Nonsymmetric Askey-Wilson polynomials as vector-valued polynomials

Tom Koornwinder

University of Amsterdam, T.H.Koornwinder@uva.nl,

http://www.science.uva.nl/~thk/

Lecture on September 17, 2010 at the conference
Symmetry, Separation, Super-integrability and Special Functions (S4),
in honor of Willard Miller,
University of Minnesota, Minneapolis, September 17–19, 2010

Work in collaboration with Fethi Bouzeffour (Bizerte, Tunisia)

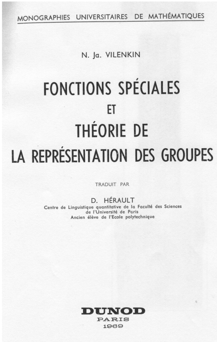
Last modified: September 28, 2010

Some personal recollections

1969–1970
Dick Askey on sabbatical at Mathematical Centre Amsterdam. Inspired by him I started working on group theoretic interpretations of special functions.

I had three heroes in this area.

Three heroes



MONOGRAPHIES UNIVERSITAIRES DE MATHÉMATIQUES

N. Ja. VILENKIN

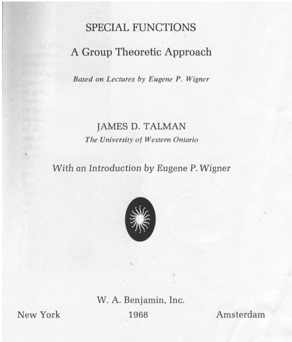
FONCTIONS SPÉCIALES
ET
THÉORIE DE
LA REPRÉSENTATION DES GROUPES

TRADUIT PAR

D. HÉRAULT
Centre de Linguistique quantitative de la Faculté des Sciences de l'Université de Paris
Ancien élève de l'Ecole polytechnique

DUNOD
PARIS
1969

Vilenkin



SPECIAL FUNCTIONS

A Group Theoretic Approach

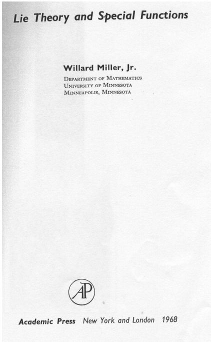
Based on Lectures by Eugene P. Wigner

JAMES D. TALMAN
The University of Western Ontario

With an Introduction by Eugene P. Wigner

W. A. Benjamin, Inc.
New York 1968 Amsterdam

Wigner (Talman)



Lie Theory and Special Functions

Willard Miller, Jr.
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA

Academic Press New York and London 1968

Willard Miller, Jr.

Different approaches

Vilenkin and Wigner started with the Lie group and identified matrix elements of irreducible representations with special functions. Only special parameter values came out.

But Miller started with the special functions and their differential recurrence relations, and he built from them a Lie algebra and next a local Lie group. This worked for all parameter values.

Symmetry and separation of variables

After a second book in 1972, Willard's third book appeared in 1977, on which I wrote a review for Bull. Amer. Math. Soc.

GIAN-CARLO ROTA, *Editor*
ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS
Volume 4

Section: Special Functions
Richard Askey, *Section Editor*

Symmetry and Separation of Variables

Willard Miller, Jr.
School of Mathematics
University of Minnesota
Minneapolis, Minnesota

With a Foreword by Richard Askey
University of Wisconsin

1977

Addison-Wesley Publishing Company
Advanced Book Program
Reading, Massachusetts

London · Amsterdam · Don Mills, Ontario · Sydney · Tokyo

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 6, November 1979

0002-9904/79/0000-0523/$02.25

Symmetry and separation of variables, by Willard Miller, Jr., Addison-Wesley Publishing Company, Reading, Massachusetts, 1977, xxx + 285 pp., $21.50.

