Algebraic methods: Lie groups, quantum groups

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These notes concentrate on the paradigm of spherical functions on Riemannian symmetric spaces. This paradigm has led to the notion of special functions associated with root systems. The first sections deal with older work on the interaction between spherical functions on rank one symmetric spaces and special functions (q=1) in one variable. The (spectacular) developments in the higher rank case and the q c.q. quantum case are summarized in two charts: one for $q = 1$ and one for the q-case. See also the related lectures by W. Miller and C. Dunkl during this program (at `http://www.ima.umn.edu/digital-age/`).

1. Spherical harmonics

In *Higher Transcendental functions* the only material about the connection between special functions and group theory occurs in the (excellent) chapter on spherical harmonics. I summarize below some of the definitions and results (not exactly following HTF).

S^{d-1} unit sphere in $\mathbb{R}^d$.

Orthogonal group $O(d)$ acts on $\mathbb{R}^d$ and acts transitively on S^{d-1}.

Subgroup $O(d-1) := \{T \in O(d) \mid Te_1 = e_1\}$ $\quad$ ($e_1, \ldots, e_d$ standard basis of $\mathbb{R}^d$).

Inner product on $L^2(S^{d-1})$:

$$\langle f_1, f_2\rangle := \frac{1}{\omega_d} \int_{S^{d-1}} f_1(\xi)\, \overline{f_2(\xi)}\, d\omega(\xi),$$

where $d\omega(\xi)$ is the surface element on S^{d-1} and ω_d is the total area of S^{d-1}.

$(Tf)(\xi) := f(T^{-1}\xi) \quad (T \in O(d))$.

$\langle Tf_1, Tf_2\rangle = \langle f_1, f_2\rangle \quad (T \in O(d))$.

Definition $\quad \mathcal{H}_n = \{$spherical harmonics of degree n on $S^{d-1}\}$
$:= \{$restrictions to S^{d-1} of harmonic homogeneous polynomials of degree n on $\mathbb{R}^d\}$.

Theorem $\quad L^2(S^{d-1}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, $\quad$ orthogonal direct sum;
is the unique orthogonal decomposition of $L^2(S^{d-1})$ into irreducible subspaces with respect to $O(d)$.
The $O(d-1)$-invariants in $\mathcal{H}_n$ form 1-dimensional subspace spanned by $\xi \mapsto C_n^{\frac{1}{2}d-1}(\langle \xi, e_1\rangle)$ (Gegenbauer polynomial).

$N_n := \dim \mathcal{H}_n = (2n+d-2)\, \dfrac{(n+d-3)!}{(d-2)!\, n!}$.

First stage of addition formula

Let $f_1, f_2, \ldots, f_N$ be orthonormal basis of $\mathcal{H}_n$. Then:

$$\frac{C_n^{\frac{1}{2}d-1}(\langle \xi, \eta\rangle)}{C_n^{\frac{1}{2}d-1}(1)} = \frac{1}{N_n} \sum_{k=1}^{N_n} f_k(\xi)\, \overline{f_k(\eta)}.$$

Second stage of addition formula
Write $\xi \in S^{d-1}$ as $\quad \xi = \cos\theta\, e_1 + \sin\theta\, \xi'$,
where $\quad 0 \le \theta \le \pi, \quad \xi' \in S^{d-2} := \{\xi \in S^{d-1} \mid \langle \xi, e_1\rangle = 0\}$.
Then $\quad \mathcal{H}_n = \bigoplus_{j=0}^n \mathcal{H}_{n,j}$, $\quad$ the unique decomposition of $\mathcal{H}_n$ into irreducible subspaces with respect to $O(d-1)$.
$\mathcal{H}_{n,j}$ consists of the functions

$$\cos\theta\, e_1 + \sin\theta\, \xi' \mapsto C_{n-j}^{\frac12 d-1+j}(\cos\theta)\,(\sin\theta)^j\, g(\xi') \quad (g \in \mathcal{H}_j(S^{d-2})).$$

For each j take an orthonormal basis $\{f_{j,k}\}$ of $\mathcal{H}_{n,j}$. Then

$$\frac{C_n^{\frac12 d-1}(\langle \xi, \eta\rangle)}{C_n^{\frac12 d-1}(1)} = \frac{1}{N_n} \sum_{j=0}^n \sum_{k=1}^{\dim \mathcal{H}_{n,j}} f_{j,k}(\xi)\, \overline{f_{j,k}(\eta)}.$$

