Syllabus Fourier analysis

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Note This syllabus is based on parts of the book "Fouriertheorie" by A. van Rooij (Epsilon Uitgaven, 1988). Many of the exercises and some parts of the text are quite literally taken from this book. Usage of this book in addition to the syllabus is recommended. The present version of the syllabus is slightly modified. Modifications were made by J. Wiegerinck and by T. H. Koornwinder.

Contents

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1 L^2 theory

1.1 The Hilbert space $L^2_{2\pi}$

1.1 *T-periodic functions.* Let $T > 0$. A function $f: \mathbb{R} \to \mathbb{C}$ is called *periodic with period T* (or *T-periodic*) if $f(t+T) = f(t)$ for all t in $\mathbb{R}$. A T-periodic function is completely determined by its restriction to some interval $[a, a+T)$, and any function on $[a, a+T)$ can be uniquely extended to a T-periodic function on $\mathbb{R}$. However, a function f on $[a, a+T]$ is the restriction of a T-periodic function iff $f(a+T) = f(a)$. When working with T-periodic functions we will usually take $T := 2\pi$.

1.2 *The Banach spaces $L^p_{2\pi}$.* Let $1 \le p < \infty$. We denote by $\mathcal{L}^p_{2\pi}$ the space of all 2π-periodic (Lebesgue) measurable functions $f: \mathbb{R} \to \mathbb{C}$ such that $\int_{-\pi}^{\pi} |f(t)|^p \, dt < \infty$. This is a complex linear space. In particular, we are interested in the cases $p = 1$ and $p = 2$. The reader may continue reading with these two cases in mind. A seminorm $\| \, . \, \|_p$ can be defined on $\mathcal{L}^p_{2\pi}$ by

$$\|f\|_p := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{1/p} \qquad (f \in \mathcal{L}^p_{2\pi}). \tag{1.1}$$

By *seminorm* we mean that $\|f\|_p \ge 0$, $\|f+g\|_p \le \|f\|_p + \|g\|_p$ and $\|\lambda f\|_p = |\lambda| \, \|f\|_p$ for $f, g \in \mathcal{L}^p_{2\pi}$, $\lambda \in \mathbb{C}$, but not necessarily $\|f\|_p > 0$ if $f \ne 0$. The factor $(2\pi)^{-1}$ is included in (1.1) just for cosmetic reasons: if f is identically 1 then $\|f\|_p = 1$.

It is known from integration theory (see syll. Integratietheorie) that, for $f \in \mathcal{L}^p_{2\pi}$, we have that $\|f\|_p = 0$ iff $f = 0$ a.e. (almost everywhere), i.e., iff $f(t) = 0$ for t outside some subset of $\mathbb{R}$ of (Lebesgue) measure 0. It follows that, for $f \in \mathcal{L}^p_{2\pi}$, the value of $\|f\|_p$ does not change if we modify f on a set of measure 0, so $\| \, . \, \|_p$ is well-defined on equivalence classes of functions, where equivalence of two functions means that they are equal almost everywhere. Also, $\|f\|_p = 0$ iff f is equivalent to the function which is identically 0 on $\mathbb{R}$.

Thus we define the space $L^p_{2\pi}$ as the set of equivalence classes of a.e. equal functions in $\mathcal{L}^p_{2\pi}$. It can also be viewed as the quotient space $\mathcal{L}^p_{2\pi} / \{f \in \mathcal{L}^p_{2\pi} \mid \|f\|_p = 0\}$. The space $L^p_{2\pi}$ becomes a normed vector space with norm $\| \, . \, \|_p$. In fact this normed vector space is complete (see syll. Integratietheorie, Chapter 6 for $p = 1$ or 2), so it is a Banach space.

If I is some interval then we can also consider the space $\mathcal{L}^p(I)$ of measurable functions on I for which $\int_I |f(t)|^p \, dt < \infty$, and the space $L^p(I)$ of equivalence classes of a.e. equal functions in $\mathcal{L}^p(I)$. A seminorm or norm $\| \, . \, \|_p$ on $\mathcal{L}^p(I)$ or $L^p(I)$, respectively, is defined by

$$\|f\|_p := \left(\int_I |f(t)|^p \, dt \right)^{1/p}. \tag{1.2}$$

The linear map $f \mapsto f|_{[-\pi,\pi)}: \mathcal{L}^p_{2\pi} \to \mathcal{L}^p([-\pi,\pi))$ is bijective and it preserves $\| \, . \, \|_p$ up to a constant factor. It naturally yields an isomorphism of Banach spaces (up to a constant factor) between $L^p_{2\pi}$ and $L^p([-\pi,\pi])$. Note that, when we work with $L^p_{2\pi}$ rather than $\mathcal{L}^p_{2\pi}$, the function values on the endpoints of the interval $[-\pi,\pi]$ do not matter, since the two endpoints form a set of measure zero.

Ex. 1.3 Show the following. If $f \in \mathcal{L}^1_{2\pi}$ then

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi+a}^{\pi+a} f(t)\,dt \qquad (a \in \mathbb{R}).$$

1.4 *The Banach space $\mathcal{C}_{2\pi}$.* Define the linear space $\mathcal{C}_{2\pi}$ as the space of all 2π-periodic continuous functions $f\colon \mathbb{R} \to \mathbb{C}$. The restriction map $f \mapsto f|_{[-\pi,\pi]}$ identifies the linear space $\mathcal{C}_{2\pi}$ with the linear space of all continuous functions f on $[-\pi,\pi]$ for which $f(-\pi) = f(\pi)$. The space $\mathcal{C}_{2\pi}$ becomes a Banach space with respect to the sup norm

$$\|f\|_\infty := \sup_{t\in\mathbb{R}} |f(t)| = \sup_{t\in[-\pi,\pi]} |f(t)| \qquad (f \in \mathcal{C}_{2\pi}).$$

We can consider the space $C([-\pi,\pi])$ of continuous functions on $[-\pi,\pi]$ as a linear subspace of $\mathcal{L}^p([-\pi,\pi])$, but also as a linear subspace of $L^p([-\pi,\pi])$.

Indeed, if two continuous functions on $[-\pi,\pi]$ are equal a.e. then they are equal everywhere. Hence each equivalence class in $\mathcal{L}^p([-\pi,\pi])$ contains at most one continuous function. Hence, if $f, g \in C([-\pi,\pi])$ are equal as elements of $L^p([-\pi,\pi])$ then they are equal as elements of $C([-\pi,\pi])$.

It follows that we can consider the space $\mathcal{C}_{2\pi}$ as a linear subspace of the space $\mathcal{L}^p_{2\pi}$, but also as a linear subspace of $L^p_{2\pi}$.

The following proposition is well known (see Rudin, Theorem 3.14):

1.5 Proposition $C([-\pi,\pi])$ is a dense linear subspace of the Banach space $L^p([-\pi,\pi])$.

Ex. 1.6 For $p \ge 1$ prove the norm inequality

$$\|f\|_p \le \|f\|_\infty \qquad (f \in \mathcal{C}_{2\pi}).$$

Ex. 1.7 Prove that the linear space $\mathcal{C}_{2\pi}$ is a dense linear subspace of the Banach space $L^p_{2\pi}$ $(p \ge 1)$.
Hint Show for $f \in C([-\pi,\pi])$ and $\varepsilon > 0$ that there exists $g \in C([-\pi,\pi])$ such that $g(-\pi) = 0 = g(\pi)$ and $\|f - g\|_p < \varepsilon$.

Ex. 1.8 Prove the following:
Let V be a dense linear subspace of $\mathcal{C}_{2\pi}$ with respect to the norm $\|\,.\,\|_\infty$. Then, for $p \ge 1$, V is a dense linear subspace of $L^p_{2\pi}$ with respect to the norm $\|\,.\,\|_p$.

1.9 For f a function on $\mathbb{R}$ and for $a \in \mathbb{R}$ define the function $T_a f$ by

$$(T_a f)(x) := f(x+a) \qquad (x \in \mathbb{R}). \tag{1.3}$$

If f is 2π-periodic then so is $T_a f$ and, if V is any of the spaces $L^p_{2\pi}$ or $\mathcal{C}_{2\pi}$ then $T_a\colon V \to V$ is a linear bijection which preserves the appropriate norm.

Proposition Let $p \ge 1$, $f \in L^p_{2\pi}$. Then the map $a \mapsto T_a f : \mathbb{R} \to L^p_{2\pi}$ is uniformly continuous.

Proof First take $g \in \mathcal{C}_{2\pi}$. Then, by the compactness of $[-\pi,\pi]$ and by periodicity, g is uniformly continuous on $\mathbb{R}$. Hence, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T_a g - T_b g\|_\infty < \varepsilon$ if $|a-b| < \delta$. Next take $f \in L^p_{2\pi}$ and let $\varepsilon > 0$. Since $\mathcal{C}_{2\pi}$ is dense in $L^p_{2\pi}$ (see Exercise 1.7), we can find $g \in \mathcal{C}_{2\pi}$ such that $\|f-g\|_p \le \frac{1}{3}\varepsilon$. Hence

$$\|T_a f - T_b f\|_p \le \|T_a(f-g)\|_p + \|T_a g - T_b g\|_p + \|T_b(f-g)\|_p \le \tfrac{2}{3}\varepsilon + \|T_a g - T_b g\|_\infty,$$

where we used Exercise 1.6. Now we can find $\delta > 0$ such that $\|T_a g - T_b g\|_\infty < \frac{1}{3}\varepsilon$ if $|a-b| < \delta$. □

1.10 *The Hilbert space* $L^2_{2\pi}$. We can say more about $L^p_{2\pi}$ if $p=2$. Then, for any two functions $f, g \in \mathcal{L}^2_{2\pi}$, we can define $\langle f, g\rangle \in \mathbb{C}$ by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\overline{g(t)}\,dt. \tag{1.4}$$

Note that $\langle f, f\rangle = (\|f\|_2)^2$. The form $\langle\ .\ ,\ .\ \rangle$ has all the properties of a hermitian inner product on the complex linear space $\mathcal{L}^2_{2\pi}$, except that it is not positive definite, only positive semidefinite: we may have $\langle f, f\rangle = 0$ while f is not identically zero. However, the right hand side of (1.4) does not change when we modify f and g on subsets of measure zero. Therefore, $\langle f, g\rangle$ is well-defined on $L^2_{2\pi}$ and it is a hermitian inner product there. In fact, the inner product space $L^2_{2\pi}$ is complete: it is a Hilbert space.

For I an interval and f, g in $\mathcal{L}^2(I)$ or $L^2(I)$ we can define $\langle f, g\rangle$ in a similar way:

$$\langle f, g\rangle := \int_I f(t)\,\overline{g(t)}\,dt. \tag{1.5}$$

The bijective linear map $f \mapsto f|_{[-\pi,\pi)} : \mathcal{L}^2_{2\pi} \to \mathcal{L}^2([-\pi,\pi))$ preserves $\langle\ .\ ,\ .\ \rangle$ up to a constant factor. It naturally yields an isomorphism of Hilbert spaces (up to a constant factor) between $L^2_{2\pi}$ and $L^2([-\pi,\pi])$.

The space $L^2_{2\pi}$ is a linear subspace of $L^1_{2\pi}$. In fact, for $f \in L^2_{2\pi}$ we have the norm inequality

$$\|f\|_1 \le \|f\|_2. \tag{1.6}$$

(Prove this by use of the Cauchy-Schwarz inequality.) Also, $L^2_{2\pi}$ is a *dense* linear subspace of $L^1_{2\pi}$. Indeed, the subspace $\mathcal{C}_{2\pi}$ of $L^2_{2\pi}$ is already dense in $L^1_{2\pi}$.

1.11 *An orthonormal basis of* $L^2_{2\pi}$. The functions $t \mapsto e^{int}$ $(n \in \mathbb{Z})$ belong to $\mathcal{C}_{2\pi}$ and they form an orthonormal system in $L^2_{2\pi}$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imt}\,\overline{e^{int}}\,dt = \delta_{m,n} \qquad (m, n \in \mathbb{Z}). \tag{1.7}$$

By a *trigonometric polynomial* on $\mathbb{R}$ (of period 2π) we mean a finite linear combination (with complex coefficients) of functions $t \mapsto e^{int}$ $(n \in \mathbb{Z})$.

The following theorem has been mentioned without proof in syll. Functionaalanalyse.

Theorem
(a) The space of trigonometric polynomials is dense in $\mathcal{C}_{2\pi}$ with respect to the norm $\| \, . \, \|_\infty$.
(b) The space of trigonometric polynomials is dense in $L^2_{2\pi}$ with respect to the norm $\| \, . \, \|_2$.
(c) The functions $t \mapsto e^{int}$ $(n \in \mathbb{Z})$ form an orthonormal basis of $L^2_{2\pi}$.

Part (b) of the Theorem follows from part (a) (why?), and part (c) follows from part (b). We will prove the theorem in a later chapter (first part (b) and hence part (c), and afterwards part (a)). However, in this Chapter we will already use the Theorem. So we have to be careful later that circular arguments will be avoided.

1.2 Generalities about orthonormal bases

1.12 We now recapitulate some generalities concerning orthonormal bases of Hilbert spaces, as given in syll. Functionaalanalyse. Let $\mathcal{H}$ be a Hilbert space. Denote the inner product by $\langle \, . \, , \, . \, \rangle$ and the norm by $\| \, . \, \|$. Let $\mathcal{A}$ be an index set and consider an *orthonormal system* $E := \{e_\alpha\}_{\alpha\in\mathcal{A}}$ in $\mathcal{H}$, i.e., $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \mathcal{A}$. For convenience we assume that the Hilbert space is *separable*. Thus the index set $\mathcal{A}$ of the orthonormal system E will be countable.

Proposition (*Bessel inequality*)

$$\sum_{\alpha\in\mathcal{A}} |\langle f, e_\alpha\rangle|^2 \le \|f\|^2 \qquad (f \in \mathcal{H}).$$

Corollary If $f \in \mathcal{H}$ then $\lim_{\alpha\to\infty}\langle f, e_\alpha\rangle = 0$, i.e., for all $\varepsilon > 0$ there is a finite subset $\mathcal{B} \subset \mathcal{A}$ such that, if $\alpha \in \mathcal{A}\backslash\mathcal{B}$ then $|\langle f, e_\alpha\rangle| < \varepsilon$.