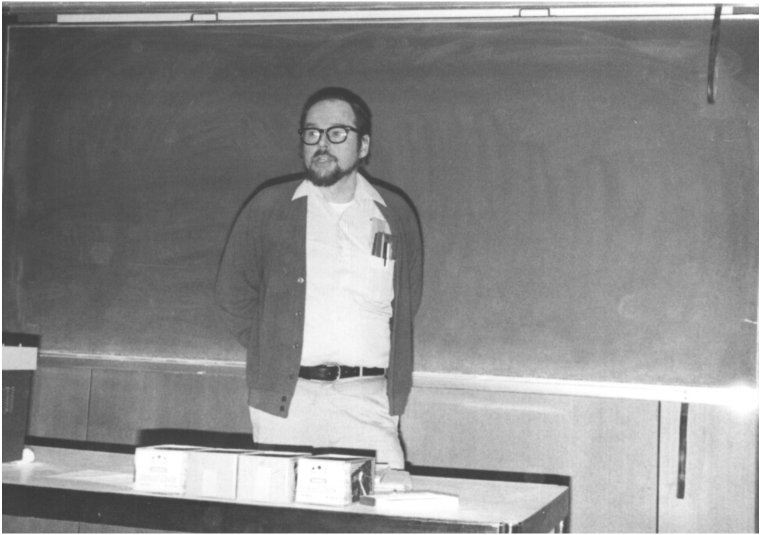
Separation of variables is a technique for solving special partial differential equations. It is taught in elementary courses on partial differential equations, but the method usually does not achieve the status of a mathematical theory.

Because most references do not give a precise definition of separation of variables, I invented a definition myself. Let us call a partial differential equation in n variables $x_1, \ldots, x_n$ *separable* if there are n ordinary differential equations in $x_1, \ldots, x_n$, respectively, jointly depending on $n - 1$ independent parameters (the separation constants), such that, for each choice of the parameters and for each set of solutions $(X_1, \ldots, X_n)$ of the o.d.e.'s, the function $u(x_1, \ldots, x_n) := X_1(x_1) \cdots X_n(x_n)$ is a solution of the p.d.e. Under the terms of this definition a converse implication often holds: If $u = X_1 \cdots X_n$ is a factorized solution of the p.d.e. then, for some choice of the parameters, the X_i's are solutions of the o.d.e.'s. The most familiar cases of separability deal with a linear second order p.d.e. which separates into n

TOM H. KOORNWINDER

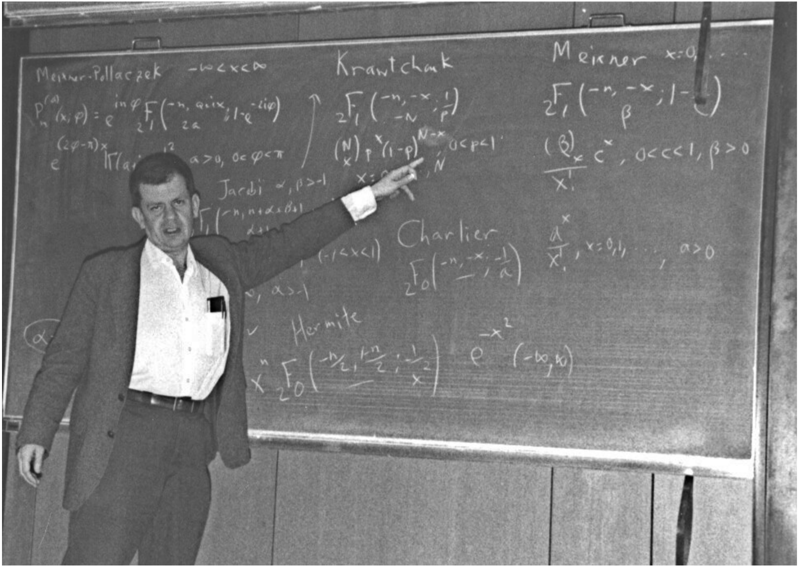
Oberwolfach 1983

Oberwolfach conference *Special functions and group theory*, March 14–18, 1983.