Third stage of the addition formula
Take $\xi = \cos\theta_1\, e_1 + \sin\theta_1\, \xi'$, $\eta = \cos\theta_2\, e_1 + \sin\theta_2\, \eta'$, and apply the first stage of the addition formula, with d replaced by $d-1$, to the inner sum in the second stage of the addition formula. Finally put $\langle \xi', \eta'\rangle = \cos\phi$. Then we get for certain constants $c_{n,j}$ (which can be computed explicitly) the *addition formula for Gegenbauer polynomials*:

$$\begin{aligned} &C_n^{\frac12 d-1}(\cos\theta_1\, \cos\theta_2 + \sin\theta_1\, \sin\theta_2\, \cos\phi) \\ &= \sum_{j=0}^n c_{n,j}\, C_{n-j}^{\frac12 d-1+j}(\cos\theta_1)\,(\sin\theta_1)^j\, C_{n-j}^{\frac12 d-1+j}(\cos\theta_2)\,(\sin\theta_2)^j\, C_j^{\frac12 d-\frac32}(\cos\phi). \end{aligned}$$

This extends analytically to real $d > 2$.
For $d = 3$ (replace $C_j^0(\cos\phi)$ by $\cos(j\phi)$) we get the addition formula for Legendre polynomials.

2. Relationship to representation theory

Let G be a compact group, K a compact subgroup, π an irreducible (finite dimensional) unitary representation of G on a complex vector space V (with hermitian inner product), and assume that *the dimension of the space of K-invariants in V is one*.
Let $e_1, \ldots, e_d$ be an orthonormal basis of V with e_1 K-invariant. Let $x, y \in G$.
$\pi_{ij}(x) := \langle \pi(x) e_j, e_i\rangle \quad$ (*matrix element of* π).
$\phi_\pi(x) := \pi_{11}(x) = \langle \pi(x) e_1, e_1\rangle \quad$ *spherical function*, is K-biinvariant).
$\phi_\pi(y^{-1}x) = \sum_{j=1}^d \pi_{j1}(x)\, \overline{\pi_{j1}(y)} \qquad$ (*first stage of addition formula for* ϕ_π).
$\phi_\pi(x)\, \phi_\pi(y) = \int_K \phi_\pi(xky)\, dk \quad$ (*product formula for* ϕ_π).
Suppose that there is a compact subgroup M of K and a subset A_+ of G such that:

(a) Each irreducible representation of K occurring in π contains the trivial representation of M precisely once.
(b) $am = ma$ ($a \in A_+$, $m \in M$) and $G = KA_+K$.

Then there is an orthonormal basis of V given by
$f_1,\ f_{21}, \ldots, f_{2d_2}, \ldots\ldots, f_{j1}, \ldots, f_{jd_j}, \ldots\ldots, f_{e1}, \ldots, f_{ed_e}$,
with f_1 K-invariant, and where for each j the vectors $f_{j1}, \ldots, f_{jd_j}$ span a subspace of V which is invariant and irreducible with respect to K, and where f_{j1} is M-invariant for each j.

Then $\psi_j(k) := \pi_{j1;j1}(k)$ $(k \in K)$ is a spherical function for K with respect to M and $\pi_{jr;1}(a) = 0$ $(a \in A_+)$ if $r \neq 1$.

Then we get from the first stage of the addition formula the second stage and next the *third stage of the addition formula for* ϕ_π:

$$\phi_\pi(a_2^{-1}ka_1) = \sum_{j=1}^{e} \pi_{j1,1}(a_1)\,\overline{\pi_{j1;1}(a_2)}\,\psi_j(k) \quad (k \in K,\ a_1, a_2 \in A_+).$$

3. Gelfand pairs and symmetric pairs

Let G be a locally compact group and K a compact subgroup.

(G,K) is called a *Gelfand pair* if each irreducible (possibly infinite dimensional) unitary representation of G contains the trivial represenation of K at most once.

If π is an irreducible unitary representation of G on a Hilbert space $\mathcal{H}(\pi)$ and if $e_\pi \in \mathcal{H}(\pi)$ is a K-fixed unit vector then

$$\phi_\pi(x) := \langle \pi(x)e_\pi, e_\pi \rangle \quad (x \in G)$$

is called a *spherical function* on G with respect to K and the spherical functions are precisely the nonzero continuous solutions ϕ of

$$\phi(x)\,\phi(y) = \int_K \phi(xky)\,dk \quad (x, y \in G)$$

which are moreover positive definite functions on G.