(The set $\mathcal{A}$, equipped with the discrete topology, is locally compact. By adding the point ∞, we obtain the one-point compactification of $\mathcal{A}$. This gives a further explanation of the notion $\lim_{\alpha\to\infty}$.)

Definition-Theorem (*orthonormal basis; Parseval's norm equality*)
The orthonormal system E is called an *orthonormal basis* of $\mathcal{H}$ if the following equivalent properties hold:
(a) $\mathrm{Span}(E)$ is dense in $\mathcal{H}$.
(b) $\sum_{\alpha\in\mathcal{A}} |\langle f, e_\alpha\rangle|^2 = \|f\|^2$ for all $f \in \mathcal{H}$ (*Parseval equality*).
(c) If $f \in \mathcal{H}$ and f is orthogonal to E then $f = 0$ (i.e., E is a *maximal* orthonormal system).

1.13 Definition-Proposition (*unconditional convergence*)
Let $\mathcal{H}$ be a separable Hilbert space. Let $\mathcal{A}$ be a countably infinite index set. Let $\{v_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathcal{H}$. We say that the sum $\sum_{\alpha\in\mathcal{A}} v_\alpha$ *unconditionally converges* to some $v \in \mathcal{H}$ if the following two equivalent properties are valid:
(a) For each way of ordering $\mathcal{A}$ as a sequence $\alpha_1, \alpha_2, \ldots$ we have that $v = \lim_{N\to\infty}\sum_{n=1}^N v_{\alpha_n}$ in the topology of $\mathcal{H}$.
(b) For each $\varepsilon > 0$ there is a finite subset $\mathcal{B} \subset \mathcal{A}$ such that for each finite set $\mathcal{C}$ satisfying $\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$ we have that $\|v - \sum_{\alpha\in\mathcal{C}} v_\alpha\| < \varepsilon$.

Theorem Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ ($\mathcal{A}$ a countable index set). Let $\{c_\alpha\}_{\alpha\in\mathcal{A}} \in \ell^2(\mathcal{A})$ (i.e. $\sum_{\alpha\in\mathcal{A}} |c_\alpha|^2 < \infty$). Then there is a unique $f \in \mathcal{H}$ such that $\langle f, e_\alpha\rangle = c_\alpha$ ($\alpha \in \mathcal{A}$). This element f can be written as $f = \sum_{\alpha\in\mathcal{A}} c_\alpha e_\alpha$ (with unconditional convergence).

Proposition (*Parseval's inner product equality*)
Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ ($\mathcal{A}$ a countable index set). Then

$$\langle f, g\rangle = \sum_{\alpha\in\mathcal{A}} \langle f, e_\alpha\rangle\, \overline{\langle g, e_\alpha\rangle} \qquad (f, g \in \mathcal{H})$$

with absolute convergence.

1.14 Theorem (*summarizing this subchapter*) Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ ($\mathcal{A}$ a countable index set). Then there is an isometry of Hilbert spaces

$$\mathcal{F}\colon f \mapsto \{c_\alpha\}_{\alpha\in\mathcal{A}}\colon \mathcal{H} \to \ell^2(\mathcal{A})$$

defined by $c_\alpha := \langle f, e_\alpha\rangle$, with inverse isometry

$$\mathcal{F}^{-1}\colon \{c_\alpha\}_{\alpha\in\mathcal{A}} \mapsto f\colon \ell^2(\mathcal{A}) \to \mathcal{H}$$

given by $f := \sum_{\alpha\in\mathcal{A}} c_\alpha e_\alpha$ (unconditionally).

1.3 Application to an orthonormal basis of $L^2_{2\pi}$

By Theorem 1.11 the functions $t \mapsto e^{int}$ ($n \in \mathbb{Z}$) form an orthonormal basis of $L^2_{2\pi}$. Let us apply the general results of the previous subsection to this particular orthonormal basis of the Hilbert space $L^2_{2\pi}$.

1.15 Definition Let $f \in L^1_{2\pi}$. The *Fourier coefficients* of f are given by the numbers

$$\widehat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \qquad (n \in \mathbb{Z}). \tag{1.8}$$

Note that the integral is absolutely convergent since $|f(t)\, e^{-int}| \le |f(t)|$.
We can consider $\widehat{f}$ as a function $\widehat{f}\colon \mathbb{Z} \to \mathbb{C}$. The map $\mathcal{F}\colon f \mapsto \widehat{f}$, sending 2π-periodic functions on $\mathbb{R}$ to functions on $\mathbb{Z}$, is called the *Fourier transform* (for periodic functions). Thus, by the *Fourier transform* of a function $f \in L^1_{2\pi}$ we mean the function $\widehat{f}$.

Remark In this subchapter we will consider the Fourier transform $f \mapsto \widehat{f}$ restricted to functions $f \in L^2_{2\pi}$. For such f we can consider the right hand side of (1.8) as an inner product. Indeed, if we put $e_n(t) := e^{int}$ ($n \in \mathbb{Z}$) then $\widehat{f}(n) = \langle f, e_n\rangle \quad (f \in L^2_{2\pi})$.

Proposition (*Bessel inequality*)

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\, dt \qquad (f \in L^2_{2\pi}).$$

Corollary (*Riemann-Lebesgue Lemma*) If $f \in L^2_{2\pi}$ then $\lim_{|n|\to\infty} \widehat{f}(n) = 0$.

1.16 The results of §1.15 did not depend on the completeness of the orthonormal system. However, the proof of the next results uses this completeness, so we cannot be sure about these results until we have proved Theorem 1.11(b).

Theorem (*Parseval*) Let $f, g \in L^2_{2\pi}$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2, \tag{1.9}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\overline{g(t)}\,dt = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\,\overline{\widehat{g}(n)}. \tag{1.10}$$

In (1.9) and (1.10), the integral on the left hand side and the sum on the right hand side are absolutely convergent.

1.17 Theorem (*Riesz-Fischer*) Let $\{\gamma_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, i.e. $\sum_{n=-\infty}^{\infty} |\gamma_n|^2 < \infty$. Then there is a unique $f \in L^2_{2\pi}$ such that $\widehat{f}(n) = \gamma_n$ $(n \in \mathbb{Z})$, and this f can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} \gamma_n\, e^{inx} \qquad \text{(unconditional convergence in } L^2_{2\pi}).$$

So, in particular,

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \gamma_n\, e^{inx} \qquad \text{(convergence in } L^2_{2\pi}),$$

i.e.,

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \Bigl| f(t) - \sum_{n=-N}^{N} \gamma_n\, e^{int} \Bigr|^2 dt = 0.$$

1.18 (*summarizing this subchapter*) The map $\mathcal{F}\colon f \mapsto \{\gamma_n\}_{n\in\mathbb{Z}}\colon L^2_{2\pi} \to \ell^2(\mathbb{Z})$ given by

$$\gamma_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\,dt$$

defines an isometry of Hilbert spaces with inverse isometry $\mathcal{F}^{-1}\colon \{\gamma_n\}_{n\in\mathbb{Z}} \mapsto f\colon \ell^2(\mathbb{Z}) \to L^2_{2\pi}$ given by

$$f(x) := \sum_{n=-\infty}^{\infty} \gamma_n\, e^{inx} \qquad \text{(unconditional convergence in } L^2_{2\pi}).$$

1.19 Without proof we mention a more recent, very important and deep result: the almost everywhere convergence of Fourier series of any function in $L^2_{2\pi}$.

Theorem (L. Carleson, Acta Mathematica 116 (1966), 135–157) If $f \in L^2_{2\pi}$ then

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{f}(n)\, e^{inx}$$

with pointwise convergence for almost all $x \in \mathbb{R}$.

1.4 Orthonormal bases with cosines and sines

The orthonormal basis of functions $t \mapsto e^{int}$ ($n \in \mathbb{Z}$) for $L^2_{2\pi}$ (see Theorem 1.11(c)) immediately gives rise to an orthonormal basis for $L^2_{2\pi}$ in terms of cosines and sines. We formulate it in two steps and leave the straightforward proofs to the reader.

1.20 Lemma For $n = 1, 2, \ldots$ the two-dimensional subspace of $L^2_{2\pi}$ spanned by the two orthonormal functions $t \mapsto e^{\pm int}$ also has an orthonormal basis given by the two functions $t \mapsto \sqrt{2}\cos(nt)$ and $t \mapsto \sqrt{2}\sin(nt)$.

Proposition The functions $t \mapsto 1$, $t \mapsto \sqrt{2}\cos(nt)$ ($n = 1, 2, \ldots$), and $t \mapsto \sqrt{2}\sin(nt)$ ($n = 1, 2, \ldots$) form an orthonormal basis for $L^2_{2\pi}$.

Note that, in the orthonormal basis of the above Proposition, the cosine functions (including 1) are even and the sine functions are odd. In fact, we can split the Hilbert space $L^2_{2\pi}$ as a direct sum of the subspace of even functions and the subspace of odd functions, such that the cosines form an orthonormal basis for the first subspace and the sines an orthonormal subspace for the second subspace. Let us first discuss the notion of direct sum decomposition.

1.21 Definition Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be closed linear subspaces of $\mathcal{H}$ (so $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces themselves). We say that $\mathcal{H}$ is the *(orthogonal) direct sum* of $\mathcal{H}_1$ and $\mathcal{H}_2$ (notation $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$) if the two following conditions are satisfied:

(i) The subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal to each other.
(ii) Each $v \in \mathcal{H}$ can be written as $v = v_1 + v_2$ with $v_1 \in \mathcal{H}_1$ and $v_2 \in \mathcal{H}_2$.

Ex. 1.22 Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and make $\mathcal{H} := \{(v_1, v_2) \mid v_1 \in \mathcal{H}_1,\ v_2 \in \mathcal{H}_2\}$ into an inner product space by the rules

$$(v_1, v_2) + (w_1, w_2) := (v_1 + w_1, v_2 + w_2), \quad \lambda(v_1, v_2) := (\lambda v_1, \lambda v_2),$$
$$\langle (v_1, v_2), (w_1, w_2)\rangle := \langle v_1, w_1\rangle + \langle v_2, w_2\rangle.$$

Show that $\mathcal{H}$ is a Hilbert space and that it is the direct sum of its two closed linear subspaces $\{(v, 0) \mid v \in \mathcal{H}_1\}$ and $\{(0, v) \mid v \in \mathcal{H}_2\}$.

An example of a direct space decomposition is given by the following Proposition.

1.23 Proposition The space $L^2_{2\pi}$ is the orthogonal direct sum of the two closed linear subspaces

$$L^2_{2\pi,\mathrm{even}} := \{f \in L^2_{2\pi} \mid f(t) = f(-t) \quad \text{a.e.}\},$$
$$L^2_{2\pi,\mathrm{odd}} := \{f \in L^2_{2\pi} \mid f(t) = -f(-t) \quad \text{a.e.}\}.$$

The maps $f \mapsto f|_{[0,\pi]}\colon L^2_{2\pi,\mathrm{even}} \to L^2([0,\pi])$ and $f \mapsto f|_{[0,\pi]}\colon L^2_{2\pi,\mathrm{odd}} \to L^2([0,\pi])$ are isomorphisms of Hilbert spaces, provided the inner product on $L^2([0,\pi])$ is normalized as

$$\langle f, g\rangle := \frac{1}{\pi}\int_0^\pi f(t)\,\overline{g(t)}\,dt. \tag{1.11}$$

Theorem The functions $t \mapsto 1$ and $t \mapsto \sqrt{2}\cos(nt)$ ($n = 1, 2, \ldots$) form an orthonormal basis of $L^2_{2\pi,\mathrm{even}}$, and hence also of $L^2([0,\pi])$ (with inner product (1.11)). The functions $t \mapsto \sqrt{2}\sin(nt)$ ($n = 1, 2, \ldots$) form an orthonormal basis of $L^2_{2\pi,\mathrm{odd}}$, and hence also of $L^2([0,\pi])$ (with inner product (1.11)).

Ex. 1.24 Give the proofs of the above Proposition and Theorem.

1.5 Further exercises

Ex. 1.25 Let f be the *sawtooth function*, i.e. the 2π-periodic function which is given on $(0, 2\pi)$ by:

$$f(x) := \pi - x \qquad (0 < x < 2\pi), \tag{1.12}$$

and which is arbitarary for $x = 0$. Show the following:

(a) $f \in L^2_{2\pi}$ and $\widehat{f}(n) = (in)^{-1}$ if $n \neq 0$ and $\widehat{f}(0) = 0$.

(b) $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$.

Hint Apply formula (1.9).

Ex. 1.26 Show that

$$\sum_{n=1}^{\infty} n^{-4} = \pi^4/90$$

by applying formula (1.9) to the function $f \in L^2_{2\pi}$ such that $f(t) := t^2$ for $t \in (-\pi, \pi)$.

Ex. 1.27 For $0 \neq a \in \mathbb{R}$ let $f_a \in L^2_{2\pi}$ such that $f_a(t) := e^{at}$ for $t \in (-\pi, \pi)$.

(a) Compute the Fourier coefficients of f_a.

(b) Let $a, b \in \mathbb{R}$ such that $a, b, a + b \neq 0$. Derive the identity

$$\coth(\pi a) + \coth(\pi b) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \left(\frac{1}{a - in} + \frac{1}{b + in} \right)$$

by applying (1.10) to the functions $f := f_a$ and $g := f_b$.

(c) Show that the case $a = b$ of this identity yields

$$\coth(\pi a) = \pi^{-1} \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{a - in}\,.$$

Does the right hand side converge? Is it allowed to replace $\lim_{N\to\infty} \sum_{n=-N}^{N}$ by $\sum_{n=-\infty}^{\infty}$? Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi \coth(a\pi)}{4a} - \frac{1}{4a^2} \qquad (a \neq 0).$$

This becomes the identity in Exercise 1.25(b) as $a \to 0$.