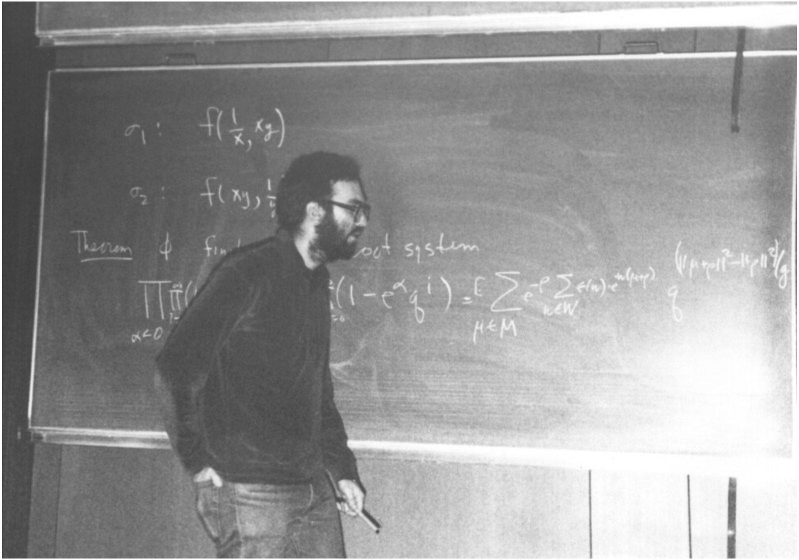
Willard Miller

Oberwolfach 1983 (cntd)



Dick Askey

Oberwolfach 1983 (cntd)



Dennis Stanton

Hankel transform

Normalized Bessel function:

$$\mathcal{J}_\alpha(x) := \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}x^2)^k}{(\alpha+1)_k\, k!} = {}_0F_1\left(\begin{matrix} - \\ \alpha+1 \end{matrix}; -\tfrac{1}{4}x^2\right).$$

$$\mathcal{J}_\alpha(x) = \mathcal{J}_\alpha(-x), \quad \mathcal{J}_\alpha(0) = 1, \quad \mathcal{J}_{-\frac{1}{2}}(x) = \cos x, \quad \mathcal{J}_{\frac{1}{2}}(x) = \frac{\sin x}{x}.$$

Eigenfunctions:

$$\left(\frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}\right)\mathcal{J}_\alpha(\lambda x) = -\lambda^2\,\mathcal{J}_\alpha(\lambda x).$$

Hankel transform pair:

$$\begin{cases} \widehat{f}(\lambda) = \displaystyle\int_0^\infty f(x)\mathcal{J}_\alpha(\lambda x)x^{2\alpha+1}\,dx, \\ f(x) = \displaystyle\frac{1}{2^{2\alpha+1}\Gamma(\alpha+1)^2}\int_0^\infty \widehat{f}(\lambda)\mathcal{J}_\alpha(\lambda x)\lambda^{2\alpha+1}\,d\lambda. \end{cases}$$

Non-symmetric Hankel transform

Non-symmetric Bessel function:

$$\mathcal{E}_\alpha(x) := \mathcal{J}_\alpha(x) + \frac{i\,x}{2(\alpha+1)}\,\mathcal{J}_{\alpha+1}(x), \quad \text{so} \quad \mathcal{E}_{-\frac{1}{2}}(x) = e^{ix}.$$

Non-symmetric Hankel transform pair:

$$\begin{cases} \widehat{f}(\lambda) = \displaystyle\int_{-\infty}^{\infty} f(x)\,\mathcal{E}_\alpha(-\lambda x)\,|x|^{2\alpha+1}\,dx, \\ f(x) = \dfrac{1}{2^{2(\alpha+1)}\Gamma(\alpha+1)^2} \displaystyle\int_{-\infty}^{\infty} \widehat{f}(\lambda)\,\mathcal{E}_\alpha(\lambda x)\,|\lambda|^{2\alpha+1}\,d\lambda. \end{cases}$$

Differential-reflection operator:

$$(Yf)(x) := f'(x) + (\alpha + \tfrac{1}{2})\,\frac{f(x) - f(-x)}{x}$$

(Dunkl operator for root system A_1).

Eigenfunctions:

$$Y(\mathcal{E}_\alpha(\lambda\,.)) = i\,\lambda\,\mathcal{E}_\alpha(\lambda\,.)\,.$$

Askey-Wilson polynomials

Assume $0 < q < 1$.

