Some remarks about Koornwinder polynomials

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Lecture at the conference *Representation Theory, Special Functions and Painlevé Equations*, RIMS, Kyoto, Japan, March 3–6, 2015,

on the occasion of the 60th birthday of Masatoshi Noumi

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Mimachi, Noumi and me in Columbus, Ohio, June 1989
(the birth of Askey-Wilson interpreted on quantum $SU(2)$)

Opdam, Mimachi, Noumi and me in Ann Arbor, Michigan, June 1989

Thesis defense by Jasper Stokman, Univ. of Amsterdam, June 1998

Thesis defense by Jasper Stokman, the committee in detail.
From left to right on front row: van der Geer (standing), Dijkgraaf, Noumi, Koornwinder, Hoogland;
on back row: Keane, Korevaar, van Emde Boas, Klaassen, Does.

Workshop in Amsterdam for my 60th birthday, August 2003; walking to the boat for the roundtrip through the canals. What is on the photo Noumi is taking?

Reference

See details of this talk in my preprint

T. H. Koornwinder,

Okounkov's BC-type interpolation Macdonald polynomials and their $q = 1$ *limit*,

http://arxiv.org/abs/1408.5993

Prelude: one variable, $q = 1$

monomial	$P_m(x)$	x^m
factorial	$P_m^{\mathrm{ip}}(x)$	$(-1)^m(-x)_m$
quadr. factorial	$P_m^{\mathrm{ip}}(x;\alpha)$	$(-1)^m(\alpha-x)_m(\alpha+x)_m$
Jacobi	$P_m(x;\alpha,\beta)$	$\frac{(-1)^m(\alpha+1)_m}{(m+\alpha+\beta+1)_m}\,{}_2F_1\left({-m,m+\alpha+\beta+1 \atop \alpha+1};x\right)$

$$P_m^{\mathrm{ip}}(x) = x(x-1)\ldots(x-m+1),$$
$$P_m^{\mathrm{ip}}(x;\alpha) = \left(x^2-\alpha^2\right)\left(x^2-(\alpha+1)^2\right)\left(x^2-(\alpha+m-1)^2\right),$$

(even) monic polynomials determined by vanishing properties.

Limits:

$$\left.\begin{matrix} P_m(x;\alpha,\beta) \\ P_m^{\mathrm{ip}}(x) \end{matrix}\right\} = P_m(x) + \text{lower degree},$$
$$P_m^{\mathrm{ip}}(x;\alpha) \qquad = P_m(x^2) + \text{lower degree};$$

$$(2\alpha)^{-m}P_m^{\mathrm{ip}}(x+\alpha,\alpha) \stackrel{\alpha\to\infty}{\longrightarrow} P_m^{\mathrm{ip}}(x),$$
$$P_m(x;\alpha,\beta) \stackrel{\alpha\to\infty}{\longrightarrow} P_m(x-1).$$

Prelude: one variable, $q = 1$, binomial formulas

monomial	$P_m(x)$	x^m
factorial	$P_m^{\mathrm{ip}}(x)$	$(-1)^m(-x)_m$
quadr. factorial	$P_m^{\mathrm{ip}}(x;\alpha)$	$(-1)^m(\alpha-x)_m(\alpha+x)_m$
Jacobi	$P_m(x;\alpha,\beta)$	$\frac{(-1)^m(\alpha+1)_m}{(m+\alpha+\beta+1)_m}\,{}_2F_1\left(\begin{matrix}-m,m+\alpha+\beta+1\\ \alpha+1\end{matrix};x\right)$

$$\frac{P_m(x;\alpha,\beta)}{P_m(0;\alpha,\beta)} = \sum_{k=0}^{m} \frac{(-m)_k(m+\alpha+\beta+1)_k}{(\alpha+1)_k k!}\,x^k$$

$$\overset{\alpha':=\frac{1}{2}(\alpha+\beta+1)}{=} \sum_{k=0}^{m} \frac{P_k^{\mathrm{ip}}(m+\alpha';\alpha')}{P_k^{\mathrm{ip}}(k+\alpha';\alpha')}\,\frac{P_k(x)}{P_k(0;\alpha)} \qquad \overset{\alpha\to\infty}{\longrightarrow}$$

$$(1-x)^m = \sum_{k=0}^{m} \binom{m}{k}(-x)^k, \quad \text{i.e.,}$$

$$\frac{P_m(1-x)}{P_m(1)} = \sum_{k=0}^{m} \frac{P_k^{\mathrm{ip}}(m)}{P_k^{\mathrm{ip}}(k)}\,\frac{P_k(-x)}{P_k(1)}\,.$$

Prelude: one variable, q-case

Throughout $0 < q < 1$. Four classes of polynomials:

monomial	$P_m(x;q)$	x^m
q-factorial	$P_m^{\mathrm{ip}}(x;q)$	$x^m(x^{-1};q)_m$
quadr. q-factorial	$P_m^{\mathrm{ip}}(x;q,a)$	$\dfrac{(ax,ax^{-1};q)_m}{(-1)^m q^{\frac{1}{2}m(m-1)}a^m}$
Askey-Wilson	$\dfrac{P_m(x;q;a_1,a_2,a_3,a_4)}{P_m(a_1;q;a_1,a_2,a_3,a_4)}$	${}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}a_1a_2a_3a_4,a_1x,a_1x^{-1} \\ a_1a_2,a_1a_3,a_1a_4 \end{matrix};q,q\right)$

Put $\langle x;y\rangle := x + x^{-1} - y - y^{-1}$.

$$P_m^{\mathrm{ip}}(x;q) = (x-1)(x-q)\ldots(x-q^{m-1}),$$

$$P_m^{\mathrm{ip}}(x;q,a) = \prod_{j=0}^{m-1}(x + x^{-1} - a\,q^j - a^{-1}q^{-j}) = \prod_{j=0}^{m-1}\langle x; a\,q^j\rangle,$$

monic polynomials resp. monic symmetric Laurent polynomials determined by their vanishing properties.