Ex. 1.28 Put $c_0(t) := 1$, $c_n(t) := \sqrt{2}\cos(nt)$ $(n = 1, 2, \ldots)$, $s_n(t) := \sqrt{2}\sin(nt)$ $(n = 1, 2, \ldots)$, and let

$$\langle c_m, s_n \rangle := \frac{1}{\pi} \int_0^{\pi} c_m(t)\, s_n(t)\, dt \qquad (m = 0, 1, 2, \ldots,\ n = 1, 2, \ldots)$$

(inner products in $L^2([0, \pi])$). Consider the $\infty \times \infty$ matrix with (real) matrix entries $\langle c_m, s_n \rangle$, (row indices m and column indices n). Show that the columns of this matrix are orthonormal, and that also the rows are orthonormal, i.e.,

$$\sum_{m=0}^{\infty} \langle c_m, s_k \rangle \langle c_m, s_l \rangle = \delta_{k,l}, \quad \sum_{n=1}^{\infty} \langle c_k, s_n \rangle \langle c_l, s_n \rangle = \delta_{k,l}.$$

Next compute the matrix entries explicitly.

Ex. 1.29 Let $f \in L^2_{2\pi}$. For $k = 1, 2, \ldots$ put $g_k(t) := f(kt)$. Then $g_k \in L^2_{2\pi}$. Express the Fourier coefficients of g_k in terms of the Fourier coefficients of f.

Ex. 1.30 Let $\sigma, \tau \in \{\pm 1\}$. Define

$$L^2_{2\pi,\sigma,\tau} := \{f \in L^2_{2\pi} \mid f(-x) = \sigma\, f(x) \quad \text{a.e.}, \quad f(\pi - x) = \tau\, f(x) \quad \text{a.e.}\}.$$

a) Show that the Hilbert space $L^2_{2\pi}$ is the direct sum of the four mutually orthogonal closed linear subspaces $L^2_{2\pi,\sigma,\tau}$ ($\sigma, \tau \in \{\pm 1\}$).
b) For each choice of σ, τ find an orthonormal basis of $L^2_{2\pi,\sigma,\tau}$. (Start with the orthonormal basis for $L^2_{2\pi,\text{even}}$ or $L^2_{2\pi,\text{odd}}$ given in §1.23.)
c) For each σ, τ give a Hilbert space isomorphism of $L^2_{2\pi,\sigma,\tau}$ with $L^2([0, \frac{1}{2}\pi])$.

Ex. 1.31 For F a function on $\mathbb{R}$ define a function f on $(0, 2\pi)$ by $f(x) := F(\cot \frac{1}{2}x)$. This establishes a one-to-one linear correspondence between functions f on $(0, 2\pi)$ and functions F on $\mathbb{R}$

(a) Show that the map $f \mapsto F$ is a Hilbert space isomorphism of $L^2((0, 2\pi); (2\pi)^{-1}\, dx)$ onto $L^2(\mathbb{R}; \pi^{-1}\, (t^2 + 1)^{-1}\, dt)$., i.e.,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} |F(t)|^2\, \frac{dt}{t^2 + 1}\,.$$

(b) Show that the orthonormal basis of $L^2((0, 2\pi); (2\pi)^{-1}\, dx)$ provided by the functions $x \mapsto e^{inx}$ ($n \in \mathbb{Z}$), is sent by the map $f \mapsto F$ to the orthonormal basis of $L^2(\mathbb{R}; \pi^{-1}\, (t^2 + 1)^{-1}\, dt)$ given by the functions $t \mapsto (\frac{t+i}{t-i})^n$.

Ex. 1.32 Let V be the linear space of piecewise linear continuous functions on the closed bounded interval $[a, b]$. So, $f \in V$ iff $f \in C([a, b])$ and there is a partition $a = a_0 < a_1 < a_2 \ldots < a_n = b$ such that the restriction of f to $[a_{i-1}, a_i]$ is linear for $i = 1, \ldots, n$. Let $V_0 := \{f \in V \mid f(a) = f(b) = 0\}$. Show that V is dense in $C([a, b])$ and that V_0 is dense in $L^p([a, b])$ ($1 \le p < \infty$).

2 L^1 theory

2.1 Growth rates of Fourier coefficients

2.1 The Fourier coefficients $\widehat{f}(n)$ $(n \in \mathbb{Z})$ of a function $f \in L^1_{2\pi}$ were defined in formula (1.8). It follows from this formula that

$$|\widehat{f}(n)| \le \|f\|_1 \qquad (f \in L^1_{2\pi},\ n \in \mathbb{Z}). \tag{2.1}$$

Hence

$$\|\widehat{f}\|_\infty \le \|f\|_1 \quad \text{where} \quad \|\widehat{f}\|_\infty := \sup_{n \in \mathbb{Z}} |\widehat{f}(n)|.$$

So $\widehat{f} \in \ell^\infty(\mathbb{Z})$ if $f \in L^1_{2\pi}$. Here $\ell^\infty(\mathbb{Z}) := \{(\gamma_n)_{n\in\mathbb{Z}} \mid \sup_{n\in\mathbb{Z}} |\gamma_n| < \infty\}$, a Banach space.

Ex. 2.2 Show that the map $f \mapsto \widehat{f}\colon L^1_{2\pi} \to \ell^\infty(\mathbb{Z})$ is a bounded linear operator. Also determine its operator norm.

2.3 In §1.15 we gave the so-called Riemann-Lebesgue Lemma in the L^2-case. It remains valid for the L^1-case:

Theorem (*Riemann-Lebesgue Lemma*) If $f \in L^1_{2\pi}$ then $\lim_{|n|\to\infty} \widehat{f}(n) = 0$.

Proof Let $f \in L^1_{2\pi}$. Let $\varepsilon > 0$. We have to look for a natural number N such that $|\widehat{f}(n)| < \varepsilon$ if $|n| \ge N$. Since $L^2_{2\pi}$ is dense in $L^1_{2\pi}$ (see §1.10), there exists a function $g \in L^2_{2\pi}$ such that $\|f - g\|_1 < \frac{1}{2}\varepsilon$. By the Riemann-Lebesgue Lemma for L^2 (see §1.15) there exists $N \in \mathbb{N}$ such that: $|n| \ge N \;\Rightarrow\; |\widehat{g}(n)| < \frac{1}{2}\varepsilon$. Hence, if $|n| \ge N$ then

$$|\widehat{f}(n)| \le |\widehat{f}(n) - \widehat{g}(n)| + |\widehat{g}(n)| \le \|f - g\|_1 + |\widehat{g}(n)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

where we used the inequality (2.1). □

Ex. 2.4 Define the linear space $c_0(\mathbb{Z})$ by

$$c_0(\mathbb{Z}) := \{\{c_n\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}) \mid \lim_{|n|\to\infty} c_n = 0\}.$$

Show that $c_0(\mathbb{Z})$ is a closed linear subspace of $\ell^\infty(\mathbb{Z})$. So, in particular, $c_0(\mathbb{Z})$ becomes a Banach space. Note also that $f \mapsto \widehat{f}\colon L^1_{2\pi} \to c_0(\mathbb{Z})$ is a bounded linear map.

2.5 In addition to the space $\mathcal{C}_{2\pi}$ we will deal, for $k = 1, 2, \ldots$ or ∞, with the linear space $\mathcal{C}^k_{2\pi}$ consisting of 2π-periodic C^k-functions on $\mathbb{R}$. By $\mathcal{C}^0_{2\pi}$ we will just mean $\mathcal{C}_{2\pi}$. Elementary integration by parts yields:

$$(f')\widehat{\ }(n) = in\,\widehat{f}(n) \qquad (f \in \mathcal{C}^1_{2\pi},\ n \in \mathbb{Z}). \tag{2.2}$$

This is an important result. Corresponding to the differentiation operator $f \mapsto f'$ acting on 2π-periodic functions we have on the Fourier transform side the operator multiplying a function of n by in. Iteration of (2.2) yields:

$$(f^{(k)})\widehat{\ }(n) = (in)^k\,\widehat{f}(n) \qquad (f \in \mathcal{C}^k_{2\pi},\ k = 0, 1, 2, \ldots,\ n \in \mathbb{Z}). \tag{2.3}$$

For $f \in \mathcal{C}_{2\pi}$ the Riemann-Lebesgue Lemma gives that $\widehat{f}(n) = o(1)$ as $|n| \to \infty$, a modest rate of decline for the Fourier coefficients. It turns out that a higher order of differentiability of a 2π-periodic function f implies a faster decline of $\widehat{f}(n)$ as $|n| \to \infty$:

Theorem (a) Let $k \in \{0,1,2,\dots\}$. If $f \in \mathcal{C}_{2\pi}^k$ then $\widehat{f}(n) = o(|n|^{-k})$ as $|n| \to \infty$.
(b) If $f \in \mathcal{C}_{2\pi}^\infty$ then $\widehat{f}(n) = \mathcal{O}(|n|^{-k})$ as $|n| \to \infty$ for all $k \in \{0,1,2,\dots\}$.

So we see that for 2π-periodic C^∞-functions the Fourier coefficients decrease faster in absolutele value to zero than any inverse power of $|n|$. We then say that the Fourier coefficients are *rapidly decreasing*.

Proof of Theorem Part (b) follows immediately from part (a). For the proof of part (a) we use equations (2.3) and the Riemann-Lebesgue Lemma:

$$|\widehat{f}(n)| = |n|^{-k}\,|(f^{(k)})\widehat{\ }(n)| = |n|^{-k}\,o(1) \quad \text{as } n \to \infty,$$

since $f^{(k)} \in \mathcal{C}_{2\pi}$ if $f \in \mathcal{C}_{2\pi}^k$. □

Ex. 2.6 Show that (2.2) still holds if $f \in \mathcal{C}_{2\pi}$ with piecewise continuous derivative.

Ex. 2.7 Let $f \in \mathcal{C}_{2\pi}$. Suppose that f can be extended to a function analytic on the open strip $\{z \in \mathbb{C} \mid |\operatorname{Im} z| < K\}$ and continuous on the closed strip $\{z \in \mathbb{C} \mid |\operatorname{Im} z| \le K\}$. Show that $|\widehat{f}(n)| = \mathcal{O}(e^{-K|n|})$ as $|n| \to \infty$.
Hint If $z \in \mathbb{C}$ and $|\operatorname{Im} z| \le K$ then $f(z+2\pi) = f(z)$. Next show by contour integration that

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{a-\pi}^{a+\pi} f(z)\,e^{-inz}\,dz \qquad (n \in \mathbb{Z},\ |\operatorname{Im} a| \le K).$$

2.8 In the previous sections we derived the behaviour of the Fourier coefficients $\widehat{f}(n)$ from the behaviour of the function f. Now we consider the inverse problem: Let coefficients γ_n with certain behaviour be given. Find a 2π-periodic function f such that $\widehat{f}(n) = \gamma_n$ and give the behaviour of f.

We say that the doubly infinite sequence $(\gamma_n)_{n\in\mathbb{Z}}$ is in $\ell^1(\mathbb{Z})$ if $\|(\gamma_n)\|_1 := \sum_{n=-\infty}^{\infty} |\gamma_n| < \infty$.

Theorem Let $(\gamma_n) \in \ell^1(\mathbb{Z})$. Put

$$f(x) := \sum_{n=-\infty}^{\infty} \gamma_n\,e^{inx} \qquad (x \in \mathbb{R}), \tag{2.4}$$

well defined because the series converges absolutely. Then $f \in \mathcal{C}_{2\pi}$ and $\widehat{f}(n) = \gamma_n$ $(n \in \mathbb{Z})$. Also $\|f\|_\infty \le \|(\gamma_n)\|_1$.

Proof The absolute convergence of the series on the right hand side of (2.4) is uniform for $x \in \mathbb{R}$ because of the Weierstrass test, since $|\gamma_n\,e^{inx}| \le |\gamma_n|$ and $\sum_{n=-\infty}^{\infty} |\gamma_n| < \infty$. The sum of a uniform convergent series of continuous functions is continuous. Since the terms of the series are 2π-periodic in x, the same holds for the sum function. Thus $f \in \mathcal{C}_{2\pi}$. The inequality $\|f\|_\infty \le \|(\gamma_n)\|_1$ follows by taking absolute values on both sides of (2.4), and by dominating the absolute value of the sum on the right hand side by the sum of the absolute values of the terms. Finally, we derive that

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} \gamma_m\,e^{imx}\,e^{-inx}\right)dx = \sum_{m=-\infty}^{\infty} \gamma_m \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx\right) = \gamma_n,$$

where the second equality is permitted because the integral over a bounded interval of a uniformly convergent sum of continuous functions equals the sum of the integrals of the terms. □

Ex. 2.9 Let $f(x)$ be given by (2.4). Prove the following statements. Use for (a) and (b) the theorem about differentiation of a series of functions on a bounded interval for which the series of derivatives is uniformly convergent.

(a) If $\sum_{n=-\infty}^{\infty} |n|\,|\gamma_n| < \infty$ then $f \in \mathcal{C}^1_{2\pi}$.

(b) Let $k \in \{0,1,2,\ldots\}$. If $\sum_{n=-\infty}^{\infty} |n|^k\,|\gamma_n| < \infty$ then $f \in \mathcal{C}^k_{2\pi}$.

(c) Let $\lambda > 1$. If $\gamma_n = \mathcal{O}(|n|^{-\lambda})$ as $|n| \to \infty$ then $f \in \mathcal{C}^k_{2\pi}$ for all integer k such that $0 \le k < \lambda - 1$.

(d) If $\gamma_n = \mathcal{O}(|n|^{-k})$ as $|n| \to \infty$ for all $k \in \{0,1,2,\ldots\}$ then $f \in \mathcal{C}^\infty_{2\pi}$.

