The Askey scheme as a four-manifold with corners

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Orthogonal polynomials

- **Orthogonal polynomials (OP's)**: A system of polynomials $\{p_n\}_{n=0,1,2,\ldots}$ (p_n of degree n) which are *orthogonal* with respect to some positive measure μ on $\mathbb{R}$.
- Special subclasses:
  - $d\mu(x) = w(x)\,dx$ (absolutely continuous measure)
  - $\mu = \sum_{i=0}^{\infty} w_i\,\delta_{x_i}$ (discrete measure)
  - $\mu = \sum_{i=0}^{N} w_i\,\delta_{x_i}$ (finite measure), only OP's $p_0, p_1, \ldots, p_N$
- Three-term recurrence relation

$$x\,p_n(x) = A_n p_{n+1}(x) + B_n\,p_n(x) + C_n\,p_{n-1}(x),$$
$$p_0(x) = 1, \quad p_{-1}(x) = 0,$$

Hence the $p_n(x)$ are eigenfunctions of a second-order difference operator in the n-variable with eigenvalue x.

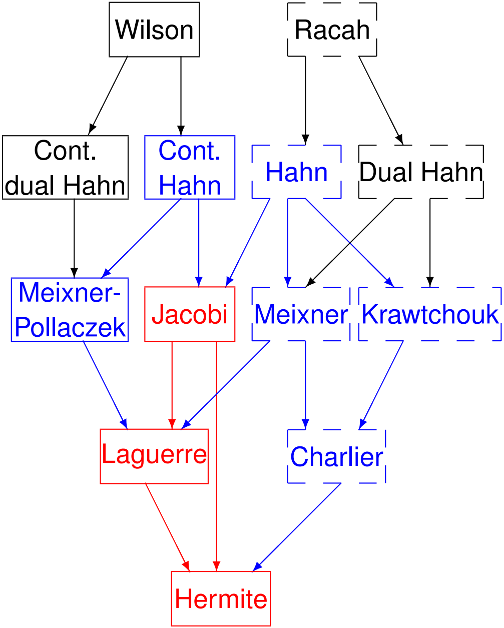
Classical orthogonal polynomials

Call a system $\{p_n\}$ **classical** if $p_n(x)$ is also an eigenfunction of some second order operator L in the x-variable with eigenvalue depending on n:

- L is second order differential operator (**very classical OP's**): Jacobi, Laguerre, Hermite
- $(Lf)(x)$ or $(Lf)(ix) = a(x)\,f(v(x+1)) + b(x)\,f(v(x)) + c(x)\,f(v(x-1))$ (v polynomial)
  - v is first degree polynomial (**Hahn class**): Hahn, Krawtchouk, Mexiner, Charlier (μ discrete); continuous Hahn, Meixner-Pollaczek
  - v is second degree polynomial (**quadratic lattice class**): Racah, dual Hahn (μ discrete); Wilson, continuous dual Hahn

All classical OP's are collected in the *Askey scheme*.

Askey scheme



Dick Askey

discrete OP

quadratic lattice

Hahn class

very classical OP

Group theoretic interpretations

All OP's in the Askey scheme have group theoretic interpretations, sometimes interpretations of various kinds for one family.

- **Jacobi** : matrix elements of irreps of $SU(2)$; spherical functions on compact symmetric spaces of rank one.
- **Laguerre**: matrix elements of irrreps of the Heisenberg group; related to discrete series reps of $SL(2, \mathbb{R})$.
- **Hermite**: related to irreps of the Heisenberg group.
- **Krawtchouk**: matrix elements of irreps of $SU(2)$; spherical functions on wreath products of the symmetric group (Wilson schemes).
- **Hahn**: Clebsch-Gordan ($3j$) coefficients for $SU(2)$; spherical functions on $S_n/(S_k \times S_{n-k})$ (Johnson schemes).
- **Racah**: Racah ($6j$) coefficients for $SU(2)$.
- ...

Limits for very classical OP's

Monic OP's: $p_n(x) = x^n +$ terms of lower degree.

Very classical OP's

- Jacobi: $p_n^{(\alpha,\beta)}(x)$, $w(x) = (1-x)^\alpha\,(1+x)^\beta$ on $(-1,1)$
- Laguerre: $\ell_n^\alpha(x)$, $w(x) = e^{-x}\,x^\alpha$ on $(0,\infty)$
- Hermite: $h_n(x)$, $w(x) = e^{-x^2}$ on $(-\infty,\infty)$

$$\alpha^{n/2} p_n^{(\alpha,\alpha)}(x/\alpha^{1/2}) \to h_n(x), \qquad (1-x^2/\alpha)^\alpha \to e^{-x^2},\ \alpha\to\infty$$

$$(-\beta/2)^n\, p_n^{(\alpha,\beta)}(1-2x/\beta) \to \ell_n^\alpha(x), \qquad x^\alpha(1-x/\beta)^\beta \to x^\alpha e^{-x},\ \beta\to\infty$$

$$(2\alpha)^{-n/2}\,\ell_n^\alpha((2\alpha)^{1/2}x+\alpha) \to h_n(x), \quad (1+(2/\alpha)^{1/2}x)^\alpha e^{-(2\alpha)^{1/2}x} \to e^{-x^2},\quad \alpha\to\infty$$

For Laguerre $\to$ Hermite see:
Palama (1939), Toscano (1939), Askey (1986).