Prelude: one variable, q-case, limits

monomial	$P_m(x;q)$	x^m
q-factorial	$P_m^{\rm ip}(x;q)$	$x^m(x^{-1};q)_m$
quadr. q-factorial	$P_m^{\rm ip}(x;q,a)$	$\dfrac{(ax,ax^{-1};q)_m}{(-1)^m q^{\frac{1}{2}m(m-1)}a^m}$
Askey-Wilson	$\dfrac{P_m(x;q;a_1,a_2,a_3,a_4)}{P_m(a_1;q;a_1,a_2,a_3,a_4)}$	${}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}a_1a_2a_3a_4,a_1x,a_1x^{-1} \\ a_1a_2,a_1a_3,a_1a_4\end{matrix};q,q\right)$

$$\left.\begin{matrix} P_m(x;q;a_1,a_2,a_3,a_4) \\ P_m^{\rm ip}(x;q,a) \\ P_m^{\rm ip}(x;q) \end{matrix}\right\} = P_m(x;q) + \text{lower degree};$$

$$a^{-m}P_m^{\rm ip}(ax;q,a) \stackrel{a\to\infty}{\longrightarrow} P_m^{\rm ip}(x;q),$$

$$a_1^{-m}P_m(a_1x;q;a_1,a_2,a_3,a_4) \stackrel{a_1\to\infty}{\longrightarrow} P_m(x;q),$$

$$P_m(x;q;a_1,q^{1-m}a_1^{-1},a_3,a_4) = P_m^{\rm ip}(x;q,a_1).$$

Prelude: one variable, q-case, binomial formulas

$$a^{-m} P_m^{\rm ip}(ax;q,a) \stackrel{a\to\infty}{\longrightarrow} P_m^{\rm ip}(x;q),$$

$$a_1^{-m}\, P_m(a_1x;q,t;a_1,a_2,a_3,a_4) \stackrel{a_1\to\infty}{\longrightarrow} P_m(x;q).$$

$$\frac{P_m(x;q;a_1,a_2,a_3,a_4)}{P_m(a_1;q;a_1,a_2,a_3,a_4)} = \sum_{k=0}^m \frac{q^k}{(a_1a_2,a_1a_3,a_1a_4,q;q)_k} \times (q^{-m},q^{m-1}a_1a_2a_3a_4;q)_k\,(a_1x,a_1x^{-1};q)_k$$

$$\stackrel{a_1':=(q^{-1}a_1a_2a_3a_4)^{\frac12}}{=} \sum_{k=0}^m \frac{P_k^{\rm ip}(q^m a_1';q,a_1')}{P_k^{\rm ip}(q^k a_1';q,a_1')}\,\frac{P_k^{\rm ip}(x;q,a_1)}{P_k(a_1;q;a_1,a_2,a_3,a_4)}$$

$$\stackrel{x:=a_1x,\ a_1\to\infty}{\longrightarrow}$$

$$x^m = {}_2\phi_0\left(\begin{matrix} q^{-m},x^{-1} \\ - \end{matrix};q,q^m x\right) = \sum_{k=0}^m \frac{(q^{-m},x^{-1};q)_k}{(-1)^k q^{\frac12 k(k-1)}(q;q)_k}\,(q^m x)^k,$$

i.e., $$\frac{P_m(x;q)}{P_m(1;q)} = \sum_{k=0}^m \frac{P_k^{\rm ip}(q^m;q)}{P_k^{\rm ip}(q^k;q)}\,\frac{P_k^{\rm ip}(x;q)}{P_k(1;q)}\,.$$

Prelude: one variable, limits for $q \uparrow 1$

monomial	$P_m(x;q)$	x^m
q-factorial	$P_m^{\mathrm{ip}}(x;q)$	$x^m(x^{-1};q)_m$
quadr. q-factorial	$P_m^{\mathrm{ip}}(x;q,a)$	$\dfrac{(ax,ax^{-1};q)_m}{(-1)^m q^{\frac{1}{2}m(m-1)} a^m}$
Askey-Wilson	$\dfrac{P_m(x;q;a_1,a_2,a_3,a_4)}{P_m(a_1;q;a_1,a_2,a_3,a_4)}$	${}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}a_1a_2a_3a_4,a_1x,a_1x^{-1} \\ a_1a_2,a_1a_3,a_1a_4 \end{matrix};q,q\right)$

$$P_m(x;q) \to P_m(x),$$

$$(q-1)^{-m} P_m^{\mathrm{ip}}(q^x;q) \to P_m^{\mathrm{ip}}(x),$$

$$P_m^{\mathrm{ip}}(x;q) \to P_m(x-1),$$

$$(1-q)^{-2m} P_m^{\mathrm{ip}}(q^x;q,q^\alpha) \to P_m^{\mathrm{ip}}(x;\alpha),$$

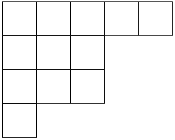
$$P_m^{\mathrm{ip}}(x;q,q^\alpha) \to (-4)^m P_m(\tfrac{1}{4}(2-x-x^{-1})),$$

$$P_m(x;q;q^{\alpha+1},-q^{\beta+1},1,-1) \to (-4)^m P_m(\tfrac{1}{4}(2-x-x^{-1});\alpha,\beta).$$

Partitions

Fixed n. $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0)$ is a *partition*. Set of all such λ is denoted by Λ_n. λ has *length* $\ell(\lambda) := |\{j \mid \lambda_j > 0\}|$ and *weight* $|\lambda| := \lambda_1 + \cdots + \lambda_n$.