(e) Let $K > 0$. If $\gamma_n = \mathcal{O}(e^{-K|n|})$ as $|n| \to \infty$ then f can be extended to an analytic function on the strip $\{z \in \mathbb{C} \mid |\operatorname{Im} z| < K\}$. Compare with Exercise 2.7

2.10 Remark Observe that the Fourier images of $L^2_{2\pi}$ and of $\mathcal{C}^\infty_{2\pi}$ can be completely characterized. Namely, for a given sequence $(\gamma_n)_{n\in\mathbb{Z}}$ we have:

$\gamma_n = \widehat{f}(n)$ $(n \in \mathbb{Z})$ for some $f \in L^2_{2\pi}$ iff $\sum_{n=-\infty}^{\infty} |\gamma_n|^2 < \infty$;

$\gamma_n = \widehat{f}(n)$ $(n \in \mathbb{Z})$ for some $f \in \mathcal{C}^\infty_{2\pi}$ iff $\gamma_n = \mathcal{O}(|n|^{-k})$ as $|n| \to \infty$ for all $k \in \{0,1,2,\ldots\}$.

However, such a characterization is not possible for the Fourier images of $\mathcal{C}_{2\pi}, \mathcal{C}^1_{2\pi}, \mathcal{C}^2_{2\pi}, \ldots$ and of $L^1_{2\pi}$.

Ex. 2.11 Let $\gamma_0 := 0$ and $\gamma_n := |n|^{-\lambda}$ for $n \in \mathbb{Z}\backslash\{0\}$. Then it is known in the literature for which real values of λ there exists $f \in L^1_{2\pi}$ such that $\widehat{f}(n) = \gamma_n$ $(n \in \mathbb{Z})$. For some values of λ this follows from a non-trivial theorem (see sections 7.17 and 7.19 in van Rooij). However, for some other values of λ the answer is immediate. Give the answer in these cases.

2.2 Fubini's Theorem

2.12 For the treatment of convolution we will need Fubini's Theorem. The Proposition below formulates Fubini's theorem for nonnegative measurable functions, see also syll. Integratietheorie, Sections 5.6–5.8. Next the Theorem below will deal with complex-valued measurable functions. Below we will work with σ-finite measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ and their *product space* $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$, again a σ-finite measure space.

It is helpful to read first the Theorem below for the case that X and Y are bounded intervals, μ and ν are Lebesgue measure, and the function $h\colon X \times Y \to \mathbb{C}$ is continuous. Then all integrals can also be considered as Riemann integrals and the assumption in the Theorem concerning the absolute convergence of a repeated integral is automatic.

The general Proposition and Theorem below are much more delicate, because measure spaces may be σ-finite rather than finite and because measurable rather than continuous functions are considered.

2.13 Proposition Let the function $h\colon X \times Y \to [0,\infty]$ be measurable with respect to the σ-algebra $\mathcal{A} \otimes \mathcal{B}$. Then:

(a) The function $y \mapsto \int_X h(x,y)\,d\mu(x)$ is measurable on $\mathcal{B}$;

(b) The function $x \mapsto \int_Y h(x,y)\,d\nu(y)$ is measurable on $\mathcal{A}$;

(c) We have

$$\int_{X\times Y} h(x,y)\,d(\mu\times\nu)(x,y) = \int_X \left(\int_Y h(x,y)\,d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X h(x,y)\,d\mu(x)\right) d\nu(y). \tag{2.5}$$

Note that all integrals considered in (2.5) are integrals of measurable functions which take values on $[0,\infty]$. The integral of such a function is well-defined and it will yield a number in $[0,\infty]$.

Consider next a function $h\colon X\times Y \to \mathbb{C}$ (so not necessarily non-negative) which is measurable with respect to the σ-algebra $\mathcal{A}\otimes\mathcal{B}$. Then the nonnegative function $|h|$ is measurable with respect to $\mathcal{A}\otimes\mathcal{B}$, so the above Proposition will hold with $h(x,y)$ replaced by $|h(x,y)|$. This we will need in the next theorem.

2.14 Theorem (see Rudin, Theorem 7.8) Let the function $h\colon X\times Y\to\mathbb{C}$ be measurable with respect to the σ-algebra $\mathcal{A}\otimes\mathcal{B}$. Suppose that one of the two inequalities below is valid.

$$\int_Y \left(\int_X |h(x,y)|\,d\mu(x)\right) d\nu(y) < \infty. \tag{2.6}$$

$$\int_X \left(\int_Y |h(x,y)|\,d\nu(y)\right) d\mu(x) < \infty. \tag{2.7}$$

(So the other inequality is also valid by the above Proposition.) Then:

(a) $\int_X |h(x,y)|\,d\mu(x) < \infty$ for y outside some set $Y_0\subset Y$ of ν-measure 0 and the function $y\mapsto \int_X h(x,y)\,d\mu(x)\colon Y\backslash Y_0\to\mathbb{C}$, arbitrarily extended to a complex-valued function on Y, is ν-integrable on Y.

(b) $\int_Y |h(x,y)|\,d\nu(y) < \infty$ for x outside some set $X_0\subset X$ of μ-measure 0 and the function $x\mapsto \int_Y h(x,y)\,d\nu(y)\colon X\backslash X_0\to\mathbb{C}$, arbitrarily extended to a complex-valued function on X, is μ-integrable on X.

(c) Formula (2.5) is valid.

The crucial condition to be checked in this Theorem is inequality (2.6) or (2.7). The important conclusion in the Theorem is that (almost) all integrals in (2.5) are well-defined and that the equalities in (2.5) (in particular the second one) hold.

2.3 Convolution

2.15 Definition The *convolution product* of two 2π-periodic functions f,g is a function $f*g$ on $\mathbb{R}$ given by

$$(f*g)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,g(x-t)\,dt \qquad (x\in\mathbb{R}), \tag{2.8}$$

provided the right hand side is well-defined. Then, clearly, $f*g$ is also 2π-periodic and $f*g = g*f$. Check these two properties by simple transformations of the integration variable in (2.8)

For deriving further properties of the convolution product, the easiest case is when f,g are continuous:

2.16 Proposition If $f, g \in \mathcal{C}_{2\pi}$ then $f * g \in \mathcal{C}_{2\pi}$ and

$$(f * g)\hat{}(n) = \widehat{f}(n)\,\widehat{g}(n) \qquad (n \in \mathbb{Z}). \tag{2.9}$$

Proof For the proof of the continuity (in fact uniform continuity) of $f * g$ observe that

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,(g(x-t) - g(y-t))\,dt\right| \le \|f\|_1 \sup_{t\in\mathbb{R}} |g(x-t) - g(y-t)|.$$

Let $\varepsilon > 0$. Since any periodic continuous function is uniformly continuous on $\mathbb{R}$ (why?), there exists $\delta > 0$ such that $|g(x-t) - g(y-t)| < \varepsilon$ for all $t \in \mathbb{R}$ if $|x - y| < \delta$. Hence $|(f * g)(x) - (f * g)(y)| < \varepsilon\,\|f\|_1$ if $|x - y| < \delta$.

For the proof of (2.9) use Fubini's theorem in the case of continuous functions on a bounded interval. Thus

$$\begin{aligned}(f * g)\hat{}(n) &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(t)\,g(x-t)\,dt\right) e^{-inx}\,dx\\ &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi} f(t)\,e^{-int}\left(\int_{-\pi}^{\pi} g(x-t)\,e^{-in(x-t)}\,dx\right) dt\\ &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi} f(t)\,e^{-int}\,\widehat{g}(n)\,dt = \widehat{f}(n)\,\widehat{g}(n).\end{aligned}$$

□

Ex. 2.17 Let $f(x) := e^{imx}$, $g(x) := e^{inx}$ $(m, n \in \mathbb{Z})$. Compute $f * g$. Also check that the result agrees with equality (2.9).

Ex. 2.18 Prove by Fubini's theorem in the case of continuous functions on a bounded interval the following. If $f, g, h \in \mathcal{C}_{2\pi}$ then

$$(f * g) * h = f * (g * h) \qquad \text{(associativity)}. \tag{2.10}$$

2.19 Definition-Theorem Let $f, g \in \mathcal{L}^1_{2\pi}$. Let $(f * g)(x)$ be defined by (2.8) for those $x \in \mathbb{R}$ for which the integral on the right hand side of (2.8) converges absolutely.

(a) For almost all $x \in \mathbb{R}$ the integral on the right hand side of (2.8) converges absolutely. Extend $f * g$ to a (2π-periodic) function on $\mathbb{R}$ by choosing arbitrary values of $(f * g)(x)$ on the set of measure zero where $f(x)$ is not yet defined by (2.8). Then $f * g \in \mathcal{L}^1_{2\pi}$ and

$$\|f * g\|_1 \le \|f\|_1\,\|g\|_1. \tag{2.11}$$

(b) The equivalence class of $f * g$ only depends on the equivalence classes of f and g. In other words, for $f, g \in L^1_{2\pi}$ the convolution product $f * g$ is well-defined as an element of $L^1_{2\pi}$.

(c) For $f, g \in L^1_{2\pi}$, formula (2.9) is valid.

(d) For $f, g, h \in L^1_{2\pi}$, the associativity property (2.10) is valid.

Proof Let $f, g \in \mathcal{L}^1_{2\pi}$. First observe that

$$\begin{aligned}&\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} |f(t)\,g(x-t)|\,dx\right) dt = \frac{1}{4\pi^2}\int_{-\pi}^{\pi} |f(t)|\left(\int_{-\pi}^{\pi} |g(x-t)|\,dx\right) dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|\,\|g\|_1\,dt = \|f\|_1\,\|g\|_1 < \infty.\end{aligned} \tag{2.12}$$

Hence, by Fubini's Theorem 2.14 we have that

$$\int_{-\pi}^{\pi} |f(t)\, g(x-t)|\, dt < \infty$$

for almost all x. Thus, $(f * g)(x)$ is well-defined by (2.8) for almost all x, and $f * g$ is measurable on $\mathbb{R}$. Again by Theorem 2.14, it follows that

$$\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} f(t)\, g(x-t)\, dt \right| dx < \infty.$$

So $f * g \in \mathcal{L}^1_{2\pi}$. For the proof of (2.11) use that

$$\|f * g\|_1 \le \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |f(t)\, g(x-t)|\, dt \right) dx = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |f(t)\, g(x-t)|\, dx \right) dt$$

by Proposition 2.13, and combine with (2.12).

For the proof of part (b) let $f, g \in \mathcal{L}^1_{2\pi}$ with $\|f\|_1 = 0$ or $\|g\|_1 = 0$. Then $\|f * g\|_1 = 0$ because of (2.11).

For the proof of (c) repeat the proof of (2.9) for the continuous case. We can now justify the interchange of integration order in the second equality of that proof by Fubini's Theorem 2.14, because

$$\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |f(t)\, g(x-t)\, e^{-inx}|\, dx \right) dt < \infty.$$

This last inequality is true because the left hand side equals $\|\, |f| * |g|\, \|_1 < \infty$.

Finally (d) can be proved similarly as in Exercise 2.18, with justification of the interchange of integration order by Theorem 2.14. □

2.20 Proposition If $f \in L^1_{2\pi}$ and $g \in \mathcal{C}_{2\pi}$ then $f * g \in \mathcal{C}_{2\pi}$ and

$$\|f * g\|_\infty \le \|f\|_1\, \|g\|_\infty. \tag{2.13}$$

Proof The proof given in Proposition 2.16 that $f * g \in \mathcal{C}_{2\pi}$ if $f, g \in \mathcal{C}_{2\pi}$, still works if the assumption on f is relaxed to $f \in L^1_{2\pi}$. The proof of inequality (2.13) is straightforward. □

Ex. 2.21 Show: If $f, g \in L^2_{2\pi}$ then $f * g \in \mathcal{C}_{2\pi}$ and

$$\|f * g\|_\infty \le \|f\|_2\, \|g\|_2. \tag{2.14}$$

Hint First show that $\sup_{x \in \mathbb{R}} |(f*g)(x)| \le \|f\|_2\, \|g\|_2$ if $f, g \in L^2_{2\pi}$. Next use this inequality and the fact that $f * g$ is continuous if $f, g \in \mathcal{C}_{2\pi}$ (see Proposition 2.16), together with the density of $\mathcal{C}_{2\pi}$ in $L^2_{2\pi}$ and the completeness of the Banach space $\mathcal{C}_{2\pi}$ with respect to the sup norm.

2.4 *Further exercises*

Ex. 2.22 Show that

$$\sum_{n=0}^{\infty} 2^{-n} \cos(nx) = \frac{4 - 2\cos x}{5 - 4\cos x} \qquad (x \in \mathbb{R}).$$

Compare with Exercise 2.9(e).

Ex. 2.23 Let $A \in \mathbb{C}$ such that $|A| \neq 1$. Let $f(x) := (A + e^{ix})^{-1} \quad (x \in \mathbb{R})$. Compute the Fourier coefficients $\widehat{f}(n)$.

Ex. 2.24 Let $f \in L^1_{2\pi}$. Express $\widehat{g}(n)$ in terms of the $\widehat{f}(m)$ if:
(a) $g(x) = f(x + a) \quad (a \in \mathbb{R})$;
(b) $g(x) = f(-x)$;
(c) $g(x) = \overline{f(x)}$;
(d) $g(x) = \overline{f(-x)}$.

Ex. 2.25 Let $f \in L^1_{2\pi}$, $g \in \mathcal{C}^1_{2\pi}$. Show that $f * g \in \mathcal{C}^1_{2\pi}$ and that $(f * g)' = f * g'$.

