Hall, we can build for each prime p a commutative, associative ring $H(p)$ with a $\mathbb{Z}$-basis $(u_\lambda(p))$ with index λ running through the set of all partitions such that the $g^{\lambda}_{\mu\nu}(p)$ are the structure coefficients for the multiplication:

$$u_\mu(p)\, u_\nu(p) = \sum_\lambda g^{\lambda}_{\mu\nu}(p)\, u_\lambda(p).$$

This ring is called the *Hall algebra*.

Hall also realized the Hall algebra (extended to an algebra over $\mathbb{Q}$) within the algebra of symmetric functions (they are called functions rather than polynomials because they are in infinitely many variables). He did this by identifying with $p^{r(r-1)/2} u_{(1^r)}(p)$ the elementary symmetric function e_r, defined by $e_r(x) := \sum_{i_1<i_2<\ldots<i_r} x_{i_1} x_{i_2} \ldots x_{i_r}$. This will implicitly determine, for each partition λ, some symmetric function corresponding to $u_\lambda(p)$ under this identification. A few years later, in 1961, D. E. Littlewood found an elementary explicit expression for this function. In fact, he obtained a symmetric function $P_\lambda(x;t)$ in $x_1, x_2, \ldots$ which is a polynomial with integer coefficients in the parameter t such that $u_\lambda(p) = p^{-n(\lambda)} P_\lambda(\,.\,; p^{-1})$, where $n(\lambda)$ is defined as $\sum_{i\geq 1}(i-1)\lambda_i$. I. G. Macdonald introduced the name *Hall-Littlewood symmetric functions* for the functions $P_\lambda(\,.\,;t)$. They interpolate between the monomial symmetric functions ($t = 1$) and the Schur functions ($t = 0$). Macdonald also found an interpretation for the functions $\lambda \mapsto P_\lambda(x_1, \ldots, x_n; p^{-1})$ as zonal spherical functions on the homogeneous space $GL_n(\mathbb{Q}_p)/GL_n(\mathbb{Z}_p)$, where $\mathbb{Q}_p$ is the field of p-adic numbers and $\mathbb{Z}_p$ the ring of p-adic integers. His monograph *Symmetric functions and Hall polynomials* appeared in 1979.

It turned out later that Hall's work was preceded in 1900 by E. Steinitz. The only source is a summary of a lecture by him, not containing any proofs.

Jack symmetric functions

Somewhat later than the work by Hall and Littlewood, and with a very differerent motivation, Henry Jack introduced in 1970 a class of symmetric polynomials with a parameter which are now called the *Jack polynomials* $J^{(\alpha)}_\lambda$. Here λ is a partition and α is the parameter. Jack was inspired by work in multivariate statistics, for instance by A. T. James. A recurring theme in Jack's work was analytic methods of group theory to evaluate certain integrals over matrix spaces. The Jack polynomials contain as special cases the Schur functions s_λ ($\alpha = 1$) and the zonal polynomials Z_λ ($\alpha = 2$) introduced by James. These latter polynomials have an interpretation as zonal spherical functions on the homogeneous space GL_n/O_n.

Macdonald polynomials

In 1987 I. G. Macdonald, in a grand synthesis, unified the Hall-Littlewood symmetric functions and Jack's symmetric functions into a class of symmetric functions $P_\lambda(x;q,t)$ depending on two parameters q,t. When $q=t$, they reduce to the Schur functions s_λ, and when $q=0$ to the Hall-Littlewood functions $P_\lambda(x;t)$. Jack's symmetric functions $J_\lambda^{(\alpha)}$ are obtained as the limit $P_\lambda(x;t^\alpha,t)$ for $t\to 1$. The symmetric functions $P_\lambda(x;q,t)$ became known as *Macdonald polynomials*. A much extended second edition of Macdonald's monograph *Symmetric functions and Hall polynomials* appeared in 1995 and incorporated these functions $P_\lambda(x;q,t)$. The two-variable case of these polynomials (homogeneous, so essentially depending on one variable) turns down to the q-ultraspherical polynomials.