Limits by 3-term recurrence relation

We will always take monic OP's.

$$
\begin{aligned}
x\,p_n(x) &= p_{n+1}(x) + B_n\,p_n(x) + C_n\,p_{n-1}(x), \quad n = 1, 2, \ldots \\
x\,p_0(x) &= p_1(x) + B_0\,p_0(x), \\
p_0(x) &= 1.
\end{aligned}
$$

The monic OP's $p_n(x)$ are completely determined by the B_n, C_n. If B_n and C_n depend continuously on some parameter, then so does $p_n(x)$.

The recurrence relation for $p_n(x) := (2\alpha)^{-n/2}\,\ell_n^\alpha((2\alpha)^{1/2}x + \alpha)$:

$$
x\,p_n(x) = p_{n+1}(x) - (2\alpha)^{-1/2}(2n+1)p_n(x) + \frac{n(n+\alpha)}{2\alpha}\,p_{n-1}(x)
$$

tends for $\alpha \to \infty$ to the Hermite recurrence relation

$$
x\,p_n(x) = p_{n+1}(x) + \tfrac{1}{2}n\,p_{n-1}(x)
$$

Very classical OP scheme with uniform limits

$p_n(x) := \rho^n\, p_n^{(\alpha,\beta)}(\rho^{-1}x - \sigma)$ (rescaled Jacobi) satisfies

$$x\, p_n(x) = p_{n+1}(x) + B_n\, p_n(x) + C_n\, p_{n-1}(x) \quad \text{with}$$

$$C_n = \frac{\rho^2\,\alpha\,\beta}{(\alpha+\beta)^3}$$
$$\times \frac{4n\,(1+n/\alpha)\,(1+n/\beta)\,(1+n/(\alpha+\beta))}{(1+(2n-1)/(\alpha+\beta))\,(1+2n/(\alpha+\beta))^2\,(1+(2n+1)/(\alpha+\beta))}\,,$$

$$B_n = \rho\left(\frac{\beta-\alpha}{\beta+\alpha}\,\frac{1}{1+2n/(\alpha+\beta)}\,\frac{1}{1+(2n+2)/(\alpha+\beta)} + \sigma\right).$$

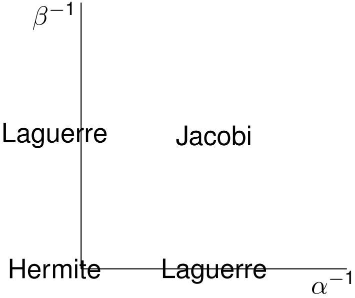
$$\text{Put}\quad \rho := \frac{(\alpha+\beta)^{3/2}}{\alpha^{1/2}\,\beta^{1/2}}\,,\quad \sigma := \frac{\alpha-\beta}{\alpha+\beta}\,.$$

Then B_n and C_n become continuous in (α^{-1},β^{-1}) for $\alpha^{-1},\beta^{-1} \ge 0$. In fact, we get for B_n and C_n :

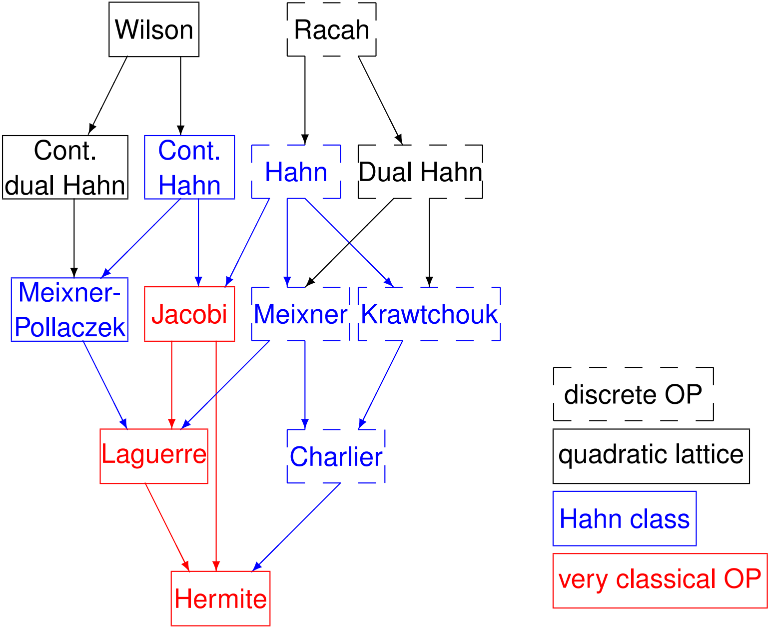
Very classical OP scheme with uniform limits (cntd.)

$$C_n = \frac{4n\,(1+n/\alpha)\,(1+n/\beta)\,(1+n/(\alpha+\beta))}{(1+(2n-1)/(\alpha+\beta))\,(1+2n/(\alpha+\beta))^2\,(1+(2n+1)/(\alpha+\beta))}\,,$$