Definition A *symmetric space* is a Riemannian manifold X such that for all $p \in X$ there is an isometry of X which leaves p fixed and which reverts the geodesics through p.

Choose $\xi_0 \in X$. Then $X = G/K$ with G the connected component of the group of isometries of X and K the stabilizer of ξ_0 in G. Then G and K are Lie groups, the pair (G,K) is called a *symmetric pair* and this pair is in particular a Gelfand pair.

There is a connected abelian Lie subgroup A of G such that

$$G = KAK \quad (\textit{Cartan decomposition}).$$

$\dim(A)$ is called the *rank* of (G,K). The abelian group A parametrizes the K-orbits in G/K (up to the action of a finite subgroup: the Weyl group or, in the compact case, an extension of the Weyl group).

There are three types of irreducible symmetric spaces:

(a) of non-compact type (e.g. $SL(2,\mathbb{R})/SO(2)$, $SO_0(d,1)/S(O(d) \times O(1))$);

(b) of compact type (e.g. $SU(2)/SO(2)$, $SO(d+1)/SO(d)$);

(c) of Euclidean type (the spaces $(SO(d) \circ \mathbb{R}^d)/SO(d)$).

In cases (a) and (b) the Lie-group G is semisimple. Moreover there is a duality between cases (a) and (b), where the dual pairs have the form G/K and U/K with the same K and with G and U non-compact respectively compact forms of the same complex semisimple Lie group.

The compact symmetric spaces of rank 1 (equivalently the compact 2-point homogeneous metric spaces) have Jacobi polynomials as spherical functions:

$$\xi \mapsto \frac{P_n^{(\alpha,\beta)}(\cos(d(\xi,\xi_0))}{P_n^{(\alpha,\beta)}(1)}\,.$$

Here $\xi \in X$ and d is the Riemannian distance on X, where d is normalized such that the maximal distance between two points on X equals π.

For these spaces we have $G = KAK$ with $A \simeq \mathbb{T}$ (the circle group).

Let $M := \{k \in K \mid \forall a \in A \quad ka = ak\}$.

Then, for all these rank one spaces, (K, M) is a Gelfand pair (but generally not a symmetric pair).

The following table gives the classification and relevant data for the compact rank one symmetric spaces.

G	K	M	α	β
$SO(d)$	$SO(d-1)$	$SO(d-2)$	$\frac{1}{2}d - \frac{3}{2}$	$\frac{1}{2}d - \frac{3}{2}$
$SO(d)$	$S(O(d-1) \times O(1)) \simeq O(d-1)$	$O(d-2)$	$\frac{1}{2}d - \frac{3}{2}$	$-\frac{1}{2}$
$SU(d)$	$S(U(d-1) \times U(1)) \simeq U(d-1)$	$U(d-2)$	$d-2$	0
$Sp(d)$	$Sp(d-1) \times Sp(1)$	$(Sp(d-1) \times Sp(1))^*$	$2d-3$	1
F_4	$Spin(9)$	$Spin(7)$	7	3

4. Addition formula on the unit sphere in a complex vector space

In view of the previous results it seems now an easy task to work out explicitly the addition formula for Jacobi polynomials for (α, β) as listed in the table above. In particular we can try the case of the *complex projective space* $SU(d)/U(d-1)$. However, remark that functions on $SU(d)/U(d-1)$ can be identified with $U(1)$-invariant functions on $U(d)/U(d-1)$, which is the sphere $S^{2d-1)}$ in $\mathbb{C}^d$, and which is a Gelfand pair, also occurring as K/M for $G = SU(d+1)$. Thus it looks both easier (in view of analogy with the case of classical spherical harmonics) and more general to treat the case of spherical harmonics on the sphere $U(d)/U(d-1)$.

S^{2d-1} unit sphere in $\mathbb{C}^d$.

Unitary group $U(d)$ acts on $\mathbb{C}^d$ and acts transitively on S^{2d-1}.

Subgroup $U(d-1) := \{T \in U(d) \mid Te_1 = e_1\}$ $\quad(e_1, \ldots, e_d$ standard basis of $\mathbb{C}^d)$.

Definition

$\mathcal{H}_{m,n} :=$ {homogeneous polynomials of degree m in $z_1, \ldots, z_d$, of degree n in $\overline{z_1}, \ldots, \overline{z_d}$, annihilated by $\frac{\partial^2}{\partial z_1\, \partial \overline{z_1}} + \cdots + \frac{\partial^2}{\partial z_d\, \partial \overline{z_d}}$, restricted to S^{2d-1}}.