Monic Askey-Wilson polynomials as symmetric Laurent polynomials):

$$P_n[z] = P_n[z; a, b, c, d \mid q] = P_n\big(\tfrac{1}{2}(z + z^{-1})\big)$$
$$:= \frac{(ab, ac, ad; q)_n}{a^n(abcdq^{n-1}; q)_n}\ {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right).$$

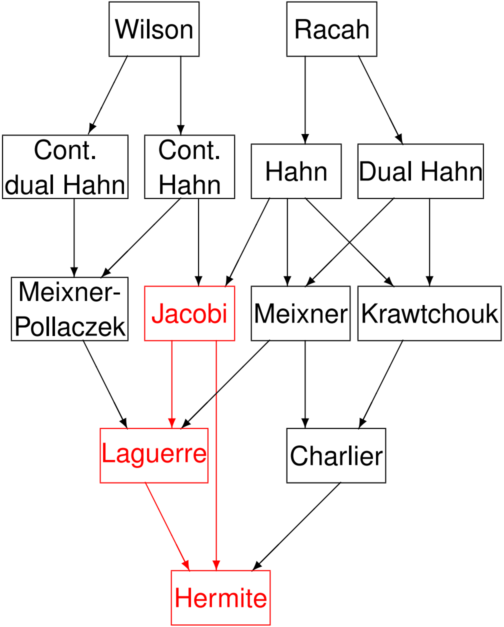
Eigenfunctions of second order q-difference operator L:

$$(LP_n)[z] := A[z]\,P_n[qz] + A[z^{-1}]\,P_n[q^{-1}z] - (A[z] + A[z^{-1}])\,P_n[z]$$
$$= (q^{-n} - 1)(1 - abcdq^{n-1})P_n[z],$$

where $\quad A[z] := \dfrac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)}.$

These are on the top level of the q-Askey scheme.

Askey scheme



Dick Askey

Jim Wilson

Non-symmetric Askey-Wilson polynomials

Further assume: $a, b, c, d \neq 0$, $abcd \neq q^{-m}$ $(m = 0, 1, 2, \ldots)$, $\{a, b\} \cap \{a^{-1}, b^{-1}\} = \emptyset$.

In terms of

$$P_n[z] \ = P_n[z; a, b, c, d \mid q],$$
$$Q_n[z] := a^{-1} b^{-1} z^{-1} (1 - az)(1 - bz)\, P_{n-1}[z; qa, qb, c, d \mid q]$$

the nonsymmetric Askey-Wilson polynomials are defined by:

$$E_{-n} := \frac{ab}{ab - 1} (P_n - Q_n) \qquad (n = 1, 2, \ldots), \qquad E_0[z] := 1,$$

$$E_n := \frac{(1 - q^n ab)(1 - q^{n-1} abcd)}{(1 - ab)(1 - q^{2n-1} abcd)} P_n - \frac{ab(1 - q^n)(1 - q^{n-1} cd)}{(1 - ab)(1 - q^{2n-1} abcd)} Q_n \qquad (n = 1, 2, \ldots).$$

Eigenfunctions of q-difference-reflection operator

Let

$$(Yf)[z] := \frac{z\big(1+ab-(a+b)z\big)\big((c+d)q-(cd+q)z\big)}{q(1-z^2)(q-z^2)}\,f[z]$$
$$+\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}f[qz]$$
$$+\frac{(1-az)(1-bz)\big((c+d)qz-(cd+q)\big)}{q(1-z^2)(1-qz^2)}\,f[z^{-1}]$$
$$+\frac{(c-z)(d-z)\big(1+ab-(a+b)z\big)}{(1-z^2)(q-z^2)}\,f[qz^{-1}],$$

Then

$$YE_{-n} = q^{-n}\,E_{-n} \qquad (n=1,2,\ldots),$$
$$YE_n = q^{n-1}abcd\,E_n \qquad (n=0,1,2,\ldots).$$

These come from I. Cherednik's theory of double affine Hecke algebras associated with root systems, extended by S. Sahi to the type $(C_l^\vee, C_l)$. Here his case $l=1$ is considered.