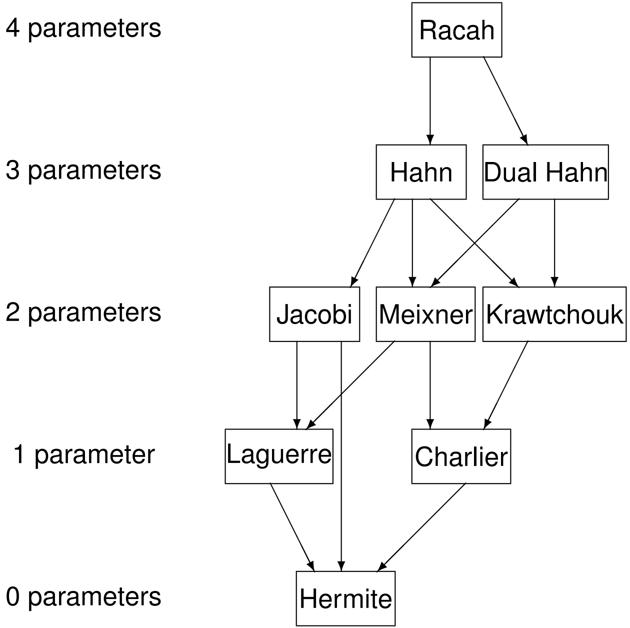
$$B_n = (\beta^{-1/2}-\alpha^{-1/2})(\beta^{-1/2}+\alpha^{-1/2})^{1/2} \times \frac{4n+2+4n\,(n+1)/(\alpha+\beta)}{(1+2n/(\alpha+\beta))\,(1+(2n+2)/(\alpha+\beta))}\,.$$



Askey scheme (recall)



Racah scheme



Manifolds with corners

Manifolds with corners were introduced by
J. Cerf (Bull. Soc. Math. France, 1961) and
A. Douady (Séminaire Henri Cartan, 1961/62).

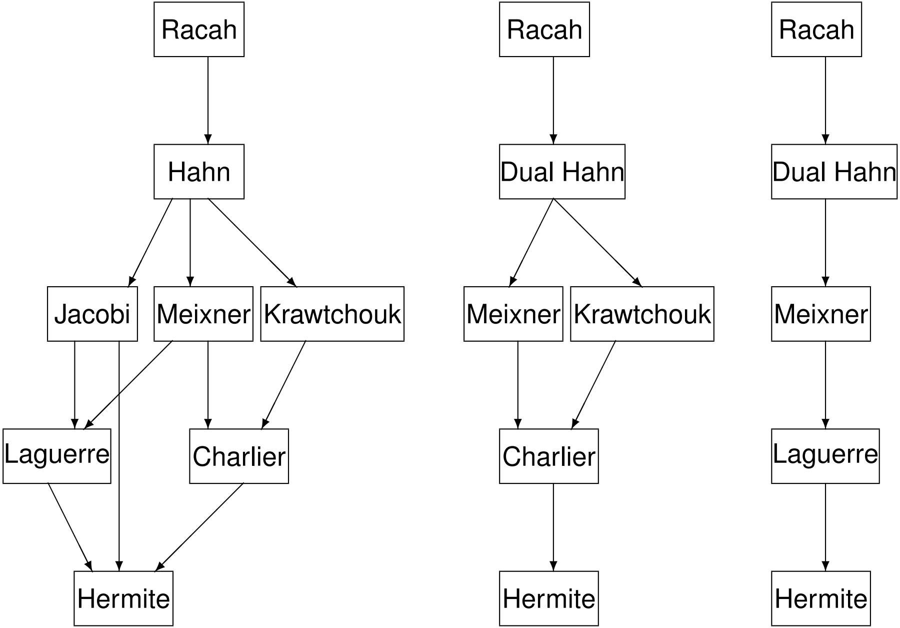
A *manifold with corners* X is defined like an ordinary manifold, except that a *chart* is now a homeomorphism from an open subset of X onto an open subset of

$$\mathbb{R}^n_{(q)} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{q+1}, \ldots, x_n \geq 0\}.$$

I will describe the Racah scheme as a (quotient space of a) four-manifold with corners, in one degree of freedom discretized, with an atlas of three charts mapping to

$$\mathbb{R}^4_{(0)} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1, x_2, x_3, x_4 \geq 0\}.$$

Racah scheme: the three charts



Racah polynomials

Let N be a positive integer, and $n = 0, 1, \ldots, N$.
Racah polynomials are given by

$$\begin{aligned} &R_n\big(y(y+\gamma+\delta+1);\alpha,\beta,-N-1,\delta\big) \\ &:= {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, -y, y+\delta-N \\ \alpha+1, \beta+\delta+1, -N\end{matrix};1\right) \\ &= \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k(-y)_k(y+\delta-N)_k}{(\alpha+1)_k(\beta+\delta+1)_k(-N)_k k!}\,. \end{aligned}$$

Here $(a)_k := a(a+1)(a+2)\ldots(a+k-1)$ is the *Pochhammer symbol*.

Racah polynomials (cntd.)

Monic Racah polynomials:

$$r_n(x;\alpha,\beta,-N-1,\delta) := \frac{(\alpha+1)_n\,(\beta+\delta+1)_n\,(-N)_n}{(n+\alpha+\beta+1)_n}\,R_n(x;\alpha,\beta,-N-1,\delta).$$

Three-term recurrence relation:

$$x\,r_n(x) = r_{n+1}(x) + B_n\,r_n(x) + C_n\,r_{n-1}(x),$$

where $B_n = a_n + c_n$, $C_n = a_{n-1}\,c_n$ with

$$a_n := \frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\,,$$

$$c_n := \frac{n(n+\alpha+\beta+N+1)(\delta-\alpha-n)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}\,.$$

If $\alpha,\beta>-1$ and $\delta>\alpha+N$ then $C_n>0$: positive orthogonality.