3 The Dirichlet kernel

3.1 Definition of the Dirichlet kernel

3.1 Let $f \in L^1_{2\pi}$. The formal doubly infinite series

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{inx} \tag{3.1}$$

is called the *Fourier series* of f. We will see that this series converges in a certain sense to $f(x)$, but the type of convergence will depend on the nature of f. Saying that the series (3.1) converges in some sense to $f(x)$ amounts to the same as saying that the sequence of *partial Fourier sums*

$$S_N(x) = (S_N[f])(x) := \sum_{n=-N}^{N} \widehat{f}(n)\, e^{inx} \qquad (N = 0, 1, 2, \ldots,\ x \in \mathbb{R}) \tag{3.2}$$

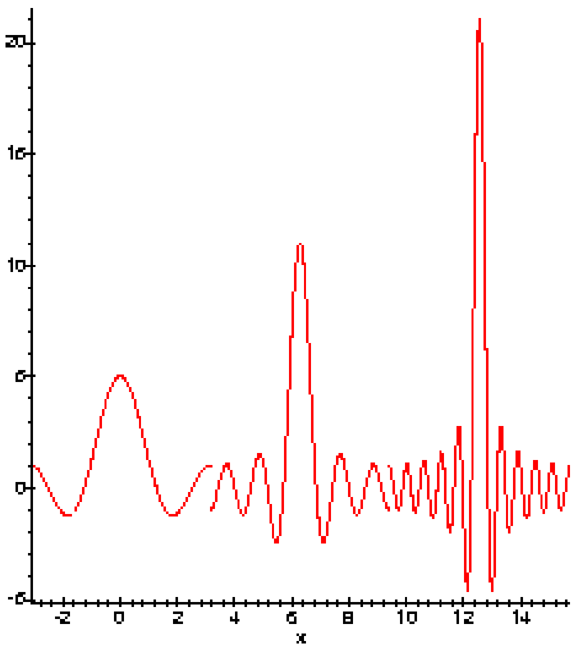
converges in some sense to $f(x)$ as $N \to \infty$. Therefore it is important to examine the functions $S_N[f]$ in more detail.

3.2 Definition-Proposition The partial Fourier sum (3.2) can be written as

$$(S_N[f])(x) = (f * D_N)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_N(x-t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\, D_N(t)\, dt, \tag{3.3}$$

where D_N is the *Dirichlet kernel*:

$$D_N(x) := \sum_{n=-N}^{N} e^{inx} = \begin{cases} \dfrac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)} & (x \notin 2\pi\mathbb{Z}), \\ 2N+1 & (x \in 2\pi\mathbb{Z}). \end{cases} \tag{3.4}$$



The Dirichlet Kernels D_2, D_5, D_{10}, centered at 0, 2π, 4π respectively.
(See also the graph of D_N in van Rooij, p.13.)

Proof Observe from (3.2) that

$$(S_N[f])(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(\sum_{n=-N}^{N} e^{in(x-t)}\right)dt.$$

From (3.4) we see that $D_N \in \mathcal{C}_{2\pi}$ and that it is an even function. The third equality in (3.3) uses these last facts. □

3.2 *Criterium for convergence of Fourier series in one point*

3.3 We will need the following three easy consequences of (3.4):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1, \tag{3.5}$$

$$|D_N(x)| \le \frac{1}{|\sin(\frac{1}{2}x)|} \le \frac{\pi}{|x|} \quad (0 < |x| \le \pi), \tag{3.6}$$

and

$$D_N(x) = \frac{e^{\frac{1}{2}ix}}{2i\sin(\frac{1}{2}x)}\,e^{iNx} - \frac{e^{-\frac{1}{2}ix}}{2i\sin(\frac{1}{2}x)}\,e^{-iNx} \quad (x \notin 2\pi\mathbb{Z}). \tag{3.7}$$

3.4 Theorem Let $f \in \mathcal{L}^1_{2\pi}$, $a \in \mathbb{R}$. Suppose that one of the two following conditions holds:

(a) There are $M, \alpha, \delta > 0$ such that

$$|f(a+t) - f(a)| \le M|t|^\alpha \quad \text{for } -\delta < t < \delta. \tag{3.8}$$

(b) f is continuous in a and also right and left differentiable in a.

Then we have:

$$\lim_{N\to\infty} (S_N[f])(a) = f(a).$$

Remark Condition (a) of Theorem 3.4 implies continuity of f in a. Functions f satisfying condition (a) are said to be *Hölder continuous of order* α in a. (The terminology *Lipschitz continuous of order* α is equally common.) If (b) holds then $\frac{|f(a+t)-f(a)|}{|t|}$ is bounded for t in some neighbourhood of 0 since it has a limit as $t \downarrow 0$ and as $t \uparrow 0$. Thus if (b) holds then (a) is satisfied with $\alpha = 1$. So, we only need to prove the Theorem under condition (a).

Proof of Theorem 3.4 Assume condition (a). Let $\varepsilon > 0$. We want to show that $|S_N(a) - f(a)| < \varepsilon$ for N sufficiently large. It follows from (3.3), (3.5) and (3.7) that

$$\begin{aligned} S_N(a) - f(a) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(a+t) - f(a))\,D_N(t)\,dt \\ &= \int_{-\pi}^{\pi} g_{a,+}(t)e^{iNt}\,dt - \int_{-\pi}^{\pi} g_{a,-}(t)e^{-iNt}\,dt, \end{aligned} \tag{3.9}$$

where

$$g_{a,\pm}(t) = \frac{1}{2\pi}\,\frac{e^{\pm\frac{1}{2}it}}{2i\sin(\frac{1}{2}t)}\,(f(a+t) - f(a)) \quad (0 < |t| < \pi). \tag{3.10}$$

For $0 < |t| < \delta$ we can estimate $|g_{a,\pm}(t)| < \frac{1}{4}M|t|^{\alpha-1}$ (use (3.6) and (3.8)). For $\delta < |t| < \pi$ we can estimate $|g_{a,\pm}(t)| < (4\pi\sin(\frac{1}{2}\delta))^{-1}|f(a+t) - f(a)|$. Hence the functions $g_{a,\pm}$ are in $\mathcal{L}^1([-\pi,\pi])$. By the Riemann-Lebesgue Lemma (Theorem 2.3) it follows that (3.9) tends to 0 as $N \to \infty$. □

3.5 Theorem Let $f \in \mathcal{L}^1_{2\pi}$, let $a \in \mathbb{R}$, and suppose that the limits

$$f(a_+) := \lim_{x \downarrow a} f(x), \qquad f(a_-) := \lim_{x \uparrow a} f(x)$$

exist. Suppose that one of the two following conditions holds:

(a) There are $M, \alpha, \delta > 0$ such that

$$|f(a+t) - f(a_+)| \le Mt^\alpha \quad \text{and} \quad |f(a-t) - f(a_-)| \le Mt^\alpha \quad \text{for } 0 < t < \delta.$$

(b) f is right and left differentiable in a in the sense that the following two limits exist:

$$f'(a_+) := \lim_{t \downarrow 0} \frac{f(a+t) - f(a_+)}{t}, \qquad f'(a_-) := \lim_{t \downarrow 0} \frac{f(a-t) - f(a_-)}{-t}.$$

Then we have:

$$\lim_{N \to \infty} (S_N[f])(a) = \tfrac{1}{2}(f(a_+) + f(a_-)).$$

Proof Since D_N is an even function, we conclude from (3.3) that

$$S_N(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tfrac{1}{2}(f(a+t) + f(a-t))\, D_N(t)\, dt.$$

Hence

$$S_N(a) - \tfrac{1}{2}(f(a_+) + f(a_-)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{f(a+t) + f(a-t)}{2} - \frac{f(a_+) + f(a_-)}{2} \right) D_N(t)\, dt.$$

Now put $g(t) := \frac{1}{2}(f(a+t) + f(a-t))$ for $0 < |t| \le \pi$ and $g(0) := \lim_{t \to 0} g(t) = \frac{1}{2}(f(a_+) + f(a_-))$. Then condition (a) or (b) for f at the point a implies the corresponding condition in Theorem 3.4 for g at the point 0. Now apply Theorem 3.4 to g at 0. □

3.6 Corollary (*localization principle*) Let $f, g \in \mathcal{L}^1_{2\pi}$, let $a \in \mathbb{R}$, and suppose that $f(x) = g(x)$ for x in a certain neighbourhood of a. Then precisely one of the following two alternatives holds for the two sequences $\big((S_N[f])(a)\big)_{N=0}^\infty$ and $\big((S_N[g])(a)\big)_{N=0}^\infty$:

(a) The two sequences both converge and have the same limit.

(b) The two sequences both diverge.

Proof Apply Theorem 3.4 to the function $f - g$ at a. Since $f - g$ is identically zero in a neighbourhood of a, we conclude that $\lim_{N \to \infty} (S_N[f-g])(a) = 0$. Hence

$$\lim_{N \to \infty} \big((S_N[f])(a) - (S_N[g])(a)\big) = 0.$$

□

Thus the convergence and possible sum of the Fourier series of a function $f \in \mathcal{L}^1_{2\pi}$ at a point a is completely determined by the restriction of f to an arbitrarily small neighbourhood of f. If we leave a given function f unchanged on such a neighbourhood, but change it elsewhere, then the Fourier coefficients $\widehat{f}(n)$ may become completely different, but convergence or divergence and the possible sum of the Fourier series at a will remain the same. This explains why the above Corollary is called localization principle.

Ex. 3.7 Let f be the *sawtooth function* defined in Exercise 1.25, formula (1.12). Put $f(x) := 0$ for $x \in 2\pi\mathbb{Z}$.

(a) Prove that

$$f(x) = \lim_{N\to\infty} (S_N[f])(x) = 2\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} \tag{3.11}$$

with pointwise convergence for all $x \in \mathbb{R}$.

(b) Show by substitution of $x := \pi/2$ or $x := \pi/4$ in (1.12), (3.11) that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \tag{3.12}$$

$$\frac{\pi}{\sqrt{8}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \cdots. \tag{3.13}$$

Ex. 3.8 Let $0 < \delta < \pi$. Let f be a 2π-periodic function which is given on $[-\pi, \pi]$ by:

$$f(x) := \begin{cases} 1 & \text{if } x \in [-\delta, \delta], \\ 0 & \text{if } \delta < |x| \le \pi. \end{cases}$$

(a) Determine $\widehat{f}(n)$ $(n \in \mathbb{Z})$.

(b) Determine

$$\frac{\delta}{\pi} + \lim_{N\to\infty} \sum_{0<|n|\le N} \frac{\sin(n\delta)}{\pi n}\, e^{inx}$$

provided the limit exists.

Ex. 3.9 Let f be a 2π-periodic function which is C^1 outside $2\pi\mathbb{Z}$ and such that $f(0_+)$, $f(0_-)$, $f'(0_+)$, $f'(0_-)$ exist in the sense of Theorem 3.5. Prove that $\widehat{f}(n) = \mathcal{O}(|n|^{-1})$ as $|n| \to \infty$, but not necessarily $\widehat{f}(n) = o(|n|^{-1})$ as $|n| \to \infty$.
Hint Use Theorem 2.5 and Exercise 1.25(a).

Ex. 3.10 Let f be a 2π-periodic function which is C^1 outside a discrete set X having the property that $X \cap [-\pi, \pi)$ is a finite set. Assume that $f(x_+), f(x_-), f'(x_+), f'(x_-)$ exist in the sense of Theorem 3.5 for each $x \in X$. Prove that $\widehat{f}(n) = \mathcal{O}(|n|^{-1})$ as $|n| \to \infty$.

3.3 Continuous functions with non-convergent Fourier series

We will now show in a “soft” way (i.e., by using functional analytic methods) that there exists a 2π-periodic continuous function for which the Fourier series does not converge everywhere. For this we need a “hard” estimate of the Dirichlet kernel in the two lemmas below. This estimate will also be useful elsewhere. We refer to Stein & Shakarchi for an explicit example of a 2π-periodic continuous function with not everywhere converging Fourier series.

3.11 Lemma Let

$$\theta(x) := \cot(\tfrac{1}{2}x) - 2x^{-1} \quad (0 < |x| < 2\pi). \tag{3.14}$$

Then

$$D_N(x) = \frac{2\sin(Nx)}{x} + \theta(x)\,\sin(Nx) + \cos(Nx) \quad (0 < |x| < 2\pi). \tag{3.15}$$

Proof It follows from (3.4) that, for $0 < |x| < 2\pi$,

$$D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)} = \sin(Nx)\,\cot(\tfrac{1}{2}x) + \cos(Nx).$$

□

Ex. 3.12 Show the following:
(a) $\lim_{x\to 0}\theta(x) = 0$;
(b) $\lim_{x\to 0}\theta'(x) = -\frac{1}{6}$;
(c) Put $\theta(x) := 0$. Then θ is differentiable and strictly decreasing on $(-2\pi, 2\pi)$.
(d) $\theta(\pi) = -2\pi^{-1}, \quad \theta(-\pi) = 2\pi^{-1}, \quad \sup_{-\pi\le x\le\pi}|\theta(x)| = 2\pi^{-1}$.

3.13 Lemma We have

$$\int_{-\pi}^{\pi} |D_N(x)|\,dx = 8\pi^{-1}\log N + \mathcal{O}(1) \quad \text{as } N \to \infty.$$

In particular,

$$\lim_{N\to\infty}\int_{-\pi}^{\pi} |D_N(x)|\,dx = \infty.$$

Proof It follows from (3.15) and Exercise 3.12(d) that

$$\big||D_N(x)| - |2x^{-1}\sin(Nx)|\big| \le |D_N(x) - 2x^{-1}\sin(Nx)| \le 2\pi^{-1} + 1 \quad (0 < |x| < \pi).$$

Hence

$$\int_{-\pi}^{\pi} |D_N(x)|\,dx = 2\int_{-\pi}^{\pi}\left|\frac{\sin(Nx)}{x}\right| dx + \mathcal{O}(1) \quad \text{as } N\to\infty.$$

Next we write

$$\begin{aligned}
&2\int_{-\pi}^{\pi}\left|\frac{\sin(Nx)}{x}\right| dx = 4\int_0^N \frac{|\sin(\pi x)|}{x}\,dx\\
&= 4\sum_{n=0}^{N-1}\int_0^1 \frac{\sin(\pi x)}{x+n}\,dx\\
&= 4\sum_{n=1}^{N-1}\int_0^1 \frac{\sin(\pi x)}{x+n}\,dx + \mathcal{O}(1) \quad \text{as } N\to\infty\\
&\overset{(1)}{=} 4\pi^{-1}\sum_{n=1}^{N-1}(n^{-1} + (n-1)^{-1}) - 4\pi^{-1}\sum_{n=1}^{N}\int_0^1 \frac{\cos(\pi x)}{(x+n)^2}\,dx + \mathcal{O}(1) \quad \text{as } N\to\infty\\
&\overset{(2)}{=} 8\pi^{-1}\sum_{n=1}^{N} n^{-1} + \mathcal{O}(1) \quad \text{as } N\to\infty\\
&\overset{(3)}{=} 8\pi^{-1}\log N + \mathcal{O}(1) \quad \text{as } N\to\infty.
\end{aligned}$$

In equality (1) we used integration by parts. In equality (2) we majorized $|\int_0^1 (x+n)^{-2}\cos(\pi x)\,dx|$ by n^{-2} and we used that $\sum_{n=1}^N n^{-2} = \mathcal{O}(1)$ as $N\to\infty$. In equality (3) we used that $\sum_{n=1}^N n^{-1} = \log N + \mathcal{O}(1)$ as $N\to\infty$ (by comparing with the corresponding Riemann integral, see the definition of Euler's constant γ). □

Ex. 3.14 Let $\phi\colon[-\pi,\pi]\to\mathbb{R}$ be continuous with only finitely many zeros. Define the linear functional $L\colon\mathcal{C}_{2\pi}\to\mathbb{C}$ by

$$L(f):=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,\phi(x)\,dx\quad (f\in\mathcal{C}_{2\pi}).\tag{3.16}$$

Prove that L is a bounded linear functional with operator norm

$$\|L\|=\|\phi\|_1=\frac{1}{2\pi}\int_{-\pi}^{\pi}|\phi(x)|\,dx.\tag{3.17}$$

3.15 Theorem For all $x\in\mathbb{R}$ there exists $f\in\mathcal{C}_{2\pi}$ such that the sequence $\big((S_N[f])(x)\big)_{N=0}^{\infty}$ does not converge to a finite limit.