Example $\lambda = (5, 3, 3, 1)$ has *Young diagram*



dominance partial ordering:

$\mu \leq \lambda$ iff $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ $(i = 1, \ldots, n)$.

inclusion partial ordering:

$\mu \subset \lambda$ iff $\mu_i \leq \lambda_i$ $(i = 1, \ldots, n)$.

$W_n := S_n \ltimes (\mathbb{Z}_2)^n$.

$x = (x_1, \ldots, x_n)$, $x^\mu := x_1^{\mu_1} \ldots x_n^{\mu_n}$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ partition.

$m_\lambda(x) := \sum_{\mu \in S_n \lambda} x^\mu$, $\widetilde{m}_\lambda(x) := \sum_{\mu \in W_n \lambda} x^\mu$.

polynomials of A and BC type

Throughout $n \geq 2, \quad 0 < q < 1, \quad 0 < t < 1, \quad \tau > 0.$

1	A	Macdonald	$P_\lambda(x;q,t)$
2	A	interpolation Macdonald	$P_\lambda^{\rm ip}(x;q,t)$
3	BC	BC_n interpolation Macdonald	$P_\lambda^{\rm ip}(x;q,t,a)$
4	BC	Koornwinder	$P_\lambda(x;q,t;a_1,a_2,a_3,a_4)$
5	A	Jack	$P_\lambda(x;\tau)$
6	A	interpolation Jack	$P_\lambda^{\rm ip}(x;\tau)$
7	BC	BC_n interpolation Jack	$P_\lambda^{\rm ip}(x;\tau,a)$ (new)
8	BC	Jacobi	$P_\lambda(x;\tau;\alpha,\beta)$

For 1,2,5,6,8 and for $P_\lambda^{\rm ip}(x^{\frac12};\tau,\alpha)$ in 7: $= m_\lambda + \sum_{\mu<\lambda} u_{\lambda,\mu} m_\mu$.
For 3,4: $= \widetilde{m}_\lambda + \sum_{\mu<\lambda} u_{\lambda,\mu} \widetilde{m}_\mu$.
Definition by orthogonality for 1,4,5,8.

Inventors of interpolation functions

interpolation Jack:
Sahi (1994), Knop & Sahi (1996); Okounkov & Olshanski (1997)

interpolation Macdonald:
Sahi (1996); Knop (1997); Okounkov (1998)

BC_n interpolation Macdonald:
Okounkov (1998), Rains (2005), Noumi (slides of a lecture in 2013)

Pictures

Macdonald

Knop

Sahi

Okounkov

Olshanskii

Orthogonality

$\Delta(z) := \Delta_+(z)\Delta_+(z^{-1})$, $\quad \oint \frac{dz}{z} := \prod_{j=1}^n \oint \frac{dz_j}{z_j}$.

Macdonald and Jack: $\displaystyle\oint P_\lambda(z)\, m_\mu(z^{-1})\, \Delta(z)\, \frac{dz}{z} = 0$ if $\mu < \lambda$.

$\Delta_+(z) = \prod_{1\le i<j\le n} \frac{(z_i z_j^{-1};q)_\infty}{(t z_i z_j^{-1};q)_\infty}$ resp. $\prod_{1\le i<j\le n}(1 - z_i z_j^{-1})^\tau$.
Homogeneous of degree $|\lambda|$.

Koornwinder: $\displaystyle\oint P_\lambda(z)\, \widetilde{m}_\mu(z)\, \Delta(z)\, \frac{dz}{z} = 0$ if $\mu < \lambda$.

$$\Delta_+(z) := \prod_{j=1}^n \frac{(z_j^2;q)_\infty}{(a_1 z_j, a_2 z_j, a_3 z_j, a_4 z_j;q)_\infty} \prod_{1\le i<j\le n} \frac{(z_i z_j, z_i z_j^{-1};q)_\infty}{(t z_i z_j, t z_i z_j^{-1};q)_\infty}.$$

Jacobi: $\int_{[0,1]^n} P_\lambda(x)\, m_\mu(x)\, \Delta(x)\, dx = 0$ if $\mu < \lambda$.

$$\Delta(x) := \prod_{j=1}^n x_j^\alpha (1-x_j)^\beta \prod_{1\le i<j\le n} |x_i - x_j|^{2\tau}.$$

In all cases **full orthogonality** can be proved.

Interpolation

Let $\delta := (n-1, n-2, \ldots, 1, 0)$.

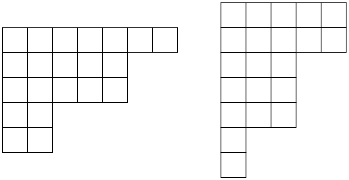
$$\left.\begin{array}{l} P_\lambda^{\mathrm{ip}}(\mu+\tau\delta;\tau) \\ P_\lambda^{\mathrm{ip}}(\mu+\tau\delta+\alpha;\tau,\alpha) \\ P_\lambda^{\mathrm{ip}}(q^\mu t^\delta;q,t) \\ P_\lambda^{\mathrm{ip}}(q^\mu t^\delta a;q,t,a) \end{array}\right\} = 0 \quad \text{if not } \lambda\subset\mu,$$

$$\text{in particular} \quad = 0 \quad \text{if } |\mu|\le|\lambda|,\ \mu\ne\lambda.$$

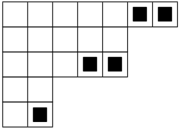
Diagrams

Young diagram of partition λ consists of boxes (i, j) with $i = 1, \ldots, \ell(\lambda)$ and $j = 1, \ldots, \lambda_i$. The *conjugate* partition λ' has transposed diagram. Then $\ell(\lambda') = \lambda_1$.

Example $\lambda = (7, 5, 5, 2, 2)$, $\lambda' = (5, 5, 3, 3, 3, 1, 1)$:



If $\mu \subset \lambda$ then $\lambda - \mu := \{s \in \lambda \mid s \notin \mu\}$ is a *skew diagram*.
$\lambda - \mu$ is a *horizontal strip* if $\lambda'_j - \mu'_j \leq 1$ $(j = 1, \ldots, \lambda_1)$.
Then write $\mu \preccurlyeq \lambda$.