Racah scheme: first chart

Now, for $t_1, t_2, t_3, t_4 > 0$, $t_1 t_3 < 1$ and $1/(t_2 t_4)$ integer, put

$$p_n(x) = p_n(x; t_1, t_2, t_3, t_4) := \rho^n \, r_n(\rho^{-1} x - \sigma; \alpha, \beta, -N-1, \delta),$$

where $\alpha = \frac{1}{t_1}, \quad \beta = \frac{1}{t_1\, t_2}, \quad N = \frac{1}{t_2\, t_4}, \quad \delta = \frac{1 + t_2\, t_3\, t_4}{t_1\, t_2\, t_3\, t_4},$

$$\rho = \frac{t_1\, t_2\, (1+t_2)^{3/2}\, t_3\, t_4^2}{(t_1 + t_4 + t_2\, t_4)^{1/2} \left(1 + (1+t_2)\, t_3\, t_4\right)^{1/2}},$$

$$\sigma = -\frac{(1+t_1)\left(1 + (1 + t_2 + t_1\, t_2)\, t_3\, t_4\right)}{t_1\, t_2\, (1 + t_2 + 2\, t_1\, t_2)\, t_3\, t_4^2}.$$

Racah scheme: first chart (cntd.)

Then

$$x\,p_n(x) = p_{n+1}(x) + B_n\,p_n(x) + C_n\,p_{n-1}(x),$$

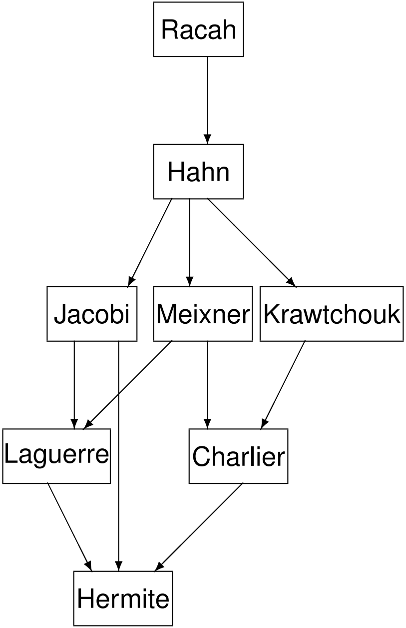
where

$$B_n = -\frac{n(1+t_2)^{3/2}\,\big(1+t_2+(n+1)\,t_1\,t_2\big)}{(1+t_2+2\,t_1\,t_2)\,(1+t_2+2\,n\,t_1\,t_2)\,(1+t_2+2\,(n+1)\,t_1\,t_2)\,(1+t_3\,t_4+t_2\,t_3\,t_4)^{1/2}}$$
$$\times\Big(2\,n\,t_1\,t_2\,t_3\,t_4^2\,(1+t_2+(n+1)\,t_1\,t_2)\,(1+t_2+2\,t_1\,t_2)$$
$$+\,t_2\,t_3\,t_4^2\,(1+t_1)\,(1+t_2)\,(1+t_2+2\,t_1\,t_2)$$
$$-\,t_4\,(1-t_2^2)\,(1+t_1\,t_3)-2t_1\,(1-t_2)\,(1+t_2\,t_4)\Big)\Big/\big(t_1+t_4+t_2\,t_4\big)^{1/2},$$

$$C_n = \frac{(1+t_2+n\,t_1\,t_2)\,\big(1+(1-n)\,t_2\,t_4\big)\,(1-n\,t_1\,t_2\,t_3\,t_4)\,(1+t_3\,t_4+t_2\,t_3\,t_4+n\,t_1\,t_2\,t_3\,t_4)}{\big(1+t_2+(2n-1)\,t_1\,t_2\big)\,(1+t_2+2\,n\,t_1\,t_2)^2\,\big(1+t_2+(2n+1)\,t_1\,t_2\big)}$$
$$\times\,n\,(1+n\,t_1)\,(1+n\,t_1\,t_2)\,\frac{(1+t_2)^3}{1+t_3\,t_4+t_2\,t_3\,t_4}\,\Big(1+(n+1)\,\frac{t_1\,t_2\,t_4}{t_1+t_4+t_2 t_4}\Big).$$

B_n and C_n, as functions of $t_1, t_2, t_3, t_4 > 0$, can be uniquely extended to continuous functions of $t_1, t_2, t_3, t_4 \geq 0$.

Racah scheme: first chart (recall)



Racah scheme: first chart (cntd.)

Hahn:

$$p_n(x;t_1,t_2,0,t_4)=\rho^n\,q_n(\rho^{-1}x-\sigma;\alpha,\beta,N),\quad \alpha=\frac{1}{t_1}\,,\quad \beta=\frac{1}{t_1\,t_2}\,,$$

$$N=\frac{1}{t_2\,t_4}\,,\quad \rho=\frac{(1+t_2)^{3/2}\,t_4}{(t_1+t_4+t_2\,t_4)^{1/2}}\,,\quad \sigma=-\frac{1+t_1}{(1+t_2+2\,t_1\,t_2)\,t_4}\,.$$

Jacobi:

$$p_n(x;t_1,t_2,t_3,0)=p_n(x;t_1,t_2,0,0)=\rho^n\,p_n^{(\alpha,\beta)}(\rho^{-1}x-\sigma),$$

$$\alpha=\frac{1}{t_1}\,,\quad \beta=\frac{1}{t_1\,t_2}\,,\quad \rho=-\frac{(1+t_2)^{3/2}}{2\,t_1^{1/2}\,t_2}\,,\quad \sigma=\frac{-1+t_2}{1+t_2+2\,t_1\,t_2}\,.$$

Racah scheme: first chart (cntd.)