Proof Without loss of generality we may take $x:=0$. Define the linear functional $L_N\colon\mathcal{C}_{2\pi}\to\mathbb{C}$ by

$$L_N(f):=(S_N[f])(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\,D_N(t)\,dt.$$

It follows from Exercise 3.14 that $\|L_N\|=\|D_N\|_1$, and it follows next from Lemma 3.13 that the sequence $(\|L_N\|)_{N=0}^{\infty}$ is unbounded. Now suppose that $\lim_{N\to\infty}L_N(f)$ exists for all $f\in\mathcal{C}_{2\pi}$. Then, for all $f\in\mathcal{C}_{2\pi}$, the sequence $(L_N(f))_{N=0}^{\infty}$ is bounded. Hence, by the Banach-Steinhaus theorem (see syll. Functionaalanalyse), the sequence $(\|L_N\|)_{N=0}^{\infty}$ is bounded. This is a contradiction. □

Ex. 3.16 Let $f\in\mathcal{L}^1_{2\pi}$ and let $a\in\mathbb{R}$. Prove that for the two sequences $\big((S_N[f])(a)\big)_{N=0}^{\infty}$ and

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f(a+t)\,\frac{\sin(Nt)}{t}\,dt,\quad N=0,1,2,\ldots$$

precisely one of the following two alternatives holds:

(a) Both sequences converge with the same limit.

(b) Both sequences diverge.

Conclude that, for f satisfying one of the conditions of Theorem 3.5, we have

$$\tfrac12(f(a_+)+f(a_-))=\pi^{-1}\lim_{N\to\infty}\int_{-\pi}^{\pi}f(a+t)\,\frac{\sin(Nt)}{t}\,dt.\tag{3.18}$$

Ex. 3.17 Prove, by using (3.18), that

$$\int_0^{\infty}\frac{\sin x}{x}\,dx=\lim_{N\to\infty}\int_0^{\pi}\frac{\sin(Nt)}{t}\,dt=\tfrac12\pi.\tag{3.19}$$

3.4 Injectivity of the Fourier transform

3.18 In this subsection we will prove that a function $f \in \mathcal{L}^1_{2\pi}$ which has all its Fourier coefficients $\widehat{f}(n) = 0$, is almost everywhere equal to zero. For the case that $f \in \mathcal{L}^2_{2\pi}$, this was already implied by Parseval's identity (1.9). However, Parseval's identity depended on Theorem 1.11, for which we postponed the proof. Let us first consider the case that $f \in \mathcal{C}_{2\pi}$.

Proposition Let $f \in \mathcal{C}_{2\pi}$ such that $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f = 0$.

Proof Let $F(x) := \int_0^x f(t)\,dt$. Then F is a C^1-function on $\mathbb{R}$ and, because $\widehat{f}(0) = 0$, the function F is 2π-periodic. Thus $F \in \mathcal{C}^1_{2\pi}$. Since $F' = f$, we have $\widehat{f}(n) = in\,\widehat{F}(n)$ (see (2.2)). Hence $\widehat{F}(n) = 0$ if $n \neq 0$. It follows that $(S_N[F])(x) = \widehat{F}(0)$. By Theorem 3.4 we conclude that $F(x) = \lim_{N\to\infty}(S_N[F])(x) = \widehat{F}(0)$. Hence $f(x) = F'(x) = 0$. □

For the similar result in the case that $f \in \mathcal{L}^1_{2\pi}$ we proceed as follows.

3.19 Lemma Let $f \in \mathcal{L}^1_{2\pi}$ such that $\widehat{f}(0) = 0$. Put $F(x) := \int_0^x f(t)\,dt$. Then $F \in \mathcal{C}_{2\pi}$ and $\widehat{f}(n) = in\,\widehat{F}(n)$.

Proof Continuity of F is a consequence of the dominated convergence theorem, applied to the functions $f\chi_{x,t}$, where $\chi_{x,t}$ is the characteristic function of the interval $[x, x+t)$ and letting $t \to 0$. Now we compute for $n \neq 0$:

$$\widehat{F}(n) = \int_0^{2\pi}\int_0^x f(t)e^{-inx}dt\,dx = \int_0^{2\pi}\int_t^{2\pi} e^{-inx}dx f(t)\,dt$$
$$= \int_0^{2\pi}\left(\frac{e^{-int}-1}{in}\right) f(t)\,dt = \frac{1}{in}(\widehat{f}(n) - \widehat{f}(0)) = \frac{1}{in}\widehat{f}(n).$$

We used Fubini's theorem for the second equality. □

Remark It is a non trivial result of Lebesgue that for almost all x $F'(x)$ exists and equals $f(x)$. (see Rudin, Theorem 8.17)

3.20 Theorem (a) Let $f \in \mathcal{L}^1_{2\pi}$ such that $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f = 0$ a.e. .

(b) Let $f, g \in \mathcal{L}^1_{2\pi}$ such that $\widehat{f}(n) = \widehat{g}(n)$ for all $n \in \mathbb{Z}$. Then $f = g$ a.e. .

Proof It is sufficient to prove part (a). Let F be as in Lemma 3.19. Then it follows that $F \in \mathcal{C}_{2\pi}$ and that $\widehat{F}(n) = 0$ for $n \neq 0$. Thus $F - \widehat{F}(0) = 0$ by Proposition 3.18. Since $F(0) = 0$, we conclude that $F = 0$. It follows that $\int_a^b f(t)\,dt = 0$ for every choice of $a, b \in \mathbb{R}$. But then $\int_E f(t)\,dt = 0$ for (bounded) open and also for (bounded) closed sets. By regularity of Lebesgue measure, we conclude that $\int_E f\,dt = 0$ for every Borel set. It follows that $f = 0$ a.e. . □

3.21 Corollary (a) The linear span of the functions $t \mapsto e^{int}$ $(n \in \mathbb{Z})$ is dense in $L^2_{2\pi}$ with respect to the norm $\|\,.\,\|_2$.

(b) The functions $t \mapsto e^{int}$ $(n \in \mathbb{Z})$ form an orthonormal basis of $L^2_{2\pi}$.

Proof Apply Definition-Theorem 1.12 to the case $f \in L^2_{2\pi}$ of Theorem 3.20. □

So we have now proved parts (b) and (c) of Theorem 1.11, therefore we have also definitely established all results in subchapter 1.2 which depended on Theorem 1.11, see the summarizing §1.18. In the next chapter we will also prove part (a) of Theorem 1.11.

As a corollary of the injectivity result of Theorem 3.20 we can also give the following addition to Theorem 2.8.

3.22 Corollary Let $f \in L^1_{2\pi}$. If $\widehat{f} \in \ell^1(\mathbb{Z})$ then $f \in \mathcal{C}_{2\pi}$ and, for almost all $x \in \mathbb{R}$:

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\, e^{inx}. \tag{3.20}$$

The right hand side of (3.20) converges absolutely and uniformly for $x \in \mathbb{R}$.

Proof Denote the right hand side of (3.20) by $g(x)$. Then, by 2.8, $g \in \mathcal{C}_{2\pi}$ and $\widehat{g}(n) = \widehat{f}(n)$ for all $n \in \mathbb{Z}$. Now apply Theorem 3.20(b). □

3.5 Uniform convergence of Fourier series

We will now consider an analogue of Theorem 3.4 such that f satisfies not just a Hölder condition at one point, but a uniform Hölder condition on some interval (a, b). Then it will turn out that $S_N[f]$ converges uniformly to f on each compact subset of (a, b). For $f \in \mathcal{C}^2_{2\pi}$ this result is almost immediate:

3.23 Proposition Let $f \in \mathcal{C}^2_{2\pi}$. Then $\lim_{N\to\infty} S_N[f] = f$, uniformly on $\mathbb{R}$.

Proof It follows from Theorem 2.5(a) that $\widehat{f}(n) = o(|n|^{-2})$ as $|n| \to \infty$. Now apply Corollary 3.22. □

3.24 The following lemma quickly follows from the well-known Arzela-Ascoli theorem (see for instance A. Browder, *Mathematical Analysis*, Springer, 1996, Theorem 6.71 and Corollary 6.73).

Lemma Let (X, d) be a compact metric set. Let $(\phi_n)_{n=1}^{\infty}$ be a sequence of complex-valued functions on X which is *equicontinuous* on X, i.e., such that, for each $\varepsilon > 0$, there is a $\delta > 0$ with the property that $|\phi_n(x) - \phi_n(y)| < \varepsilon$ for al $n \in \mathbb{N}$ if $x, y \in X$ and $d(x, y) < \delta$. Suppose that $\lim_{n\to\infty} \phi_n(x) = 0$ for $x \in X$. Then $\lim_{n\to\infty} \phi_n = 0$ uniformly on X.

Proof Suppose that the convergence $\phi_n \to 0$ is not uniform on X. Then there exist $\varepsilon > 0$, an increasing sequence of positive integers $n_1, n_2, \ldots$, and a sequence $x_1, x_2, \ldots$ in X such that $|\phi_{n_k}(x_k)| \geq \varepsilon$. By compactness of X the sequence $x_1, x_2, \ldots$ has a subsequence converging to some $x_0 \in X$. Without loss of generality we may assume that the sequence $x_1, x_2, \ldots$ already converges to x_0. Then

$$|\phi_{n_k}(x_k)| \leq |\phi_{n_k}(x_k) - \phi_{n_k}(x_0)| + |\phi_{n_k}(x_0)|.$$

By the assumptions, there exists $K \in \mathbb{N}$ such that $|\phi_{n_k}(x_0)| < \frac{1}{2}\varepsilon$ if $k \geq K$ and $|\phi_m(x_k) - \phi_m(x_0)| < \frac{1}{2}\varepsilon$ for all $m \in \mathbb{N}$ if $k \geq K$. Hence $\phi_{n_k}(x_k) < \varepsilon$ if $k \geq K$. This is a contradiction. □

3.25 Theorem Let $f \in \mathcal{L}^1_{2\pi}$. Suppose that there exist an interval (a,b) and positive real numbers M,α such that

$$|f(x)-f(y)| \le M\,|x-y|^\alpha \quad \text{for all } x,y\in(a,b) \tag{3.21}$$

(a *uniform Hölder condition* on (a,b)). Then $\lim_{N\to\infty} S_N[f]=f$ uniformly on each subinterval $[c,d]$ with $a<c<d<b$.

Proof By the proof of Theorem 3.4 we can write

$$S_N(x)-f(x)=\int_{-\pi}^{\pi} g_{x,+}(t)e^{iNt}\,dt-\int_{-\pi}^{\pi} g_{x,-}(t)e^{-iNt}\,dt \tag{3.22}$$

with the functions $g_{x,\pm}$ being given by (3.10). Put

$$\phi_{N,\pm}(x):=\int_{-\pi}^{\pi} g_{x,\pm}(t)\,e^{\pm iNt}\,dt \quad (x\in[c,d]).$$

By (3.21) and the proof of Theorem 3.4, the functions $g_{x,\pm}$ belong to $\mathcal{L}^1_{2\pi}$ for $x\in[c,d]$. So $\lim_{N\to\infty}\phi_{N,\pm}(x)=0$ for $x\in[c,d]$ by the Riemann-Lebesgue Lemma (Theorem 2.3). We will show that the functions $\phi_{N,\pm}$ are equicontinuous on $[c,d]$. Then we can apply Lemma 3.24 and thus prove the Theorem.

Let $0<\delta<\pi$ such that $\delta<c-a$ and $\delta<b-d$. Then we can estimate for $x\in[c,d]$ and $0<|t|<\delta$ that $|g_{x,\pm}(t)|<\frac14 M|t|^{\alpha-1}$ (use (3.21) and (3.6)). We can estimate for $x,y\in[c,d]$ and $\delta<|t|<\pi$ that

$$|g_{x,\pm}(t)-g_{y,\pm}(t)|<(4\pi\sin(\tfrac12\delta))^{-1}\big(|f(x+t)-f(y+t)|+|f(x)-f(y)|\big).$$

Next, for $x,y\in[c,d]$:

$$\begin{aligned}|\phi_{N,\pm}(x)-\phi_{N,\pm}(y)| &\le \Big(\int_{|t|<\delta}+\int_{\delta<|t|<\pi}\Big)|g_{x,\pm}(t)-g_{y,\pm}(t)|\,dt\\ &\le M\int_0^\delta |t|^{\alpha-1}\,dt+\frac{1}{4\pi\sin(\frac12\delta)}\int_{-\pi}^{\pi}|f(x+t)-f(y+t)|\,dt+\frac{|f(x)-f(y)|}{2\sin(\frac12\delta)}\,.\end{aligned}$$

Let $\varepsilon>0$. We will find $\gamma>0$ such that the last expression is dominated by ε if $x,y\in[c,d]$ and $|x-y|<\gamma$. First take δ such that $M\int_0^\delta |t|^{\alpha-1}\,dt<\varepsilon/3$. Then we can find $\gamma>0$ such that the second and third term are dominated by $\varepsilon/3$ if $|x-y|<\gamma$. For the third term this follows because f is uniformly continuous on $[c,d]$. For the second term this follows from Proposition 1.9. This settles the equicontinuity of the functions $\phi_{N,\pm}$ on $[c,d]$. Thus we can apply Lemma 3.24 and the Theorem will follow. □

3.26 Corollary Let $f\in\mathcal{C}^1_{2\pi}$, or let $f\in\mathcal{C}_{2\pi}$ with piecewise continuous derivative. Then $\lim_{N\to\infty}S_N[f]=f$, uniformly on $\mathbb{R}$.