Theorem $\quad L^2(S^{2d-1}) = \bigoplus_{m,n=0}^{\infty} \mathcal{H}_{m,n}, \quad$ orthogonal direct sum;
is the unique orthogonal decomposition of $L^2(S^{2d-1})$ into irreducible subspaces with respect to $U(d)$.

The $U(d-1)$-invariants in $\mathcal{H}_{m,n}$ form 1-dimensional subspace spanned by *disk polynomials* $\xi \mapsto R^{d-2}_{m,n}(\langle \xi, e_1\rangle)$, which are expressed in terms of Jacobi polynomials as follows.

$$R^{\alpha}_{m,n}(re^{i\theta}) := \frac{P^{(\alpha,|m-n|)}_{\min(m,n)}(2r^2-1)}{P^{(\alpha,|m-n|)}_{\min(m,n)}(1)}\, r^{|m-n|}\, e^{i(m-n)\theta}.$$

These are orthogonal on the unit disk with respect to the measure $(1-x^2-y^2)^{\alpha}\, dx\, dy$.

Addition formula for disk polynomials (R. L. Sapiro, 1968)

$$\begin{aligned} R^{\alpha}_{m,n}(\cos\theta_1\, e^{i\phi_1}\, \cos\theta_2\, e^{i\phi_2} + \sin\theta_1\, \sin\theta_2\, r\, e^{i\psi}) &= \sum_{k=0}^{m}\sum_{l=0}^{n} c^{\alpha}_{m,n,k,l} \\ \times (\sin\theta_1)^{k+l}\, R^{\alpha+k+l}_{m-k,n-l}(\cos\theta_1\, e^{i\phi_1})\, (\sin\theta_2)^{k+l}\, & R^{\alpha+k+l}_{m-k,n-l}(\cos\theta_2\, e^{i\phi_2})\, R^{\alpha-1}_{k,l}(r\, e^{i\psi}). \end{aligned}$$

The above result was independently obtained by the author in 1972. By easy manipulations (differentiation and analytic continuation) this formula yields:

Addition formula for Jacobi polynomials (Koornwinder (1972))

$$\begin{aligned} &P^{(\alpha,\beta)}_{n}(2\cos^2\theta_1\, \cos^2\theta_2 + 2\sin^2\theta_1\, \sin^2\theta_2\, r^2 + \sin 2\theta_1\, \sin 2\theta_2\, r\, \cos\phi - 1) \\ &\quad = \sum_{k=0}^{n}\sum_{l=0}^{k} c^{(\alpha,\beta)}_{n,k,l}\, (\sin\theta_1)^{k+l}\, (\cos\theta_1)^{k-l}\, P^{(\alpha+k+l,\beta+k-l)}_{n-k}(\cos 2\theta_1) \\ &\qquad\qquad \times (\sin\theta_2)^{k+l}\, (\cos\theta_2)^{k-l}\, P^{(\alpha+k+l,\beta+k-l)}_{n-k}(\cos 2\theta_2) \\ &\qquad\qquad \times P^{(\alpha-\beta-1,\beta+k-l)}_{l}(2r^2-1)\, r^{k-l}\, P^{(\beta-\frac12,\beta-\frac12)}_{k-l}(\cos\psi) \end{aligned}.$$

5. Positivity results

Let $\alpha \ge \beta \ge -\frac12$. Then

$$\frac{P^{(\alpha,\beta)}_{n}(x)}{P^{(\alpha,\beta)}_{n}(1)}\, \frac{P^{(\alpha,\beta)}_{n}(y)}{P^{(\alpha,\beta)}_{n}(1)} = \int_{-1}^{1} \frac{P^{(\alpha,\beta)}_{n}(z)}{P^{(\alpha,\beta)}_{n}(1)}\, d\mu_{x,y}(z) \quad (x,y \in [-1,1])$$

with positive measure $\mu_{x,y}$. Furthermore,

$$P^{(\alpha,\beta)}_{m}(x)\, P^{(\alpha,\beta)}_{n}(x) = \sum_{l=|m-n|}^{m+n} a_{m,n}(l)\, P^{(\alpha,\beta)}_{l}(x) \quad \text{with } a_{m,n}(l) \ge 0.$$

Both these results of Gasper are implied by the addition formula for Jacobi polynomials (the first already by the product formula).

Morally, an addition formula for an orthogonal system of special functions encodes all information for nice harmonic analysis in terms of this system (similar to harmonic analysis for K-biinvariant functions on G in case of a Gelfand pair (G,K)).