Double affine Hecke algebra of type $(C_1^\vee, C_1)$

This is the algebra $\tilde{\mathfrak{H}}$ generated by Z, Z^{-1}, T_1, T_0 with relations $ZZ^{-1} = 1 = Z^{-1}Z$ and

$$(T_1 + ab)(T_1 + 1) = 0, \qquad (T_0 + q^{-1}cd)(T_0 + 1) = 0,$$
$$(T_1 Z + a)(T_1 Z + b) = 0, \quad (qT_0 Z^{-1} + c)(qT_0 Z^{-1} + d) = 0.$$

This algebra acts faithfully on the space of Laurent polynomials:

$$(Zf)[z] := z\, f[z],$$
$$(T_1 f)[z] := \frac{(a+b)z - (1+ab)}{1-z^2}\, f[z] + \frac{(1-az)(1-bz)}{1-z^2}\, f[z^{-1}],$$
$$(T_0 f)[z] := \frac{q^{-1}z((cd+q)z - (c+d)q)}{q-z^2}\, f[z] - \frac{(c-z)(d-z)}{q-z^2}\, f[qz^{-1}].$$

Then $Y = T_1 T_0$.

Eigenspaces of T_1

- T_1 acting on Laurent polynomials has eigenvalues $-ab$ and -1.
- $T_1 f = -ab\, f \iff f$ is symmetric.
- $T_1 f = -f \iff f[z] = z^{-1}(1-az)(1-bz)g[z]$ for some symmetric Laurent polynomial g.

Let A be an operator acting on the Laurent polynomials. Write $f[z] = f_1[z] + z^{-1}(1-az)(1-bz)f_2[z]$ (f_1, f_2 symmetric Laurent polynomials). Then we can write

$$(Af)[z] = (A_{11} f_1 + A_{12} f_2)[z] + z^{-1}(1-az)(1-bz)(A_{21} f_1 + A_{22} f_2)[z],$$

where the A_{ij} are operators acting on the symmetric Laurent polynomials. So

$$f \leftrightarrow (f_1, f_2), \quad A \leftrightarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Rewriting the eigenvalue equation for E_n in matrix form

$$\left(\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} - q^{-n}\right)\begin{pmatrix} P_n[z;a,b,c,d \mid q] \\ -a^{-1}b^{-1}P_{n-1}[z;qa,qb,c,d \mid q] \end{pmatrix} = 0,$$

$$\left(\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} - q^{n-1}abcd\right) \\ \times \begin{pmatrix} (1-q^n ab)(1-q^{n-1}abcd)\,P_n[z;a,b,c,d \mid q] \\ -(1-q^n)(1-q^{n-1}cd)\,P_{n-1}[z;qa,qb,c,d \mid q] \end{pmatrix} = 0.$$

Here

$$Y_{11} = q^{-1}abcd - \frac{ab}{1-ab}\,L_{a,b,c,d;q},$$
$$Y_{22} = \frac{1-abcd-abq+abcdq+L_{aq,bq,c,d;q}}{q(1-ab)},$$

where $L_{a,b,c,d;q}$ is the second order q-difference operator L having the $p_n[z;a,b,c,d \mid q]$ as eigenfunctions.

On the next sheet the shift operators Y_{21}, Y_{12}.

The shift operators

$$(Y_{21}g)[z] = \frac{z(c-z)(d-z)\left(g[q^{-1}z]-g[z]\right)}{(1-ab)(1-z^2)(1-qz^2)} + \frac{z(1-cz)(1-dz)\left(g[qz]-g[z]\right)}{(1-ab)(1-z^2)(1-qz^2)},$$

$$\begin{aligned}(Y_{12}h)[z] = {} & \frac{ab(a-z)(b-z)(1-az)(1-bz)}{(1-ab)z(q-z^2)(1-qz^2)} \\ & \times \left((cd+q)(1+z^2)-(1+q)(c+d)z\right) h[z] \\ & - \frac{ab(a-z)(b-z)(c-z)(d-z)(aq-z)(bq-z)}{q(1-ab)z(1-z^2)(q-z^2)}\, h[q^{-1}z] \\ & - \frac{ab(1-az)(1-bz)(1-cz)(1-dz)(1-aqz)(1-bqz)}{q(1-ab)z(1-z^2)(1-qz^2)}\, h[qz].\end{aligned}$$