Ex. 3.27 Let the 2π-periodic function f be determined by $f(x):=|x|$ for $-\pi\le x\le\pi$. Determine the Fourier coefficients $\widehat{f}(n)$. The uniform convergence of $(S_N[f])_{N=0}^\infty$ on $\mathbb{R}$ is implied by Corollary 3.26. Check this uniform convergence independently by using the explicit values of $\widehat{f}(n)$.

3.6 The Gibbs phenomenon

The approximation of a piecewise continuous 2π-periodic function (for instance the saw-tooth function) by its partial Fourier sum shows a remarkable "overshooting" behaviour near the jumps of the function. This behaviour is called the *Gibbs phenomenon*.

3.28 Let the 2π-periodic function f be given by $f(x) := \pi - x$ for $0 < x < 2\pi$. This function was studied in Exercises 1.25 and 3.7. We have

$$\begin{aligned} S_N(x) &= (S_N[f])(x) = 2\sum_{n=1}^{N} \frac{\sin(nx)}{n} = \int_0^x D_N(t)\,dt - x \\ &= 2\int_0^x \frac{\sin(Nt)}{t}\,dt + \int_0^x \theta(t)\,\sin(Nt)\,dt + N^{-1}\sin(Nx) - x \quad (0 < |x| < 2\pi), \end{aligned} \tag{3.23}$$

where we used (3.15) and where the function θ is given by (3.14).

Ex. 3.29 Let (a, b) be a bounded interval and let $c \in (a, b)$. Let $g \in C^1((a, b))$ and define $G_n(x) := \int_c^x g(t)\,e^{-int}\,dt$ $(n \in \mathbb{Z},\ x \in (a, b))$. Prove that $G_n(x) = \mathcal{O}(|n|^{-1})$ as $|n| \to \infty$, uniformly for x in a compact subinterval of (a, b).
Hint Use integration by parts.

3.30 Let $0 < r < 2\pi$. We conclude from Exercises 3.29 and 3.12 that

$$\int_0^x \theta(t)\,\sin(Nt)\,dt = \mathcal{O}(N^{-1}) \text{ as } N \to \infty, \text{ uniformly for } |x| \le r. \tag{3.24}$$

The *integral sine* is the function defined by

$$\mathrm{Si}(x) := \int_0^x \frac{\sin y}{y}\,dy \quad (x \in \mathbb{R}). \tag{3.25}$$

This is an even C^1-function (in fact it is an analytic function). It follows from (3.19) that

$$\mathrm{Si}(\infty) := \lim_{x\to\infty} \mathrm{Si}(x) = \tfrac{1}{2}\pi.$$

The absolute maximum of $\mathrm{Si}(x)$ for $x \in [0, \infty)$ is reached for $x = \pi$. A numerical computation shows that, approximately,

$$\frac{\mathrm{Si}(\pi)}{\mathrm{Si}(\infty)} \approx 1.179\ldots\,.$$

Let $0 < r < 2\pi$. It follows from (3.23), (3.25) and (3.24) that

$$\left.\begin{aligned} S_N(x) - f(x) &= 2\mathrm{Si}(Nx) - \pi + \mathcal{O}(N^{-1}) \\ S_N(-x) - f(-x) &= -2\mathrm{Si}(Nx) + \pi + \mathcal{O}(N^{-1}) \end{aligned}\right\} \text{ as } N \to \infty, \text{ uniformly for } 0 < x \le r.$$

This implies:

$$\lim_{N\to\infty} \big(S_N(N^{-1}\pi) - f(N^{-1}\pi)\big) = 2\mathrm{Si}(\pi) - \pi \approx \pi \times 0.179\ldots\,,$$
$$\lim_{N\to\infty} \big(S_N(-N^{-1}\pi) - f(-N^{-1}\pi)\big) = -2\mathrm{Si}(\pi) + \pi \approx -\pi \times 0.179\ldots\,.$$

Also, for $0 < \delta < \pi$, we have

$$\lim_{N\to\infty}\left(\sup_{0<x\le\delta}(S_N(x)-f(x))\right) = 2\mathrm{Si}(\pi)-\pi, \tag{3.26}$$

$$\lim_{N\to\infty}\left(\inf_{-\delta\le x<0}(S_N(x)-f(x))\right) = -2\mathrm{Si}(\pi)+\pi, \tag{3.27}$$

but

$$\lim_{N\to\infty}\left(\sup_{\delta\le x\le 2\pi-\delta}|S_N(x)-f(x)|\right) = 0$$

by Theorem 3.25.

Ex. 3.31 Let $h \in C^1([0,2\pi])$ Let g be a 2π-periodic function which coincides with h on $(0,2\pi)$. Write $g(0_+) := h(0)$ and $g(0_-) := h(2\pi)$. Assume that $g(0_+) \ge g(0_-)$. Let $0 < \delta < \pi$. Prove that

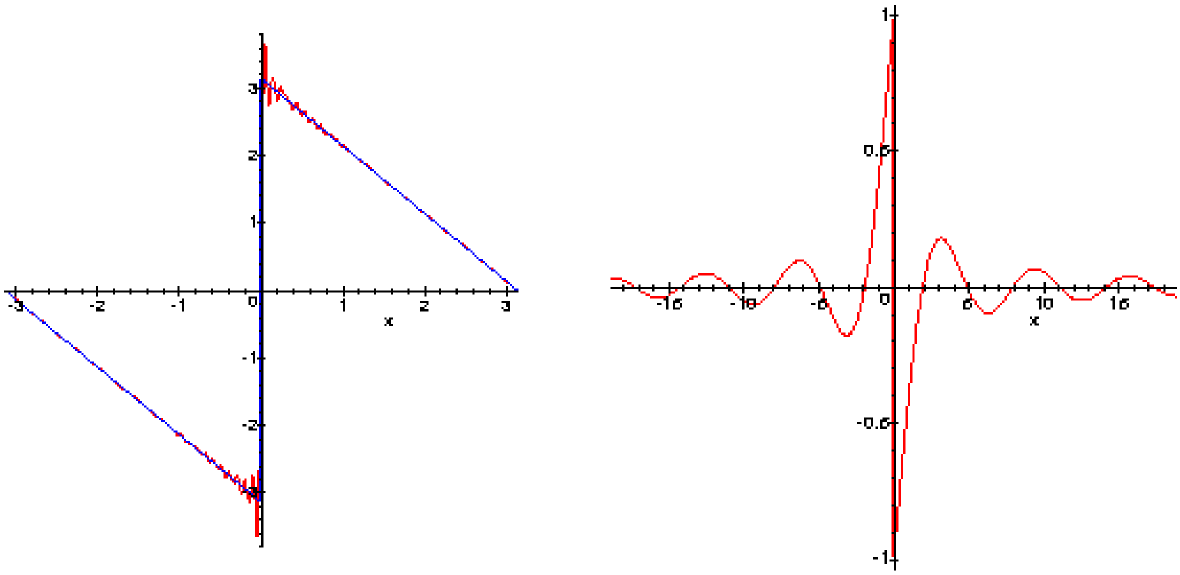
$$\lim_{N\to\infty}\left(\sup_{0<x\le\delta}((S_N[g])(x)-g(x))\right) = \tfrac{1}{2}(g(0_+)-g(0_-))\,(2\pi^{-1}\mathrm{Si}(\pi)-1),$$

$$\lim_{N\to\infty}\left(\inf_{-\delta\le x<0}((S_N[g])(x)-g(x))\right) = -\tfrac{1}{2}(g(0_+)-g(0_-))\,(2\pi^{-1}\mathrm{Si}(\pi)-1).$$

Hint Let the 2π-periodic function f be given by $f(x) := \pi - x$ for $0 < x < 2\pi$. Put

$$p(x) := g(x) - \frac{g(0_+)-g(0_-)}{2\pi}\,f(x) \qquad (0 < x < 2\pi).$$

Then $p \in \mathcal{C}_{2\pi}$ with piecewise continuous derivative. Now use (3.26), (3.27), and Corollary 3.26.



The Gibbs phenomenon.
Left: the sawtooth and S_{64}, Right: $(S_N(N^{-1}x) - f(N^{-1}x))/\pi$ for $N = 1024$

3.7 *Further exercises*

Ex. 3.32 Prove that

$$\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots = \pi/4 \quad \text{if } 0 < x < \pi.$$

Hint Use Exercise 3.8 or use (1.12), (3.11).

Ex. 3.33 Let $\phi \in L^1_{2\pi}$. Define the linear functional $L\colon \mathcal{C}_{2\pi} \to \mathbb{C}$ by (3.16). Prove that L is a bounded linear functional with norm given by (3.17).
Hint Use Exercises 3.14 and 1.32.

Ex. 3.34 Prove that $D_N(x)$ can be written as a polynomial of degree $2N$ in $\cos(\frac{1}{2}x)$.

Ex. 3.35 Prove that between each two successive zeros of $D_N(x)$ there is precisely one zero of $D'_N(x)$.

Ex. 3.36 1) Let $\{a_n\}_1^\infty$, $\{a_n\}_1^\infty$ be sequences of complex numbers. Prove (for $N = 1, 2, \ldots$) the *Summation by parts formula*

$$\sum_{n=1}^{N} a_n(b_{n+1} - b_n) = a_{N+1}b_{N+1} - a_1 b_1 - \sum_{n=1}^{N} b_{n+1}(a_{n+1} - a_n).$$

2) Write down a version of this formula for the case where $b_n = \sum_{j=1}^{n} c_j$.

3) Give a direct proof of the convergence of the series in exercise 3.32 (without using that this is a Fourier series of a smooth function)

4) Show that the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}$$

converges pointwise for every x.

5) What can you say about uniform convergence?

4 The Fejér kernel

4.1 Cesàro convergence

4.1 Definition Let $(a_n)_{n=1}^{\infty}$ be a sequence of complex numbers. If the sequence does not have a limit then we may try to find a limit in a generalized sense by considering a new sequence $(b_n)_{n=1}^{\infty}$ with b_n being the mean of the first n elements of the sequence (a_k), i.e.,

$$b_n := n^{-1}(a_1 + a_2 + \cdots + a_n).$$

If $a := \lim_{n\to\infty} b_n$ exists as a finite limit then we say that the sequence (a_n) is *Cesàro convergent* with *Cesàro limit* a. In that case we use the notation

$$\text{(C)} \lim_{n\to\infty} a_n = a. \tag{4.1}$$

Similarly, we say that the series $\sum_{n=1}^{\infty} a_n$ is *Cesàro convergent* with *Cesàro sum* s if the sequence (s_n) of partial sums $s_n := \sum_{k=1}^{n} a_k$ is Cesàro convergent with Cesàro limit s. In that case we use the notation

$$\text{(C)} \sum_{n=1}^{\infty} a_n = s. \tag{4.2}$$

More generally, Cesàro convergence of the sum $\sum_{n=n_0}^{\infty} a_n$ with Cesàro sum s means Cesàro convergence of the sequence $(s_n)_{n=n_0}^{\infty}$ of partial sums $s_n := \sum_{k=n_0}^{n} a_k$ to Cesàro limit s, i.e., that $\lim_{n\to\infty}(n - n_0 + 1)^{-1}(s_{n_0} + \cdots + s_n) = s$.

4.2 Example
The sequence $1, 0, 1, 0, 1, 0, \ldots$ does not converge but has Cesàro limit $\frac{1}{2}$.
The sum $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ does not converge but has Cesàro sum $\frac{1}{2}$.

Ex. 4.3 Show the following. The sequence $a_0, a_1, a_2, \ldots$ has Cesàro limit a iff the sequence $a_1, a_2, \ldots$ has Cesàro limit a.
Conclude that it is not necessary in the notation (4.1) to mention where the sequence (a_n) starts.

4.4 Proposition If the sequence $(a_n)_{n=1}^{\infty}$ has limit a then it has also Cesàro limit a.

Proof Let $M, N \in \mathbb{N}$ with $M < N$. For $n \geq N$ we have

$$\begin{aligned} |n^{-1}(a_1 + \cdots + a_n) - a| &\leq n^{-1}(|a_1 - a| + \cdots + |a_M - a|) + n^{-1}(|a_{M+1} - a| + \cdots + |a_n - a|) \\ &\leq MN^{-1} \sup_{k=1,2,\ldots} |a_k - a| + \sup_{k\geq M+1} |a_k - a|. \end{aligned}$$

Let $\varepsilon > 0$. Because the sequence (a_n) converges, we have $A := \sup_{k=1,2,\ldots} |a_k - a| < \infty$ and we can choose M such that $|a_k - a| < \frac{1}{2}\varepsilon$ if $k \geq M$. Hence

$$|n^{-1}(a_1 + \cdots + a_n) - a| \leq N^{-1}MA + \tfrac{1}{2}\varepsilon \quad \text{if } n \geq N > M.$$

Choose $N > M$ such that $N^{-1}MA < \frac{1}{2}\varepsilon$. We conclude that $|n^{-1}(a_1 + \cdots + a_n) - a|$ if $n \geq N$. □

Ex. 4.5 Give an example of an unbounded sequence which is Cesàro convergent.

Ex. 4.6 Which of the following statements are true?
(i) If (C) $\lim_{n\to\infty} a_n = a$ then (C) $\lim_{n\to\infty} (a_n)^3 = a^3$.
(ii) If (C) $\sum_{n=1}^{\infty} a_n = s$ then (C) $\lim_{n\to\infty} a_n = 0$.