Example $\mu = (5, 5, 3, 2, 1)$, $\lambda - \mu =$ 

Tableaux

The diagram of $\lambda \in \Lambda_n$ becomes a *tableau* T if the boxes (i,j) of λ are filled by numbers $T(i,j)$.
Then T is called a *reverse semistandard* tableau of *shape* λ with entries in $\{1,\ldots,n\}$ if $T(i,j) \in \{1,2,\ldots,n\}$ is weakly decreasing in j and strongly decreasing in i.

Example (reverse semistandard tableau)
$\lambda = (7,5,5,2,2)$, $\ell(\lambda) = 5$, $n = 6$:

6	6	6	4	3	1	1
5	5	5	2	2		
4	4	2	1	1		
3	2					
2	1					

Tableaux (cntd.)

In general, for T reverse semistandard, let:
$\lambda^{(i)} := \{s \in \lambda \mid T(s) > i\}$, this is a partition.
Then: $0^n = \lambda^{(n)} \subset \lambda^{(n-1)} \subset \ldots \subset \lambda^{(1)} \subset \lambda^{(0)} = \lambda$ and
$\lambda^{(i-1)} - \lambda^{(i)} = \{s \in \lambda \mid T(s) = i\}$.
T restricted to the diagram of $\lambda^{(i)}$ is a semistandard tableau of shape $\lambda^{(i)}$ with entries in $\{i+1, \ldots, n\}$.

In the example

6	6	6	4	3	1	1
5	5	5	2	2		
4	4	2	1	1		
3	2					
2	1					

we have

$$() \subset (3) \subset (3,3) \subset (4,3,2) \subset (5,3,2,1) \subset (5,5,3,2,1) \subset (7,5,5,2,2).$$

Combinatorial formula for Macdonald polynomials

$$P_\lambda(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} x_{T(s)}\,,$$ a sum over all reverse semistandard tableaux T of shape λ with entries in $\{1,\ldots,n\}$, so $T(s) = i$ for s in the horizontal strip $\lambda^{(i-1)} - \lambda^{(i)}$.

$$\psi_T(q,t) = \prod_{i=1}^n \psi_{\lambda^{(i-1)}/\lambda^{(i)}}(q,t),$$

$$\psi_{\mu/\nu}(q,t) = \prod_{s\in(R\backslash C)_{\mu/\nu}} \frac{b_\nu(s;q,t)}{b_\mu(s;q,t)}$$ for $\mu-\nu$ a horizontal strip.

Here $s\in(R\backslash C)_{\mu/\nu}$ iff $s\in\nu$ in a row of μ intersecting with $\mu-\nu$ but outside each column of μ intersecting with $\mu-\nu$.

$$b_\lambda(s;q,t) := \frac{1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}}{1-q^{a_\lambda(s)+1}t^{l_\lambda(s)}}\,,$$

$$a_\lambda(i,j) := \lambda_i - j,\quad l_\lambda(i,j) := |\{k>i \mid \lambda_k \ge j\}|.$$

Equivalent formulation as branching formula

$P_{\lambda/\mu}(z;q,t) := \psi_{\lambda/\mu}(q,t)\, z^{|\lambda|-|\mu|}$ if $\lambda \in \Lambda_n$, $\mu \in \Lambda_{n-1}$, $\mu \preccurlyeq \lambda$,
and $:= 0$ otherwise.

Then the *branching formula*

$$P_\lambda(x_1,\ldots,x_{n-1},x_n;q,t)) = \sum_\mu P_{\lambda/\mu}(x_n;q,t)\, P_\mu(x_1,\ldots,x_{n-1};q,t).$$

$\psi_{\lambda/\mu}(q,t) = \psi'_{\lambda'/\mu'}(t,q)$, where

$$e_r(x)\, P_\lambda(x;q,t) = \sum_\mu \psi'_{\mu/\lambda}(q,t) P_\mu(x;q,t) \quad (\textit{Pieri formula}).$$

Combinatorial formulas for interpolation polynomials

Macdonald polynomials:

$$P_\lambda(x;q,t)=\sum_T \psi_T(q,t)\prod_{s\in\lambda} x_{T(s)}\,.$$

Interpolation Macdonald polynomials:

$$P_\lambda^{\rm ip}(x;q,t)=\sum_T \psi_T(q,t)\prod_{s\in\lambda}\big(x_{T(s)}-q^{a'_\lambda(s)}t^{n-T(s)-l'_\lambda(s)}\big)\,.$$

BC_n interpolation Macdonald polynomials:

$$P_\lambda^{\rm ip}(x;q,t,a)=\sum_T \psi_T(q,t)\prod_{s\in\lambda}\langle x_{T(s)};q^{a'_\lambda(s)}t^{n-T(s)-l'_\lambda(s)}a\rangle.$$

$$a'_\lambda(i,j):=j-1,\quad l'_\lambda(i,j):=i-1,\quad \langle x;y\rangle:=x+x^{-1}-y-y^{-1}.$$

The T-sums are over all reverse tableaux T of shape λ with entries in $\{1,\ldots,n\}$.