An equivalent form for the eigenvalue equations

The eigenvalue equations for E_n and for E_{-n} are equivalent to the four equations

$$
\begin{aligned}
&L_{a,b,c,d;q} P_n[\,.\,; a, b, c, d \mid q] \\
&\qquad = (q^{-n} - 1)(1 - abcdq^{n-1}) P_n[\,.\,; a, b, c, d \mid q], \\
&L_{qa,qb,c,d;q} P_{n-1}[\,.\,; qa, qb, c, d \mid q] \\
&\qquad = (q^{-n+1} - 1)(1 - abcdq^{n}) P_{n-1}[\,.\,; qa, qb, c, d \mid q], \\
&Y_{21} P_n[\,.\,; a, b, c, d \mid q] \\
&\qquad = -\frac{(q^{-n} - 1)(1 - cdq^{n-1})}{1 - ab}\, P_{n-1}[\,.\,; qa, qb, c, d \mid q], \\
&Y_{12} P_{n-1}[\,.\,; qa, qb, c, d \mid q] \\
&\qquad = -\frac{ab(q^{-n} - ab)(1 - abcdq^{n-1})}{1 - ab}\, P_n[\,.\,; a, b, c, d \mid q].
\end{aligned}
$$

Non-symmetric Bessel functions in vector-valued form

We can rewrite the equations

$$\mathcal{E}_\alpha(x) = \mathcal{J}_\alpha(x) + \frac{i\,x}{2(\alpha+1)}\,\mathcal{J}_{\alpha+1}(x),$$

$$(Yf)(x) = f'(x) + (\alpha + \tfrac{1}{2})\,\frac{f(x) - f(-x)}{x},$$

$$Y(\mathcal{E}_\alpha(\lambda\,.)) = i\,\lambda\,\mathcal{E}_\alpha(\lambda\,.)\,.$$

in the form

$$\left(\begin{pmatrix} 0 & x\frac{d}{dx} + 2\alpha + 2 \\ x^{-1}\frac{d}{dx} & 0 \end{pmatrix} - i\lambda\right)\begin{pmatrix} \mathcal{J}_\alpha(\lambda x) \\ \frac{i\lambda}{2(\alpha+1)}\,\mathcal{J}_{\alpha+1}(\lambda x) \end{pmatrix} = 0.$$

Applications of the vector-valued approach

By writing the nonsymmetric Askey-Wilson polynomials in vector-valued form, we can obtain two results which would have been impossible or much harder in the Laurent polynomial form:

- Orthogonality relations with positive definite inner product
- Limit to nonsymmetric little q-Jacobi polynomials

Orthogonality relations: scalar case

Let $\langle\,.\,,\,.\,\rangle_{a,b,c,d;q}$ be the Hermitian inner product on the space of symmetric Laurent polynomials such that the $P_n[\,.\,;a,b,c,d\mid q]$ are orthogonal in the familiar way:

$$\langle P_n[\,.\,;a,b,c,d\mid q], P_m[\,.\,;a,b,c,d\mid q]\rangle_{a,b,c,d;q} = h_n^{a,b,c,d;q}\,\delta_{n,m}\,,$$

where

$$h_n^{a,b,c,d;q} := \frac{(q,ab,ac,ad,bc,bd,cd;q)_n}{(abcd;q)_{2n}(q^{n-1}abcd;q)_n}\,.$$

(Assume that a,b,c,d are such that $\langle\,.\,,\,.\,\rangle_{a,b,c,d;q}$ and $\langle\,.\,,\,.\,\rangle_{qa,qb,c,d;q}$ are positive definite.) Then

$$\frac{h_n^{a,b,c,d;q}}{h_{n-1}^{qa,qb,c,d;q}} = \frac{(1-q^n)(1-q^{n-1}cd)}{(1-q^n ab)(1-q^{n-1}abcd)} \times \frac{(1-ab)(1-qab)(1-ac)(1-ad)(1-bc)(1-bd)}{(1-abcd)(1-qabcd)}\,.$$

Orthogonality relations: vector-valued case

For g_1, h_1, g_2, h_2 symmetric Laurent polynomials define an hermitian inner product

$$\langle (g_1, h_1), (g_2, h_2) \rangle := \langle g_1, g_2 \rangle_{a,b,c,d;q} - ab(1-ab)(1-qab) \times \frac{(1-ac)(1-ad)(1-bc)(1-bd)}{(1-abcd)(1-qabcd)} \langle h_1, h_2 \rangle_{qa,qb,c,d;q} .$$

Then the E_n ($n \in \mathbb{Z}$) in vector-valued form are orthogonal with respect to this inner product. (Need only to check that E_n is orthogonal to E_{-n}.)