Meixner:

$$p_n(x;t_1,0,t_3,t_4)=p_n\left(x;t_1,0,0,\frac{t_4(1-t_1t_3)}{1+t_3t_4}\right)=\rho^n\,m_n(\rho^{-1}x-\sigma;\beta,c),$$

$$\beta=\frac{1+t_1}{t_1}\,,\quad c=\frac{t_1\,(1+t_3\,t_4)}{t_1+t_4}\,,$$

$$\rho=\frac{(1-t_1\,t_3)\,t_4}{(t_1+t_4)^{1/2}\,(1+t_3\,t_4)^{1/2}}\,,\quad \sigma=-\frac{(1+t_1)\,(1+t_3\,t_4)}{(1-t_1\,t_3)\,t_4}\,.$$

Racah scheme: first chart (cntd.)

Krawtchouk:

$$p_n(x;0,t_2,t_3,t_4) = p_n\left(x;0,t_2\,(1+t_3\,t_4+t_2\,t_3\,t_4),0,\frac{t_4}{1+t_3\,t_4+t_2\,t_3\,t_4}\right)$$
$$:= \rho^n\,k_n(\rho^{-1}x-\sigma;p,N),$$

$$p = \frac{t_2\,(1+t_3\,t_4+t_2\,t_3\,t_4)}{(1+t_2)\,(1+t_2\,t_3\,t_4)}\,,\quad N = \frac{1}{t_2\,t_4}\,,$$

$$\rho = \frac{t_4^{1/2}\,(1+t_2)\,(1+t_2\,t_3\,t_4)}{(1+t_3\,t_4+t_2\,t_3\,t_4)^{1/2}},\quad \sigma = -\,\frac{1+t_3\,t_4+t_2\,t_3\,t_4}{t_4\,(1+t_2)\,(1+t_2\,t_3\,t_4)}\,.$$

Racah scheme: first chart (cntd.)

Laguerre:

$$p_n(x;t_1,0,t_3,0)=p_n(x;t_1,0,0,0)=\rho^n\,l_n^{(\alpha)}(\rho^{-1}x-\sigma),$$
$$\alpha=\frac{1}{t_1},\quad \rho=t_1^{1/2},\quad \sigma=-\frac{1+t_1}{t_1}\,.$$

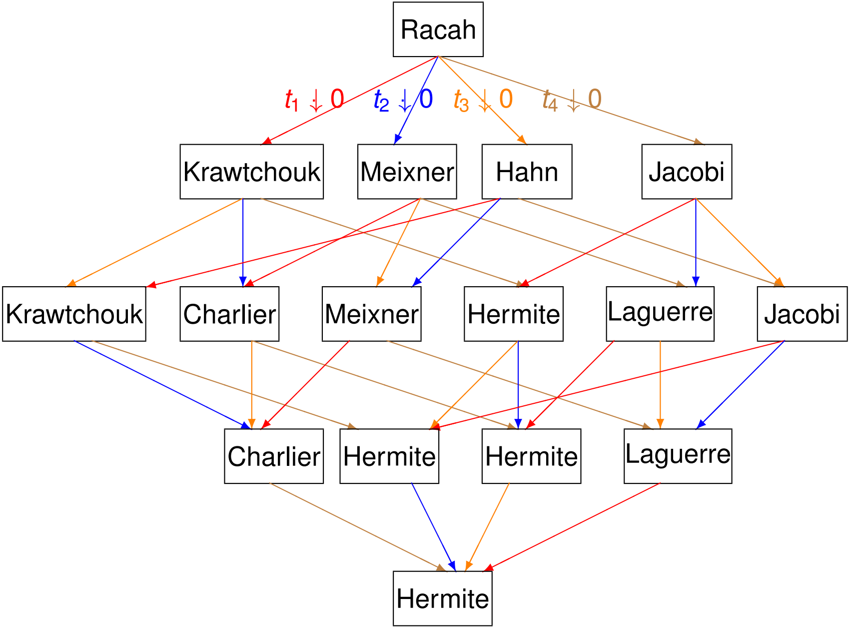
Charlier:

$$p_n(x;0,0,t_3,t_4)=p_n\left(x;0,0,0,\frac{t_4}{1+t_3\,t_4}\right)=\rho^n\,c_n(\rho^{-1}x-\sigma;a),$$
$$a=\frac{1+t_3\,t_4}{t_4}\,,\quad \rho=\frac{t_4^{1/2}}{(1+t_3\,t_4)^{1/2}}\,,\quad \sigma=-\frac{1+t_3\,t_4}{t_4}\,.$$

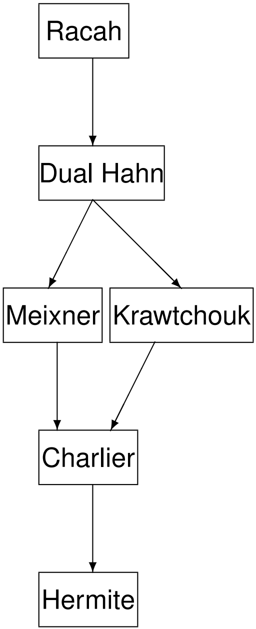
Hermite:

$$p_n(x;0,t_2,t_3,0)=p_n(x;0,t_2,0,0)=p_n(x;0,0,t_3,0)$$
$$=p_n(x;0,0,0,0)=\rho^n\,h_n(\rho^{-1}x-\sigma),$$
$$\rho=2^{1/2},\quad \sigma=0.$$

Racah scheme: summary of first chart



Racah scheme: second chart (recall)