Combinatorial formulas ($q = 1$)

$$\psi_T(\tau) := \prod_{i=1}^{n} \prod_{s\in(R\backslash C)_{\lambda^{(i-1)}/\lambda^{(i)}}} \frac{b_{\lambda^{(i)}}(s;\tau)}{b_{\lambda^{(i-1)}}(s;\tau)}\,.$$

$$b_\lambda(s;\tau) := \frac{a_\lambda(s)+\tau(l_\lambda(s)+1)}{a_\lambda(s)+\tau l_\lambda(s)+1}\,.$$

Jack polynomials: $\quad P_\lambda(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} x_{T(s)}\,.$

Interpolation Jack polynomials:

$$P_\lambda^{\rm ip}(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x_{T(s)} - a'_\lambda(s) - \tau(n - T(s) - l'_\lambda(s))\right).$$

BC_n interpolation Jack polynomials: (new)

$$P_\lambda^{\rm ip}(x;\tau,\alpha) := \lim_{q\uparrow 1}(1-q)^{-2|\lambda|} P_\lambda^{\rm ip}(q^x;q,q^\tau,q^\alpha) = \sum_T \psi_T(\tau) \times \prod_{s\in\lambda}\left(x_{T(s)}^2 - \left(a'_\lambda(s)+\tau(n-T(s)-l'_\lambda(s))+\alpha\right)^2\right).$$

The T-sums over all reverse tableaux T of shape λ with entries in $\{1,\ldots,n\}$.

Limits, $q \uparrow 1$

$$P_\lambda(x;q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1) \to (-4)^{|\lambda|}\,P_\lambda(\tfrac14(2-x-x^{-1});\tau;\alpha,\beta),$$

$$(1-q)^{-2|\lambda|}P_\lambda^{\mathrm{ip}}(q^x;q,q^\tau,q^\alpha) \to P_\lambda^{\mathrm{ip}}(x;\tau,\alpha),$$

$$P_\lambda^{\mathrm{ip}}(x;q,q^\tau,q^\alpha) \to (-4)^{|\lambda|}\,P_\lambda(\tfrac14(2-x-x^{-1});\tau),$$

$$(q-1)^{-|\lambda|}\,P_\lambda^{\mathrm{ip}}(q^x;q,q^\tau) \to P_\lambda^{\mathrm{ip}}(x;\tau),$$

$$P_\lambda^{\mathrm{ip}}(x;q,q^\tau) \to P_\lambda(x-1^n,\tau),$$

$$P_\lambda(x;q,q^\tau) \to P_\lambda(x;\tau).$$

Limits, q fixed

Highest term:

$$\left.\begin{array}{l} P_\lambda(x;q,t;a_1,a_2,a_3,a_4) \\ P_\lambda^{\rm ip}(x;q,t,a) \\ P_\lambda^{\rm ip}(x;q,t) \end{array}\right\} = P_\lambda(x;q,t) + \text{degree lower than } |\lambda|;$$

$$\left.\begin{array}{l} P_\lambda(x;\tau;\alpha,\beta) \\ P_\lambda^{\rm ip}(x,\tau) \end{array}\right\} = P_\lambda(x;\tau) + \text{degree lower than } |\lambda|,$$

$$P_\lambda^{\rm ip}(x;\tau,\alpha) = P_\lambda(x^2;\tau) + \text{degree in } x^2 \text{ lower than } |\lambda|.$$

parameter to ∞:

$$a^{-|\lambda|} P_\lambda^{\rm ip}(ax;q,t,a) \stackrel{a\to\infty}{\longrightarrow} P_\lambda^{\rm ip}(x;q,t),$$

$$a_1^{-|\lambda|} P_\lambda(a_1x;q,t;a_1,a_2,a_3,a_4) \stackrel{a_1\to\infty}{\longrightarrow} P_\lambda(x;q,t) \quad \text{(new)}.$$

$$(2\alpha)^{-|\lambda|} P_\lambda^{\rm ip}(x+\alpha,\alpha,\tau) \stackrel{\alpha\to\infty}{\longrightarrow} P_\lambda^{\rm ip}(x,\tau),$$

$$P_\lambda(x;\tau;\alpha,\beta) \stackrel{\alpha\to\infty}{\longrightarrow} P_\lambda(x-1^n;\tau).$$

Binomial formula for Koornwinder polynomials

$$a_1' := (q^{-1}a_1a_2a_3a_4)^{\frac{1}{2}},$$
$$a_1'a_2' = a_1a_2, \quad a_1'a_3' = a_1a_3, \quad a_1'a_4' = a_1a_4.$$

Binomial formula (Okounkov, Transf. Groups, 1998):

$$\frac{P_\lambda(x;q,t;a_1,a_2,a_3,a_4)}{P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)} = \sum_{\mu\subset\lambda} \frac{P_\mu^{\rm ip}(q^\lambda t^\delta a_1';q,t,a_1')}{P_\mu^{\rm ip}(q^\mu t^\delta a_1';q,t,a_1')} \, \frac{P_\mu^{\rm ip}(x;q,t,a_1)}{P_\mu(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}.$$

This implies the duality

$$\frac{P_\lambda(q^\nu t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}{P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)} = \frac{P_\nu(q^\lambda t^\delta a_1';q,t;a_1',a_2',a_3',a_4')}{P_\nu(t^\delta a_1';q,t;a_1',a_2',a_3',a_4')}$$

if $\quad \dfrac{P_\mu(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}{P_\mu^{\rm ip}(q^\mu t^\delta a_1;q,t,a_1)} = \dfrac{P_\mu(t^\delta a_1';q,t;a_1',a_2',a_3',a_4')}{P_\mu^{\rm ip}(q^\mu t^\delta a_1';q,t,a_1')}.$

Clear if $a_1' = a_1$. In general clear from evaluation formulas.