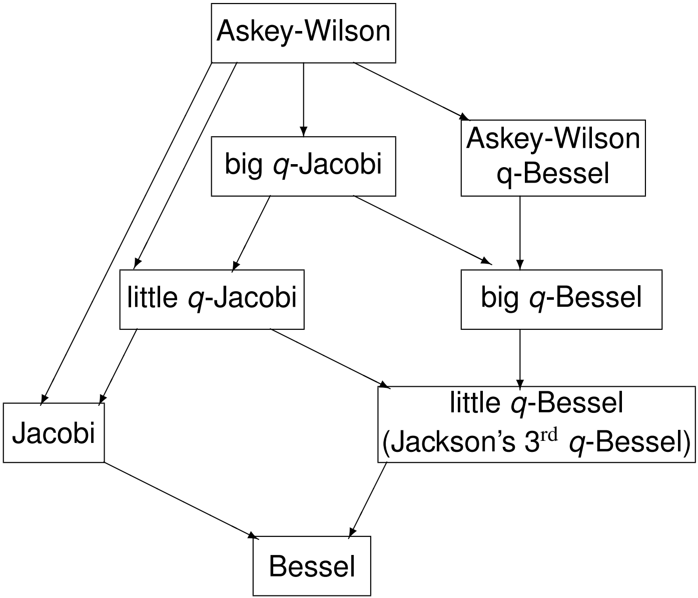
Theorem

If moreover $ab < 0$ then the inner product is positive definite.

In earlier papers (Sahi, Noumi & Stokman, Macdonald's 2003 book) a biorthogonality was given in the form of a contour integral, and there were no results on positive definiteness of the inner product.

The (q-)Askey-Bessel scheme

See Koelink & Stokman, NATO, Tempe, 2000.



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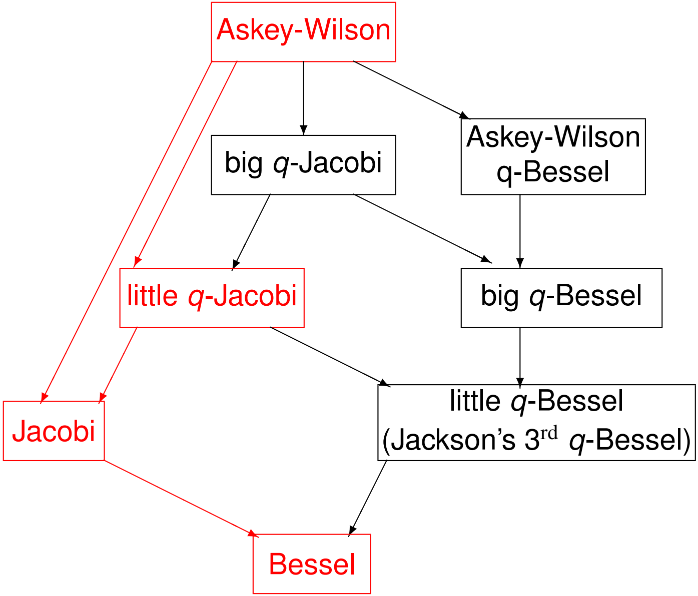

Little q-Jacobi polynomials

Monic little q-Jacobi polynomials as ordinary polynomials:

$$P_n^{\mathrm{lqJ}}(x;a,b;q) := \frac{(-1)^n q^{n(n-1)/2}(aq;q)_n}{(abq^{n+1};q)_n}\ {}_2\phi_1\left(\begin{matrix} q^n, abq^{n+1} \\ aq \end{matrix}; q, qx\right).$$