Racah scheme: second chart

For $s_1, s_2, s_3, s_4 > 0$ and $1/(s_2^2 s_4)$ integer, put

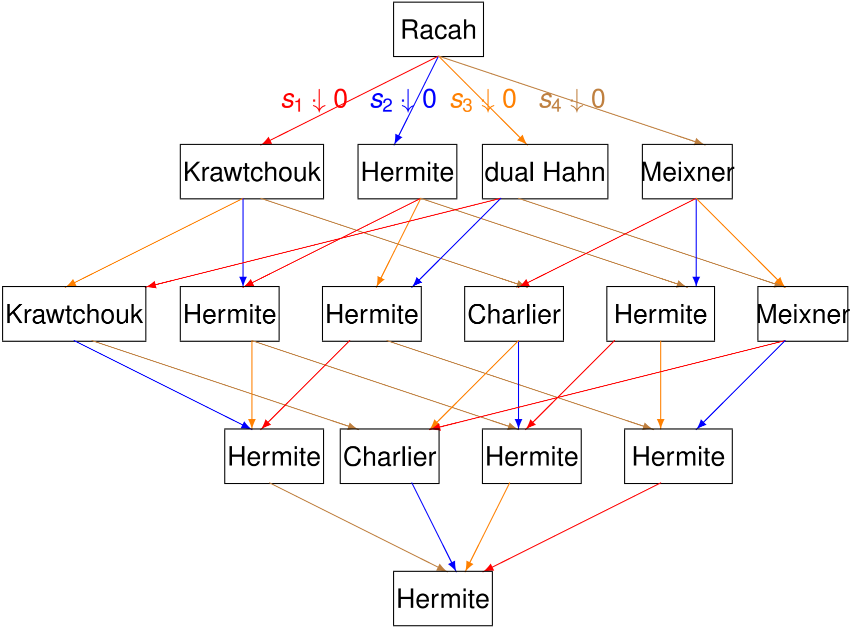
$$p_n(x; s_1, s_2, s_3, s_4) := \rho^n \, r_n(\rho^{-1} x - \sigma; \alpha, \beta, -N-1, \delta),$$

where
$$\alpha = \frac{1+s_1}{s_1\, s_2}, \quad \beta = \frac{1+s_1}{s_1\, s_2^2\, s_3\, s_4}, \quad N = \frac{1}{s_2^2\, s_4},$$

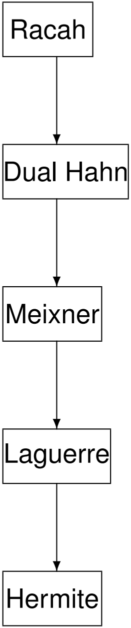
$$\delta = \frac{1 + s_1 + s_2\, s_4\,(1 + s_1 + s_1\, s_2)}{s_1\, s_2^2\, s_4},$$

$$\rho = \frac{s_1\, s_2^{5/2}\, s_4}{2^{1/2}\,(1+s_1)}, \quad \sigma = -\frac{(1+s_1)(1+s_3-s_2^2\, s_4) + s_1\, s_2}{s_1\, s_2^3\, s_4}.$$

Racah scheme: summary of second chart



Racah scheme: third chart (recall)



Racah scheme: third chart

For $u_1, u_2, u_3, u_4 > 0$, $\quad 1 + u_2^2 u_3 > u_1 u_2 u_3$
and $\quad 1/(u_2^2 u_3 u_4)$ integer, put

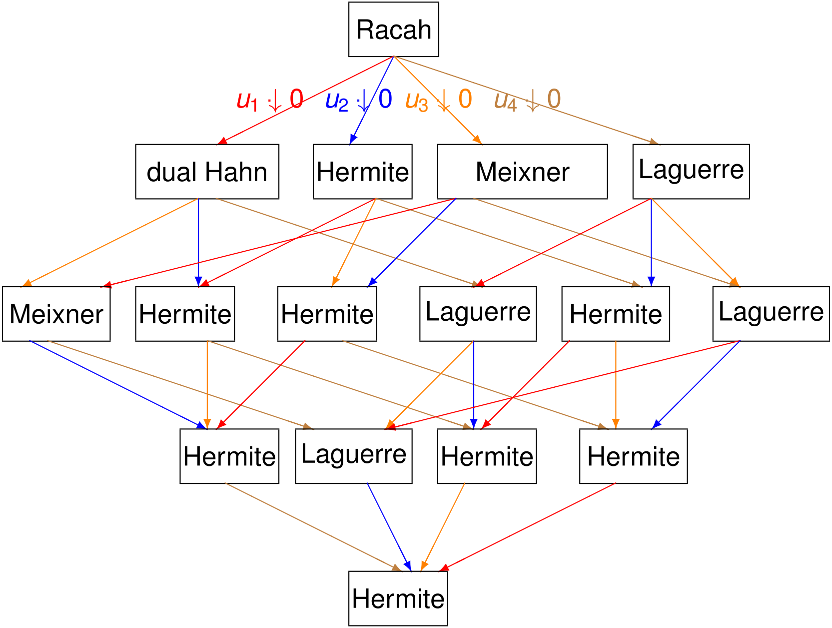
$$p_n(x; u_1, u_2, u_3, u_4) := \rho^n\, r_n(\rho^{-1} x - \sigma; \alpha, \beta, -N-1, \delta),$$

where $\quad \alpha = \dfrac{1+u_1}{u_2}, \quad \beta = \dfrac{1}{u_1\, u_2^2\, u_3^2\, u_4}, \quad N = \dfrac{1}{u_2^2\, u_3\, u_4},$

$$\delta = \frac{1 + u_4 + u_2\, u_3\, u_4 + u_2^2\, u_3\, u_4}{u_2^2\, u_3\, u_4},$$

$$\rho = 2^{-1/2}\, u_2^{5/2}\, u_3\, u_4\,, \quad \sigma = -\frac{(1+u_1)\,(1 + u_1\, u_3 + u_1\, u_3\, u_4)}{u_2^3\, u_3\, u_4}.$$