Binomial formula for Macdonald polynomials

Binomial formula (Okounkov, Math. Research Letters, 1997):

$$\frac{P_\lambda(x;q,t)}{P_\lambda(t^\delta;q,t)} = \sum_{\mu\subset\lambda} \frac{P_\mu^{\rm ip}(q^\lambda t^\delta;q,t)}{P_\mu^{\rm ip}(q^\mu t^\delta;q,t} \, \frac{P_\mu^{\rm ip}(x;q,t)}{P_\mu(t^\delta;q,t)}\,.$$

This implies the duality $\displaystyle \frac{P_\lambda(q^\nu t^\delta;q,t)}{P_\lambda(t^\delta;q,t)} = \frac{P_\nu(q^\lambda t^\delta;q,t)}{P_\nu(t^\delta;q,t)}\,.$

Compare with binomial formula for Koornwinder polynomials.

$$\frac{P_\lambda(x;q,t;a_1,a_2,a_3,a_4)}{P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)} = \sum_{\mu\subset\lambda} \frac{P_\mu^{\rm ip}(q^\lambda t^\delta a_1';q,t,a_1')}{P_\mu^{\rm ip}(q^\mu t^\delta a_1';q,t,a_1')} \, \frac{P_\mu^{\rm ip}(x;q,t,a_1)}{P_\mu(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}\,.$$

Theorem

$$\lim_{a_1\to\infty} a_1^{-|\lambda|} P_\lambda(a_1 x;q,t;a_1,a_2,a_3,a_4) = P_\lambda(x;q,t)$$

Proof Verify for $x = q^\delta$ in evaluation formulas.

Binomial formula for BC_n Jacobi polynomials

Let $\alpha' := \frac{1}{2}(\alpha+\beta+1)$.

Theorem

$$\frac{P_\lambda(x;\tau;\alpha,\beta)}{P_\lambda(0;\tau;\alpha,\beta)} = \sum_{\mu\subset\lambda} \frac{P_\mu^{\mathrm{ip}}(\lambda+\tau\delta+\alpha';\tau,\alpha')}{P_\mu^{\mathrm{ip}}(\mu+\tau\delta+\alpha';\tau,\alpha')} \, \frac{P_\mu(x;\tau)}{P_\mu(0;\tau;\alpha,\beta)}\,.$$

Proof Use $\dfrac{P_\lambda(z;q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1)}{P_\lambda(q^{\tau\delta+\alpha+1};q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1)} =$

$$\sum_{\mu\subset\lambda} \frac{P_\mu^{\mathrm{ip}}(q^{\lambda+\tau\delta+\alpha'};q,q^\tau,q^{\alpha'})}{P_\mu^{\mathrm{ip}}(q^{\mu+\tau\delta+\alpha'};q,q^\tau,q^{\alpha'})} \, \frac{P_\mu^{\mathrm{ip}}(z;q,q^\tau,q^{\alpha+1})}{P_\mu(q^{\tau\delta+\alpha+1};q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1)}$$

together with (for $q\uparrow 1$) the limits

$$P_\lambda(z;q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1) \to (-4)^{|\lambda|}\,P_\lambda(\tfrac14(2-z-z^{-1});\tau;\alpha,\beta),$$

$$P_\lambda^{\mathrm{ip}}(z;q,q^\tau,q^\alpha) \to (-4)^{|\lambda|}\,P_\lambda(\tfrac14(2-z-z^{-1});\tau),$$

$$(1-q)^{-2|\lambda|}P_\lambda^{\mathrm{ip}}(q^x;q,q^\tau,q^\alpha) \to P_\lambda^{\mathrm{ip}}(x;\tau,\alpha).$$

Binomial formula for BC_n Jacobi polynomials (cntd.)

So $$P_\lambda(x;\tau;\alpha,\beta)=\sum_{\mu\subset\lambda} c_{\lambda,\mu}P_\mu(x;\tau)$$

with $c_{\lambda,\mu}$, up to products of elementary factors, given by $P_\mu^{\rm ip}(\lambda+\tau\delta+\alpha';\tau,\alpha')$, which can be expressed by a certain sum over all reverse tableaux of shape μ with entries in $\{1,\ldots,n\}$.

On the other hand Macdonald (manuscript, 1987; arXiv:1309.4568) gives this as a certain sum over all standard tableaux of shape λ/μ and with entries in $1,\ldots,|\lambda-\mu|$.

The relationship between both expressions is not clear.

Binomial formula for Jack polynomials

$$\frac{P_\lambda(1+x,\tau)}{P_\lambda(1;\tau)} = \sum_{\mu\subset\lambda} \frac{P_\mu^{\rm ip}(\lambda+\tau\delta;\tau)}{P_\mu^{\rm ip}(\mu+\tau\delta;\tau)}\,\frac{P_\mu(x;\tau)}{P_\mu(1;\tau)}\,.$$

Hence $\quad \dfrac{P_\mu^{\rm ip}(\lambda+\tau\delta;\tau)}{P_\mu^{\rm ip}(\mu+\tau\delta;\tau)} = \dbinom{\lambda}{\mu}_\tau$ (Lassalle, 1990).

Theorem (Beerends & K., unpublished; Rösler, K. & Voit, 2013)

$\displaystyle\lim_{\alpha\to\infty} P_\lambda(x;\tau;\alpha,\beta) = P_\lambda(x-1;\tau).$

Proof Let $\alpha\to\infty$ in

$$\frac{P_\lambda(x;\tau;\alpha,\beta)}{P_\lambda(0;\tau;\alpha,\beta)} = \sum_{\mu\subset\lambda} \frac{P_\mu^{\rm ip}(\lambda+\tau\delta+\alpha';\tau,\alpha')}{P_\mu^{\rm ip}(\mu+\tau\delta+\alpha';\tau,\alpha')}\,\frac{P_\mu(x;\tau)}{P_\mu(0;\tau;\alpha,\beta)}$$

and use that $\quad (2\alpha)^{-|\lambda|}P_\lambda^{\rm ip}(x+\alpha;\tau,\alpha)\to P_\lambda^{\rm ip}(x;\tau), \quad \alpha\to\infty.$

It is now sufficient to verify the Theorem for $x=0$ by using the evaluation formulas.