They are limits of Askey-Wilson polynomials:

$$P_n^{\mathrm{lqJ}}(x;a,b;q) = \lim_{\lambda\downarrow 0} \lambda^n P_n^{\mathrm{AW}}(\lambda^{-1}x; -q^{1/2}a, qb\lambda, -q^{1/2}, \lambda^{-1} \mid q),$$

$$P_{n-1}^{\mathrm{lqJ}}(x;qa,qb;q) = \lim_{\lambda\downarrow 0} \lambda^{n-1} P_{n-1}^{\mathrm{AW}}(\lambda^{-1}x; -q^{3/2}a, q^2b\lambda, -q^{1/2}, \lambda^{-1} \mid q)$$

Nonsymmetric little q-Jacobi polynomials

$$\left(\begin{pmatrix} Y_{11}^{\mathrm{AW}} & Y_{12}^{\mathrm{AW}} \\ Y_{21}^{\mathrm{AW}} & Y_{22}^{\mathrm{AW}} \end{pmatrix} - q^{-n}\right) \begin{pmatrix} P_n^{\mathrm{AW}}[z; a, b, c, d \mid q] \\ -a^{-1}b^{-1}P_{n-1}^{\mathrm{AW}}[z; qa, qb, c, d \mid q] \end{pmatrix} = 0,$$

$$\left(\begin{pmatrix} Y_{11}^{\mathrm{AW}} & Y_{12}^{\mathrm{AW}} \\ Y_{21}^{\mathrm{AW}} & Y_{22}^{\mathrm{AW}} \end{pmatrix} - q^{n-1}abcd\right) \times \begin{pmatrix} (1-q^n ab)(1-q^{n-1}abcd)\, P_n^{\mathrm{AW}}[z; a, b, c, d \mid q] \\ -(1-q^n)(1-q^{n-1}cd)\, P_{n-1}^{\mathrm{AW}}[z; qa, qb, c, d \mid q] \end{pmatrix} = 0.$$

Substitute: $a \to -q^{1/2}a$, $b \to qb\lambda$, $c \to -q^{1/2}$, $d \to \lambda^{-1}$, $z \to \lambda^{-1}x$ and let $\lambda \downarrow 0$:

$$\left(\begin{pmatrix} Y_{11}^{\mathrm{lqJ}} & Y_{12}^{\mathrm{lqJ}} \\ Y_{21}^{\mathrm{lqJ}} & Y_{22}^{\mathrm{lqJ}} \end{pmatrix} - q^{-n}\right) \begin{pmatrix} P_n^{\mathrm{lqJ}}(x; a, b; q) \\ a^{-1}b^{-1}q^{-3/2}P_{n-1}^{\mathrm{lqJ}}(x; qa, qb; q) \end{pmatrix} = 0,$$

$$\left(\begin{pmatrix} Y_{11}^{\mathrm{lqJ}} & Y_{12}^{\mathrm{lqJ}} \\ Y_{21}^{\mathrm{lqJ}} & Y_{22}^{\mathrm{lqJ}} \end{pmatrix} - q^{n+1}ab\right) \begin{pmatrix} (1-q^{n+1}ab)\, P_n^{\mathrm{lqJ}}(x; a, b; q) \\ -(1-q^n)q^{n-1/2}\, P_{n-1}^{\mathrm{lqJ}}(x; qa, qb; q) \end{pmatrix} = 0.$$

Some literature

1. T. H. Koornwinder and F. Bouzeffour, *Nonsymmetric Askey-Wilson polynomials as vector-valued polynomials*, `arXiv:1006.11400`; to appear in Applicable Analysis.
2. T. H. Koornwinder, *The relationship between Zhedanov's algebra AW(3) and the double affine Hecke algebra in the rank one case*, SIGMA 3 (2007), 063; `arXiv:math/0612730v4`.
3. I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge University Press, 2003.
4. M. Noumi and J. V. Stokman, *Askey-Wilson polynomials: an affine Hecke algebraic approach*, in *Laredo Lectures on Orthogonal Polynomials and Special Functions*, Nova Sci. Publ., 2004; `arXiv:math.QA/0001033`.
5. S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. of Math. (2) 150 (1999), 267–282; `arXiv:q-alg/9710032`.
6. S. Sahi, *Some properties of Koornwinder polynomials*, Contemporary Math. 254, Amer. Math. Soc., 2000, pp. 395–411.