It should be emphasized, certainly for a readership of analysts, that Macdonald polynomials form an orthogonal system, as do the Hall-Littlewood and Jack polynomials. However, in the usual algebraic-combinatorial approach the scalar product $\langle\,.\,,\,.\,\rangle$ for which the polynomials are orthogonal is defined in an indirect way in terms of the symmetric functions $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$, where $p_r := \sum_i x_i^r$ is the rth power sum. Then one defines the inner product by requiring that $\langle p_\lambda, p_\mu\rangle = 0$ if $\lambda \neq \mu$ and by specifying $\langle p_\lambda, p_\lambda\rangle$. Certainly, and important for the analytic approach, there is another expression for the scalar product in the form of an integral with explicit weight function. Then one has to fix a finite number of variables $(x_1,\ldots,x_n)$ and one has to require that $0<q<1$ and $-1<t<1$.

Macdonald's symmetric functions had a great impact in many different fields such as combinatorics, representation theory, special functions and theoretical physics. At the moment, "Macdonald polynomials" in the *Anywhere* field of the MathSciNet search form yields 242 hits. Many of these hits concern Macdonald's symmetric funtions, but a considerable amount deals with the more general Macdonald polynomials associated with root systems, also introduced by Macdonald in 1987. The functions $P_\lambda(x;q,t)$ are the Macdonald polynomials for root system A_n. By a similar limit as above to Jack's symmetric functions, one arrives for each root system from the Macdonald polynomials to the Jacobi polynomials for that root systems, which were introduced by Heckman and Opdam in 1987.

In particular, much attention has been paid to the root system BC_n. The Macdonald polynomials for that root system have three parameters, apart from q. For $n=1$, only two parameters remain and one arrives at the continuous q-Jacobi polynomials, a subclass of the four-parameter Askey-Wilson polynomials. In 1992 the reviewer introduced a five-parameter class of orthogonal polynomials associated with root system BC_n which reduce to the Askey-Wilson polynomials for $n=1$ and which contain the Macdonald polynomials for BC_n as a subclass. In later work it turned out that probably a full analogue of the (q-)Askey scheme of (q-)hypergeometric orthogonal polynomials exists for BC_n.

There were two other important developments during the nineties. First, Macdonald polynomials for special root systems and parameters were interpreted as zonal spherical functions on special quantum symmetric pairs (work by Noumi and others, later followed up by G. Letzter). Second, Macdonald polynomials were given by Cherednik a new setting on double affine Hecke algebras. There it was possible to consider non-symmetric (i.e., not Weyl group invariant) Macdonald polynomials which are joint eigenfunctions of certain q-difference reflection operators.

These operators, in its turn, form a far-reaching generalization of the Dunkl operators (introduced by Dunkl in 1989). By this new approach, Cherednik was able to prove Macdonald's conjectures for the Macdonald polynomials in a unified way, at once for all root systems.

The research papers

Eight of the 13 research papers deal with various aspects, mostly analytic, of Macdonald and Jack polynomials. A newer development which is nowadays very active, is the elliptic generalizations of the various families of polynomials. This started in the one-variable case in 1997 with the paper by Frenkel and Turaev. The power series coefficients of elliptic hypergeometric functions involve products of theta functions rather than (q-)shifted factorials. Three of the research papers in the volume address the multivariable elliptic case. Two papers of quite algebraic or algebraic topological nature deal with Cherednik algebras. These are a "rational" degeneration of Cherednik's double affine Hecke algebra and they have become in the present century an object of intensive study. Some of the research papers are quite technical, but for those with some knowledge of the subject much of interest can be read here.

Conclusion

The volume editors are Vadim B. Kuznetsov and Siddharta Sahi. Very sadly, Vadim Kuznetsov died in 2005, still before the volume was published. The volume concludes with an obituary about him.

Altogether, I can recommend the book very much for a wide audience because of the first historical part. For a smaller readership there is a lot of important material in the research papers in the second part.