Racah scheme: summary of third chart



first chart $\leftrightarrow$ second chart

$$t_1 = \frac{s_1\, s_2}{1+s_1}\,, \quad t_2 = s_2\, s_3\, s_4, \quad t_3 = \frac{1+s_1}{s_2(1+s_1+s_1\, s_2^2\, s_4)}\,, \quad t_4 = \frac{s_2}{s_3}\,;$$

$$s_1 = \frac{t_1\, t_3}{1-t_1\, t_3 - t_1\, t_2\, t_3\, t_4}\,, \quad s_2 = \frac{1-t_1\, t_2\, t_3\, t_4}{t_3}\,, \quad s_3 = \frac{1-t_1\, t_2\, t_3\, t_4}{t_3\, t_4}\,,$$

$$s_4 = \frac{t_2\, t_3^2\, t_4}{(1-t_1\, t_2\, t_3\, t_4)^2}\,.$$

This is a homeomorphism between

$$\{(t_1,t_2,t_3,t_4) \mid t_1 \ge 0,\ t_2 \ge 0,\ t_3 > 0,\ t_4 > 0,\\ t_1 t_3(1+t_2 t_4) < 1,\ t_2\, t_4 < 1\}$$

and

$$\{(s_1,s_2,s_3,s_4) \mid s_1 \ge 0,\ s_2 > 0, s_3 > 0,\ s_4 \ge 0,\ s_2^2\, s_4 < 1\}.$$

second chart ↔ third chart

$$s_1 = \frac{1}{u_4\,(1-u_1\,u_2\,u_3)}\,,\qquad s_2 = \frac{u_2\,\big(1+u_4\,(1-u_1\,u_2\,u_3)\big)}{1+u_1}\,,$$

$$s_3 = u_1\,u_3\,\big(1+u_4\,(1-u_1\,u_2\,u_3)\big),\qquad s_4 = \frac{(1+u_1)^2\,u_3\,u_4}{\big(1+u_4\,(1-u_1\,u_2\,u_3)\big)^2}\,;$$

$$u_1 = \frac{-2s_1^2\,s_3 + s_4\,(1+s_1)\,(1+2\,s_1+s_1^2-s_1^2\,s_2\,s_3) - (1+s_1)\,S^{1/2}}{2\,s_1^2\,s_3\,(1+s_2\,s_4+s_1\,s_2\,s_4)}\,,$$

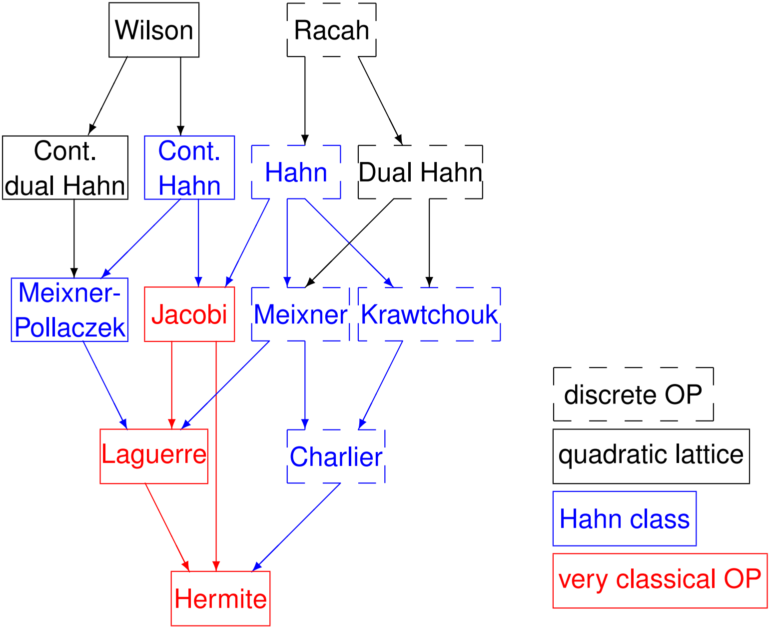
$$u_2 = \frac{s_1\,s_2^2\,s_4}{1+s_2\,s_4+s_1\,s_2\,s_4} + s_2\,\frac{s_4\,(1+2\,s_1+s_1^2-s_1^2\,s_2\,s_3) - S^{1/2}}{2\,s_1\,s_3\,(1+s_2\,s_4+s_1\,s_2\,s_4)}\,,$$

$$u_3 = \frac{-2s_1^2\,s_3 + s_4\,(1+s_1)\,(1+2\,s_1+s_1^2-s_1^2\,s_2\,s_3) + (1+s_1)\,S^{1/2}}{2\,s_1\,(1+s_1)}\,,$$

$$u_4 = \frac{1}{s_1} + s_2\,\frac{s_4\,(1+2\,s_1+s_1^2-s_1^2\,s_2\,s_3) - S^{1/2}}{2\,s_1\,(1+s_1)}\,,$$

where $S := s_4^2\,(1+2\,s_1+s_1^2-s_1^2\,s_2\,s_3)^2 - 4\,s_1^2\,s_3\,s_4\,(1+s_1)$.