More on the Koornwinder → Macdonald limit

$$P_\lambda(x;q,t;a_1,a_2,a_3,a_4) = \sum_{\mu\le\lambda} u_{\lambda,\mu}(q,t;a_1,a_2,a_3,a_4)\,\widetilde{m}_\mu(x),$$
$$P_\lambda(x;q,t) = \sum_{\mu\le\lambda} u_{\lambda,\mu}(q,t)\,m_\mu(x).$$

Consider the statements

(1) $\lim_{r\to\infty} r^{-|\lambda|} P_\lambda(rx;q,t;a_1,a_2,a_3,a_4) = P_\lambda(x;q,t).$

(2) $\lim_{a_1\to\infty} a_1^{-|\lambda|} P_\lambda(a_1x;q,t;a_1,a_2,a_3,a_4) = P_\lambda(x;q,t).$

(3) $u_{\lambda,\mu}(q,t;a_1,a_2,a_3,a_4) = u_{\lambda,\mu}(q,t)$ if $|\lambda| = |\mu|$.

(4) $u_{\lambda,\mu}(q,t;a_1,a_2,a_3,a_4)$ is bounded as $|a_i|\to\infty$ if $|\lambda| > |\mu|$.

(1) and (3) are equivalent, while (3) and (4) imply (2).
For (3) and (4) consider the binomial formula for Koornwinder polynomials.

More on the Koornwinder → Macdonald limit (cntd.)

$$P_\lambda(x;q,t;a_1,a_2,a_3,a_4)=\sum_{\mu\subset\lambda}\frac{P^{\rm ip}_\mu(q^\lambda t^\delta a_1';q,t,a_1')}{P^{\rm ip}_\mu(q^\mu t^\delta a_1';q,t,a_1')}$$

$$\times\frac{P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}{P_\mu(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}\,P^{\rm ip}_\mu(x;q,t,a_1).$$

$$P_\lambda(x;q,t;a_1,a_2,a_3,a_4)=\sum_{\mu\le\lambda}u_{\lambda,\mu}(q,t;a_1,a_2,a_3,a_4)\,\widetilde m_\mu(x),$$

$$P^{\rm ip}_\lambda(x;q,t,a)=\sum_{\mu\le\lambda}u_{\lambda,\mu}(q,t,a)\,\widetilde m_\mu(x),$$

$$P_\lambda(x;q,t)=\sum_{\mu\le\lambda}u_{\lambda,\mu}(q,t)\,m_\mu(x).$$

Hence $u_{\lambda,\mu}(q,t;a_1,a_2,a_3,a_4)=u_{\lambda,\mu}(q,t,a_1)\quad$ if $|\lambda|=|\mu|$.
Then (3) and (4) follow by the combinatorial formula for $P^{\rm ip}_\lambda(x;q,t,a)$. For (4) also use the symmetry in the a_i.

Reduction formula for BC_n interpolation Macdonald

$$P^{\mathrm{ip}}_{\mu}(z;q,t,a)=(-a)^{n\mu_n}q^{-\frac{1}{2}n\mu_n(\mu_n-1)}\prod_{j=1}^{n}\left((z_ja;q)_{\mu_n}(z_j^{-1}a;q)_{\mu_n}\right)$$
$$\times P^{\mathrm{ip}}_{\mu-\mu_n1^n}(z;q,t,q^{\mu_n}a).$$

In particular for $\mu=(\mu_1,\mu_2)$.

Also use that a sum over all reverse tableaux T of shape $(m,0)$ with entries in $\{1,2\}$ is a single sum.

Thus the combinatorial formula for $P^{\mathrm{ip}}_{m,0}(z_1,z_2;q,t,a)$ gives an explicit expression for $P^{\mathrm{ip}}_{m_1,m_2}(z_1,z_2;q,t,a)$.

Explicit formula for BC_2 interpolation Macdonald

$$P^{\mathrm{ip}}_{m_1,m_2}(z_1,z_2;q,t,a)=\frac{(-1)^{m_1-m_2}}{t^{m_1-m_2}a^{m_1+m_2}q^{\frac12 m_1(m_1-1)}q^{\frac12 m_2(m_2-1)}}$$
$$\times\,(z_1a,z_1^{-1}a,z_2a,z_2^{-1}a;q)_{m_2}\,(q^{m_2}z_1ta,q^{m_2}z_1^{-1}ta;q)_{m_1-m_2}$$
$$\times\;{}_4\phi_3\left(\begin{matrix}q^{-m_1+m_2},t,q^{m_2}z_2a,q^{m_2}z_2^{-1}a\\ q^{1-m_1+m_2}t^{-1},q^{m_2}z_1ta,q^{m_2}z_1^{-1}ta\end{matrix};q,q\right).$$

Thus, by Okounkov's binomial formula, this gives an explicit formula for Macdonald-Koornwinder polynomials $P_{n_1,n_2}(z_1,z_2;q,t;a_1,a_2,a_3,a_4)$.

Explicit formula for BC_2 interpolation Jack

$$P^{\rm ip}_{m_1,m_2}(x_1,x_2;\tau,\alpha)=(\alpha+x_1,\alpha-x_1,\alpha+x_2,\alpha-x_2)_{m_2}$$
$$\times\,(\tau+\alpha+x_1+m_2,\tau+\alpha-x_1+m_2)_{m_1-m_2}$$
$$\times\;{}_4F_3\left(\begin{matrix}-m_1+m_2,\tau,m_2+\alpha+x_2,m_2+\alpha-x_2\\ 1-m_1+m_2-\tau,\tau+\alpha+x_1+m_2,\tau+\alpha-x_1+m_2\end{matrix};1\right).$$

Together with ($C^\tau_{m_1-m_2}$ is Gegenbauer polynomial)

$$P_{m_1,m_2}(x_1,x_2;\tau)=\frac{(m_1-m_2)!}{(\tau)_{m_1-m_2}}(x_1x_2)^{\frac12(m_1+m_2)}C^\tau_{m_1-m_2}\left(\frac{x_1+x_2}{2(x_1x_2)^{\frac12}}\right)$$

we have by the binomial formula for BC_2 Jacobi polynomials an explicit expression for these polynomials.