6. Spherical functions on non-compact symmetric spaces of rank one

Let $\alpha \geq \beta \geq -\frac{1}{2}$ and let $\rho := \alpha + \beta + 1$. *Jacobi functions* are defined by

$$\begin{aligned}\phi_\lambda^{(\alpha,\beta)}(t) :=& {}_2F_1\left[\begin{matrix}\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda) \\ \alpha + 1\end{matrix}; -\sinh^2 t\right] \\ =& c_{\alpha,\beta}(\lambda)\, \mathcal{O}(e^{(i\lambda-\rho)t}) + c_{\alpha,\beta}(-\lambda)\, \mathcal{O}(e^{(-i\lambda-\rho)t}) \quad \text{as } t \to \infty.\end{aligned}$$

Then we have a generalized Fourier transform pair

$$\begin{cases} \widehat{f}(\lambda) = \displaystyle\int_0^\infty f(t)\, \phi_\lambda^{(\alpha,\beta)}(t)\, (\sinh t)^{2\alpha+1}\, (\cosh t)^{2\beta+1}\, dt, \\ f(t) = \text{const.} \displaystyle\int_0^\infty \widehat{f}(\lambda)\, \phi_\lambda^{(\alpha,\beta)}(t)\, \frac{d\lambda}{|c_{\alpha,\beta}(\lambda)|^2}\,. \end{cases}$$

The Jacobi function transform $f \mapsto \widehat{f}$ can be factorized as $\widehat{f} = (\mathcal{F} \circ \mathcal{A}_{\alpha,\beta})(f)$, where $\mathcal{A}_{\alpha,\beta}$ is the *generalized Abel transform* and $\mathcal{F}$ is the classical Fourier-cosine transform. The generalized Abel transform can be written as the composition of two Weyl-type fractional integral transforms.

All these results can be obtained in a meaningful group theoretical context on a rank one symmetric space of the non-compact type if (α, β) is as listed above.

See for various aspects of and approaches to the above results H. Weyl, Mehler-Fok, Harish-Chandra, Helgason, Flensted-Jensen, Koornwinder.

7. Spherical functions on finite groups and on p-adic groups

There are significant examples of Gelfand pairs (G, K) with G a finite group. For instance, G is the symmetric group or it is defined in terms of symmetric groups or G is a classical group over a finite field (a Chevalley group). For such Gelfand pairs Krawtchouk and Hahn polynomials and their q-analogues can be obtained as spherical functions, and addition formulas can be derived for them. See D. Stanton and Dunkl.

One can also obtain Gelfand pairs with G a semisimple group over the field of p-adic numbers. For $G = SL(2)$ the space G/K can be simply described as a homogeneous tree. In that case the spherical functions are functions $n \mapsto p_n(x)$, where p_n is a special Bernstein-Szegö polynomial, namely the $q = 0$ limit of a q-ultraspherical polynomial. The degree n of $p_n(x)$ has an interpretation as the graph distance between a fixed point and an arbitrary point on the homogeneous tree.

8. From K-biinvariant analysis on G to W-invariant analysis on $\mathbb{R}^l$

Let (G, K) be a symmetric pair of the noncompact type. Then G is a noncompact semisimple Lie group, K is a maximal compact subgroup, and $G = KAK$ with $A \simeq \mathbb{R}^l$ an abelian subgroup.
Let M' be the normalizer of A in K and M the centralizer of A in K, i.e.,

$$M' := \{k \in K \mid kAk^{-1} = A\}, \quad M := \{k \in K \mid \forall a \in A \quad kak^{-1} = a\}.$$

Then M is a normal subgroup of M' and the (finite) quotient group $W := M'/M$ is called the *Weyl group* associated with the root system R for (G, K). The group W acts on A.

Example $G = SL(3, \mathbb{R})$, $K = SO(3)$, $A = \{\mathrm{diag}(e^{t_1}, e^{t_2}, e^{t_3}) \mid t_1 + t_2 + t_3 = 0\} \simeq \mathbb{R}^2$. Then R is a root system of type A_2, this is a set of 6 vectors in $\mathbb{R}^2$ (the root vectors) of equal length, with neighbours making angles $\pi/3$ with each other. The Weyl group W is isomorphic to the symmetric group S_3. A fundamental domain for the action of W on $\mathbb{R}^2$ is the so-called positive Weyl chamber, a region bounded by two half lines from the origin, making an angle $\pi/3$ with each other. The two half lines (the walls of the Weyl chamber) are each perpendicular to some root vector. The Weyl group is generated by the reflections in the walls of the positive Weyl chamber.