Askey scheme (recall)



Monic Wilson polynomials

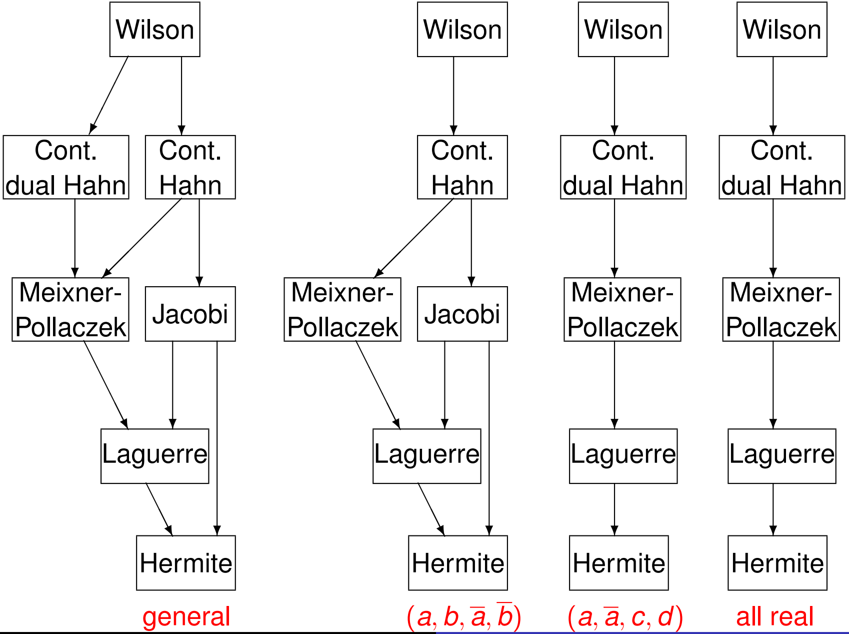
$$w_n(y^2; a,b,c,d) := \frac{(-1)^n(a+b)_n(a+c)_n(a+d)_n}{(n+a+b+c+d-1)_n}$$
$$\times\ {}_4F_3\left(\begin{matrix}-n, n+a+b+c+d-1, a+iy, a-iy\\ a+b, a+c, a+d\end{matrix}; 1\right) = \sum_{k=0}^{n}\binom{n}{k}$$
$$\times\ \frac{(a+b+k)_{n-k}(a+c+k)_{n-k}(a+d+k)_{n-k}(a+iy)_k(a-iy)_k}{(n+k+a+b+c+d-1)_{n-k}}.$$

Three possibilities for a, b, c, d:

- two pairs of complex conjugate parameters
- one pair of complex conjugate parameters and two real parameters
- four real parameters

.

Wilson scheme and its subschemes



first Wilson subscheme

$$p_n(x; a_1, a_2, a_3, a_4) := \rho^n \, w_n(\rho^{-1} x - \sigma; a, b, c, d),$$

where

$$a = \overline{c} = a_1^{-1} - \frac{1 - a_1^{1/2}\, a_2\, a_4}{2\, a_1^{3/2}\, a_2^2\, a_3\, a_4}\, i,$$

$$b = \overline{d} = a_1^{-1}\, a_2^{-1} - \frac{1 + a_1^{1/2}\, a_2\, a_4}{2\, a_1^{3/2}\, a_2^2\, a_3\, a_4}\, i,$$

$$\rho = 2^{3/2}\, a_1^2\, a_2^2\, a_3^2\, a_4,$$

$$\sigma = -\frac{1}{4\, a_1^3\, a_2^4\, a_3^2\, a_4^2} + \frac{1 - a_2}{2\, a_1^{5/2}\, a_2^3\, (1 + a_2 - a_1\, a_2)\, a_3^2\, a_4}\,.$$

second Wilson subscheme

$$p_n(x; b_1, b_2, b_3, b_4) := \rho^n\, w_n(\rho^{-1}x - \sigma; a, b, c, d),$$

where

$$a = \overline{b} = \frac{1 + b_1}{2\, b_1} + \frac{1 + 4\, b_1\, b_2}{2\, b_1^3\, b_2\, b_3}\, i,$$

$$c = \frac{1}{b_1^6\, b_2^2\, b_3^3\, b_4} + \frac{2 + b_1\, b_3 + b_1^3\, b_2\, b_3^2 + 3\, b_1^4\, b_2\, b_3^2}{2\, b_1^4\, b_2\, b_3^2},$$

$$d = -\frac{2 + b_1\, b_3 + b_1^3\, b_2\, b_3^2 + b_1^4\, b_2\, b_3^2}{2\, b_1^4\, b_2\, b_3^2}, \quad \rho = 2^{-1/2}\, b_1^{9/2}\, b_2\, b_3^2,$$

$$\sigma = -\Big(1 + 4\, b_1\, b_2 + 4\, b_1^2\, b_2\, b_4 + 4\, b_1^3\, b_2^2\, b_3\, b_4 + 4\, b_1^4\, b_2^2\, b_3^2\, b_4 + 4\, b_1^4\, b_2^2\, b_3\, b_4^2 + 4\, b_1^5\, b_2^2\, b_3^2\, b_4^2\Big)/(4\, b_1^6\, b_2^2\, b_3^2).$$

Literature

- T. H. Koornwinder, The Askey scheme as a four-manifold with corners, *Ramanujan J.* **20** (2009), 409–439; `arXiv:0909.2822`
- related Mathematica notebooks on `http://staff.science.uva.nl/~thk/art/`
- special case (first Racah chart) in 1993:
  T. H. Koornwinder, *Uniform multi-parameter limit transitions in the Askey tableau*, `arXiv:math/9309213`.
- Some special limit cases already in Ferreira, Lopez & Mainar (2003) and in Ferreira, Lopez & Pagola (2008).