Much earlier (1978) in a very different way obtained by K. & Sprinkhuizen.

Quadratic transformations

Let $\{p_m(x)\}$ be monic OP's on $\mathbb{R}$ with respect to **even** weight function $w(x) = w(-x) = v(x^2)$. Then $p_m(-x) = (-1)^m p_m(x)$.

Put $\quad q_m(x^2) := p_{2m}(x), \quad x\, r_m(x^2) := p_{2m+1}(x)$.

Then $\{q_m(x)\}$ and $\{r_m(x)\}$ are monic OP's on $[0,\infty)$:

- the q_m with respect to weight function $x^{-\frac{1}{2}}v(x)$,
- the r_m with respect to weight function $x^{\frac{1}{2}}v(x)$.

Example 1. Recall Jacobi polynomials $P_m(x;\alpha,\beta)$ with respect to weight function $x^\alpha(1-x)^\beta$ on $[0,1]$.

Let $\quad w(x) := (1-x^2)^\alpha, \quad x^{\pm\frac{1}{2}}v(x) = x^{\pm\frac{1}{2}}(1-x)^\alpha$. Then

$$P_{2m}(\tfrac{1}{2}(1-x);\alpha,\alpha) = \text{const.}\, P_m(1-x^2;\alpha,-\tfrac{1}{2}),$$
$$P_{2m+1}(\tfrac{1}{2}(1-x);\alpha,\alpha) = \text{const.}\, x\, P_m(1-x^2;\alpha,\tfrac{1}{2}),$$

Quadratic transformations (cntd.)

Example 2. Recall monic Askey-Wilson polynomials $P_m(z;q;a_1,a_2,a_3,a_4)$ as orthogonal polynomials in $x=\frac{1}{2}(z+z^{-1})$ $(|z|=1 \Leftrightarrow x\in[-1,1])$ with respect to the orthogonality measure $\Delta_+(z)\Delta_-(z)\,z^{-1}\,dz$ $(|z|=1)$, where

$$\Delta_+(z)=\Delta_+(z;q;a_1,a_2,a_3,a_4):=\frac{(z^2;q)_\infty}{(a_1z,a_2z,a_3z,a_4z;q)_\infty}$$

Since $\Delta_+(z;q;a,b,-a,-b)=\Delta_+(z^2;q^2;a^2,b^2,-1,-q)$ we have (Singh, 1959; Askey & Wilson, 1985):

$$P_{2m}(z;q;a,b,-a,-b)=P_m(z^2;q^2;a^2,b^2,-1,-q),$$
$$P_{2m+1}(z;q;a,b,-a,-b)=(z+z^{-1})P_m(z^2;q^2;a^2,b^2,-q,-q^2).$$

With $ab=q^{\alpha+1}$ a two-parameter q-analogue of Example 1.

Quadratic transformations for $n = 2$

In an essentially similar way obtain, if $n = 2$, quadratic tranformations for Jacobi (Sprinkhuizen, 1976) and Koornwinder (maybe new). These exist because B_2 and C_2, while special cases of BC_2, are isomorphic root systems.

$n = 2$ **Jacobi.** Let $m_1 + m_2$ be even.

$$P_{m_1,m_2}(\sin^2\theta_1, \sin^2\theta_2; \tau; a - \tfrac{1}{2}, a - \tfrac{1}{2}) = \text{const.}$$
$$\times P_{\frac{1}{2}(m_1+m_2),\frac{1}{2}(m_1-m_2)}(\sin^2(\theta_1+\theta_2), \sin^2(\theta_1-\theta_2); a; \tau - \tfrac{1}{2}, -\tfrac{1}{2}),$$
$$P_{m_1+1,m_2}(\sin^2\theta_1, \sin^2\theta_2; \tau; a - \tfrac{1}{2}, a - \tfrac{1}{2}) = \text{const.}\,(1 - \sin^2\theta_1 - \sin^2\theta_2)$$
$$\times P_{\frac{1}{2}(m_1+m_2),\frac{1}{2}(m_1-m_2)}(\sin^2(\theta_1+\theta_2), \sin^2(\theta_1-\theta_2); a; \tau - \tfrac{1}{2}, \tfrac{1}{2}).$$

$$\Delta_+(z_1, z_2; q, t; a_1, a_2, a_3, a_4) = \prod_{j=1}^{2} \frac{(z_j^2; q)_\infty}{(a_1 z_j, a_2 z_j, a_3 z_j, a_4 z_j; q)_\infty} \times \frac{(z_1 z_2, z_1 z_2^{-1}; q)_\infty}{(t z_1 z_2, t z_1 z_2^{-1}; q)_\infty}.$$

$$\Delta_+(z_1, z_2; q, t; a, -a, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = \Delta_+(z_1 z_2, z_1 z_2^{-1}; q^2, a^2; t, qt, -1, -q).$$

Let $m_1 + m_2$ be even.

$$P_{m_1,m_2}(z_1, z_2; q, t; a, -a, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = P_{\frac{1}{2}(m_1+m_2),\frac{1}{2}(m_1-m_2)}(z_1 z_2, z_1 z_2^{-1}; q^2, a^2; t, qt, -1, -q),$$

$$P_{m_1+1,m_2}(z_1, z_2; q, t; a, -a, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = (z_1 + z_2 + z_1^{-1} + z_2^{-1}) \times P_{\frac{1}{2}(m_1+m_2),\frac{1}{2}(m_1-m_2)}(z_1 z_2, z_1 z_2^{-1}; q^2, a^2; t, qt, -q, -q^2).$$

60th birthday Haiku

Sixty flowers bloom
In a bed of great wisdom
With a few loose springs

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