The mapping $F \mapsto f = F|_A$ identifies the K-biinvariant functions F on G with the W-invariant functions on A. Thus the harmonic analysis for K-biinvariant functions on G can be reformulated as harmonic analysis for W-invariant functions on A. One of the data about the pair (G, K) which is thus transfered to the pair (A, W) is a *multiplicity function*, i.e., a W-invariant function on the root system R assuming nonnegative integer values. In particular, the Casimir operator on G (equivalently the Laplace-Beltrami operator on G/K) and the Haar measure on G give rise to a second order differential operator respectively a positive measure on the positive Weyl chamber (both explicit) which depend on this multiplicity function. The multiplicity functions induced from symmetric pairs are quite special.

One can try to do the harmonic analysis for W-invariant functions on A more generally with respect to an arbitrary multiplicity function, W-invariant and assuming nonnegative real values. The same explicit expressions for the second order differential operator and for the measure can then be used, but with the more general multiplicity function.

For G compact, similar things can be said, but now $A \simeq T^l$ is compact, a torus. Also, instead of a positive Weyl chamber we have a so-called Weyl alcove within A on which our functions will live. For instance, in our example above, if G is replaced by $SU(3)$ then A is replaced by $\mathbb{T}^2$ and the Weyl alcove becomes a region within $\mathbb{T}^2$ which is an equilateral triangle.

9. Special functions associated with root systems: two charts

On the next two pages I give the historical development of special functions associated with root systems in two charts: first for the case $q = 1$ and next for the q-case. Of course, not all names, influences and interrelations can be squeezed in a one-page chart. However, I welcome suggestions about possible omissions. Eventually, I intend to add references to the names and results in these charts.

Concerning the Macdonald polynomials I want to observe that Macdonald, in 1987 introduced the A_n Macdonald polynomials in a quite different way as the Macdonald polynomials for general root systems (including A_n). The A_n polynomials were defined in a combinatorial-algebraic way, in the framework of symmetric functions of infinitely many variables, while the definition of the Macdonald polynomials for general root systems has a more analytic flavour. (For A_n he proved the equivalence of both definitions.) The approach to the A_n case has led to very deep results, for which the analogues in other root systems remain open, because the analogue of the combinatorial-algebraic definition is missing.

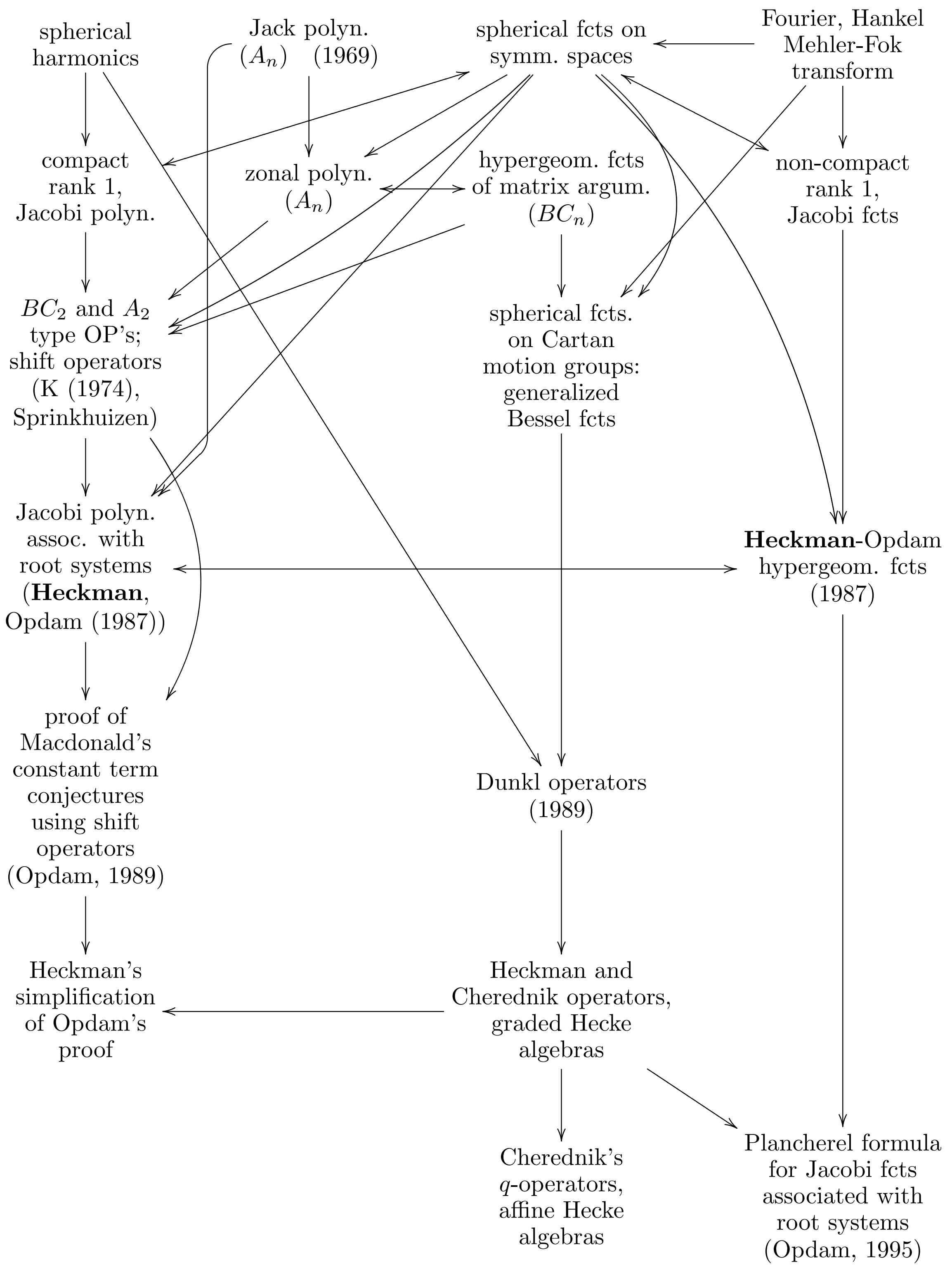
Before giving the charts, I mention some developments not covered by these charts:

Other developments for $q = 1$

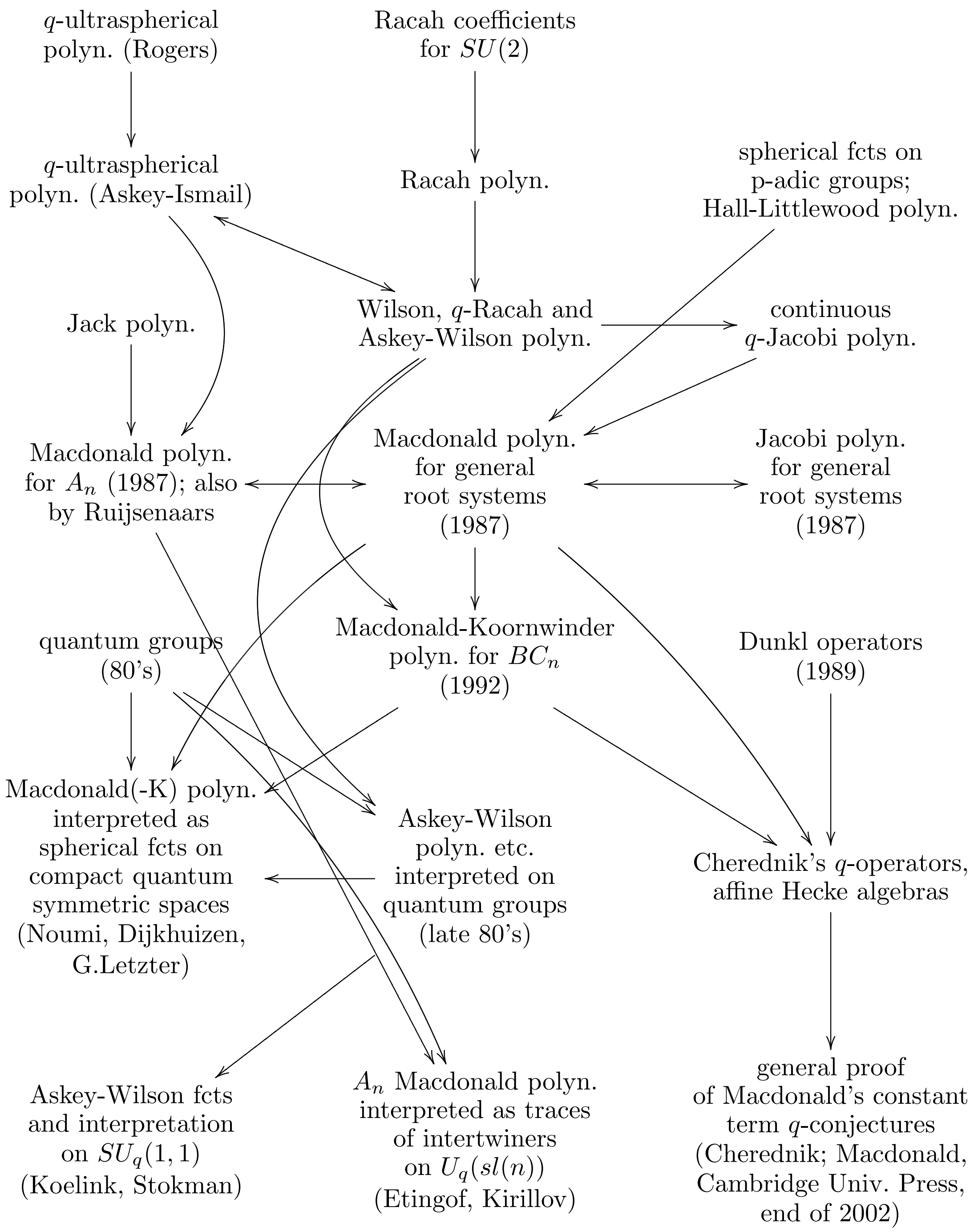
1. Gelfand's hypergeometric functions;
2. hypergeometric series associated with root systems (Biedenharn, Louck, Milne, Gustafson);
3. matrix-valued spherical functions (recent development by Grünbaum, Pacharoni, Tirao);
4. solutions of Painlevé equations (see Clarkson's lecture).

Recent developments in the q-case

1. quantum dynamical Yang-Baxter equation, exchange construction, dynamical quantum groups and related interpretations of q-special functions (Etingof & Varchenko, Etingof & Schiffmann, Koelink & Rosengren, Koornwinder & Touhami);
2. elliptic quantum groups and elliptic hypergeometric functions (Frenkel & Turaev, Etingof & Varchenko & Schiffmann, Spiridonov), see Spiridonov's lecture;
3. spherical functions on non-compact quantum groups and non-terminating analogues of Macdonald polynomials (Stokman).



spherical harmonics
Jack polyn. (A_n) (1969)
spherical fcts on symm. spaces
Fourier, Hankel Mehler-Fok transform
compact rank 1, Jacobi polyn.
zonal polyn. (A_n)
hypergeom. fcts of matrix argum. (BC_n)
non-compact rank 1, Jacobi fcts
BC_2 and A_2 type OP's; shift operators (K (1974), Sprinkhuizen)
spherical fcts. on Cartan motion groups: generalized Bessel fcts
Jacobi polyn. assoc. with root systems (**Heckman**, Opdam (1987))
Heckman-Opdam hypergeom. fcts (1987)
proof of Macdonald's constant term conjectures using shift operators (Opdam, 1989)
Dunkl operators (1989)
Heckman's simplification of Opdam's proof
Heckman and Cherednik operators, graded Hecke algebras
Cherednik's q-operators, affine Hecke algebras
Plancherel formula for Jacobi fcts associated with root systems (Opdam, 1995)



q-ultraspherical polyn. (Rogers)
Racah coefficients for $SU(2)$
q-ultraspherical polyn. (Askey-Ismail)
Racah polyn.
spherical fcts on p-adic groups; Hall-Littlewood polyn.
Jack polyn.
Wilson, q-Racah and Askey-Wilson polyn.
continuous q-Jacobi polyn.
Macdonald polyn. for A_n (1987); also by Ruijsenaars
Macdonald polyn. for general root systems (1987)
Jacobi polyn. for general root systems (1987)
quantum groups (80's)
Macdonald-Koornwinder polyn. for BC_n (1992)
Dunkl operators (1989)
Macdonald(-K) polyn. interpreted as spherical fcts on compact quantum symmetric spaces (Noumi, Dijkhuizen, G.Letzter)
Askey-Wilson polyn. etc. interpreted on quantum groups (late 80's)
Cherednik's q-operators, affine Hecke algebras
Askey-Wilson fcts and interpretation on $SU_q(1,1)$ (Koelink, Stokman)
A_n Macdonald polyn. interpreted as traces of intertwiners on $U_q(sl(n))$ (Etingof, Kirillov)
general proof of Macdonald's constant term q-conjectures (Cherednik; Macdonald, Cambridge Univ. Press, end of 2002)