4.2 Definition of Fejér kernel

4.7 In (3.2) we introduced the partial Fourier sums $S_n[f]$ of a function $f \in L^1_{2\pi}$. In Chapter 3 we examined convergence of the sequence $\big((S_N[f])(x)\big)_{N=0}^{\infty}$ for fixed x. More generally we may examine Cesàro convergence of this sequence, i.e., convergence of the sequence $\big((\sigma_N[f])(x)\big)_{N=0}^{\infty}$, where

$$\sigma_N[f] := \frac{S_0[f] + S_1[f] + \cdots + S_N[f]}{N+1} \qquad (f \in L^1_{2\pi},\ N = 0, 1, 2, \ldots). \tag{4.3}$$

The function $\sigma_N[f]$ is called the N^{th} Cesàro mean of f.

4.8 Definition-Proposition The function in (4.3) can be written as

$$(\sigma_N[f])(x) = (f * K_N)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, K_N(x-t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\, K_N(t)\, dt, \tag{4.4}$$

where $K_N(x)$ is the *Fejér kernel*:

$$\begin{aligned} K_N(x) :=& \frac{1}{N+1}\sum_{n=0}^{N} D_n(x) = \sum_{n=-N}^{N} \frac{N+1-|n|}{N+1}\, e^{inx} \\ =& \begin{cases} \dfrac{1}{N+1}\left(\dfrac{\sin(\frac{1}{2}(N+1)x)}{\sin(\frac{1}{2}x)}\right)^2 & (x \notin 2\pi\mathbb{Z}), \\ N+1 & (x \in 2\pi\mathbb{Z}). \end{cases} \end{aligned} \tag{4.5}$$

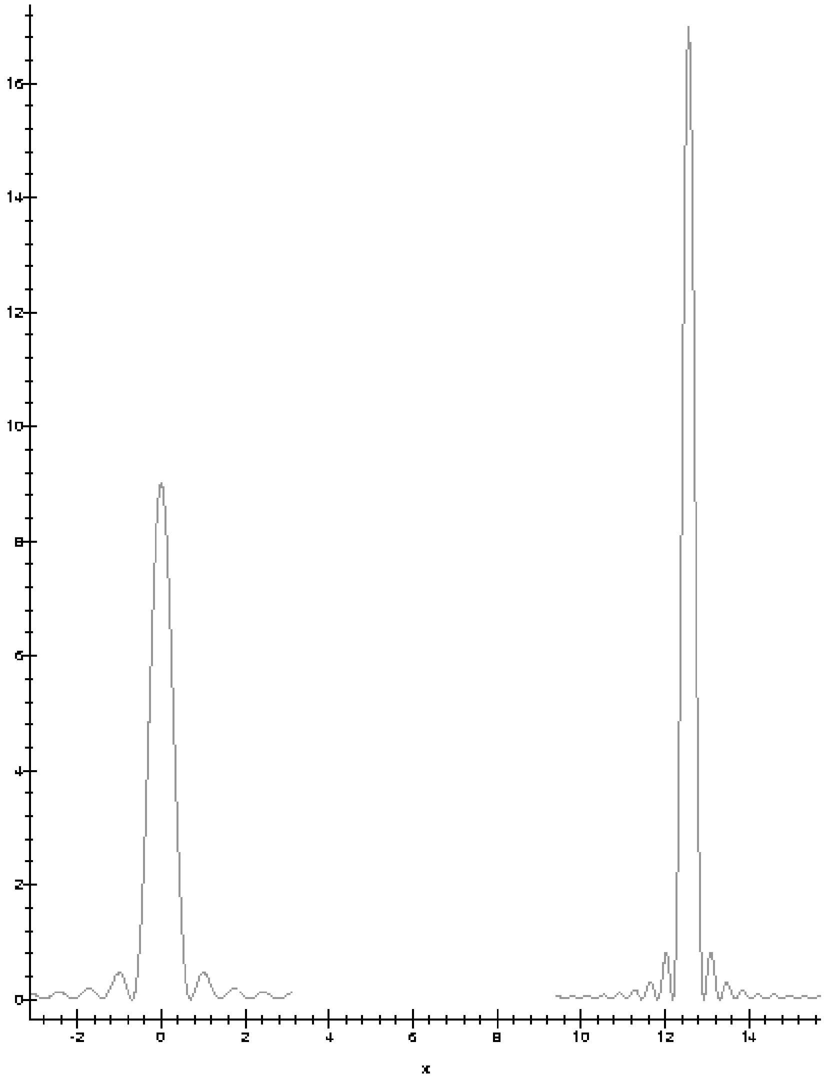
Proof The second equality in (4.5) follows from the definition of $D_n(x)$ in (3.4). From (3.4) we get also that

$$K_N(x) = \frac{\sin(\frac{1}{2}x) + \sin(\frac{3}{2}x) + \cdots \sin((N+\frac{1}{2})x)}{(N+1)\,\sin(\frac{1}{2}x)}\,.$$

Now multiply numerator and denominator of this last expression by $\sin(\frac{1}{2}x)$ and use that

$$\sin(\tfrac{1}{2}x)\,\sin((k+\tfrac{1}{2})x) = \tfrac{1}{2}(\cos(kx) - \cos((k+1)x)) \quad\text{and}\quad 1 - \cos((N+1)x) = 2\sin^2(\tfrac{1}{2}(N+1)x).$$

This proves the last equality in (4.5). From (4.5) we see that $K_N \in \mathcal{C}_{2\pi}$ and that it is an even function. The third equality in (4.4) uses these last facts. $\square$



The Fejér Kernels K_8, K_{16}, centered at 0, 4π *respectively.*

4.3 Cesàro convergence of Fourier series

4.9 We need the following three easy consequences of (4.5):

$$K_N(x) \geq 0 \qquad (x \in \mathbb{R}), \tag{4.6}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(t)\, dt = 1, \tag{4.7}$$

$$K_N(x) \leq \frac{1}{(N+1)\, \sin^2(\frac{1}{2}x)} \leq \frac{\pi^2}{(N+1)\, x^2} \quad (0 < |x| \leq \pi). \tag{4.8}$$

4.10 Theorem (Fejér) Let $f \in \mathcal{L}^1_{2\pi}$, $a \in \mathbb{R}$. Suppose that f is continuous in a. Then

$$\lim_{N\to\infty} (\sigma_N[f])(a) = f(a). \tag{4.9}$$

Moreover, if $f \in \mathcal{C}_{2\pi}$ then the convergence in (4.9) is uniform for $a \in \mathbb{R}$.

Proof Let $\varepsilon > 0$. We want to show that $|\sigma_N(a) - f(a)| < \varepsilon$ for N sufficiently large. It follows from (4.4) and (4.7) that

$$\sigma_N(a) - f(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(a+t) - f(a))\, K_N(t)\, dt.$$

For any $\delta \in [0, \pi)$ we split up the above integral as $\int_{-\pi}^{\pi} = \int_{|t|<\delta} + \int_{\delta<|t|<\pi}$. Then, by use of (4.6),

$$|\sigma_N(a) - f(a)| \le \frac{1}{2\pi} \int_{-\delta}^{\delta} |f(a+t) - f(a)|\, K_N(t)\, dt + \frac{1}{2\pi} \int_{\delta<|t|<\pi} |f(a+t) - f(a)|\, K_N(t)\, dt.$$

Denote the two successive terms on the right hand side by I_1 and I_2. First we estimate I_1. By the continuity of f in a we can choose δ such that $|f(a+t) - f(a)| \le \frac{1}{2}\varepsilon$ if $|t| \le \delta$. Keep this value of δ. Then, by use of (4.7), $I_1 \le \frac{1}{2}\varepsilon$. Next we estimate I_2. It follows from (4.8) that

$$\begin{aligned} I_2 &\le \frac{\pi}{2(N+1)} \int_{\delta<|t|<\pi} |f(a+t) - f(a)|\, t^{-2}\, dt \le \frac{\pi}{2(N+1)\delta^2} \int_{-\pi}^{\pi} (|f(a+t)| + |f(a)|)\, dt \\ &\le \frac{\pi^2}{(N+1)\delta^2} (\|f\|_1 + |f(a)|). \end{aligned}$$

Hence $I_2 \le \frac{1}{2}\varepsilon$ for N sufficiently large. This proves the first part of the theorem.

Next assume that $f \in \mathcal{C}_{2\pi}$. Then f is uniformly continuous on $\mathbb{R}$ (by the compactness of $[-\pi, \pi]$ and by the 2π-periodicity). Thus we can choose δ such that $|f(a+t) - f(a)| \le \frac{1}{2}\varepsilon$ for all $a \in \mathbb{R}$ if $|t| \le \delta$. With this choice of δ we have $I_1 \le \frac{1}{2}\varepsilon$ for all a. Then

$$I_2 \le \frac{\pi^2}{(N+1)\delta^2} (\|f\|_1 + \|f\|_\infty).$$

Hence $I_2 \le \frac{1}{2}\varepsilon$ for all a if N is sufficiently large. We conclude that $|\sigma_N(a) - f(a)| < \varepsilon$ for all a if N is sufficiently large. □

4.11 Corollary (Fejér) Let $f \in \mathcal{L}^1_{2\pi}$, let $a \in \mathbb{R}$, and suppose that the limits

$$f(a_+) := \lim_{x \downarrow a} f(x), \qquad f(a_-) := \lim_{x \uparrow a} f(x)$$

exist. Then we have:

$$\lim_{N\to\infty} (\sigma_N[f])(a) = \tfrac{1}{2}(f(a_+) + f(a_-)).$$

Ex. 4.12 Give the proof of Corollary 4.11 (compare with the proof of Theorem 3.5).

4.13 Theorem 1.11(a) is an immediate corollary of the second statement in Theorem 4.10:

Corollary (Weierstrass) The space of trigonometric polynomials is dense in $\mathcal{C}_{2\pi}$ with respect to the norm $\| \, . \, \|_\infty$.

Proof Let $f \in \mathcal{C}_{2\pi}$. Let $\varepsilon > 0$. Then, by the uniform convergence stated in Theorem 4.10, we can take N such that $\|\sigma_N[f]-f\|_\infty < \varepsilon$. Now observe that $\sigma_N[f]$ is a trigonometric polynomial. □

4.14 Corollary Let $f \in \mathcal{L}^1_{2\pi}$ and $a \in \mathbb{R}$. Suppose that f is continuous in a. If $\lim_{N\to\infty}(S_N[f])(a)$ exists then this limit is equal to $f(a)$.

Proof By Proposition 4.4 the limit is equal to $\lim_{N\to\infty}(\sigma_N[f])(a)$ and hence, by Theorem 4.10 to $f(a)$.

4.15 Remark Observe that $\sigma_N[f] := f * K_N$ $(f \in \mathcal{L}^1_{2\pi})$, where K_N $(N = 0, 1, 2, \ldots)$ is an even nonnegative function belonging to $\mathcal{L}^1_{2\pi}$ with properties $(2\pi)^{-1} \int_{-\pi}^{\pi} K_N(t)\, dt = 1$ and:

$$\text{For each } \delta \in (0, \pi): \quad \lim_{N\to\infty} K_N(x) = 0 \quad \text{uniformly for } \delta \le |x| \le \pi.$$

Observe that the proof of Theorem 4.10 only uses these properties of K_N and $\sigma_N[f]$. Hence Theorem 4.10 remains valid for any choice of the functions K_N satisfying the above properties.

Ex. 4.16 Let $f \in L^1_{2\pi}$. Show that $\lim_{N\to\infty} \sigma_N[f] = f$ in $L^1_{2\pi}$.
Hint Show that $\lim_{N\to\infty} \|\sigma_N[f] - f\|_1 = 0$ for $f \in \mathcal{C}_{2\pi}$ and use that $\mathcal{C}_{2\pi}$ is dense in $L^1_{2\pi}$.

4.4 Another approximation theorem of Weierstrass

Corollary 4.13 is an approximation theorem involving trigonometric polynomials. From this we can derive an even more famous approximation theorem of Weierstrass involving ordinary polynomials. (There exist many other proofs of this theorem.) As a tool for the proof we introduce Chebyshev polynomials.

4.17 Definition-Proposition For each nonnegative integer n there is a unique polynomial T_n in one variable such that

$$T_n(\cos t) = \cos(nt) \quad (t \in \mathbb{R}). \tag{4.10}$$

The polynomial T_n has degree n. It is called a *Chebyshev polynomial* (of the first kind).

Proof For $x \in [-1, 1]$ $T_n(x)$ is uniquely determined by (4.10). Hence, if T_n is a polynomial then it is unique. Since

$$\cos t \, \cos nt = \tfrac{1}{2} \cos(n+1)t + \tfrac{1}{2} \cos(n-1)t,$$

we have

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x) \quad (n = 1, 2, \ldots),$$

while $T_0(x) = 1$, $T_1(x) = x$. Now it follows by induction with respect to n that $T_n(x)$ is a polynomial of degree n in x. □

Ex. 4.18 Show that

$$\int_{-1}^{1} T_n(x)\, T_m(x)\, \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0, & n \neq m, \\ \pi/2, & n = m \neq 0, \\ \pi, & n = m = 0. \end{cases}$$

Thus the T_n form an orthogonal system in $L^2((-1,1); (1-x^2)^{-\frac{1}{2}}\, dx)$. Show also that this orthogonal system is complete.

Ex. 4.19 Show that for each nonnegative integer n there is a unique polynomial $U_n(x)$ of degree n in x such that

$$U_n(\cos t) = \frac{\sin(n+1)t}{\sin t} \quad (0 \neq t \in \mathbb{R}). \tag{4.11}$$

Show that

$$U_{n+1}(x) = 2x\, U_n(x) - U_{n-1}(x) \quad (n = 1, 2, \ldots),$$

while $U_0(x) = 1$, $U_1(x) = 2x$. The polynomial U_n is called a *Chebyshev polynomial* (of the second kind). Show that

$$\int_{-1}^{1} U_n(x)\, U_m(x)\, \sqrt{1-x^2}\, dx = \begin{cases} 0, & n \neq m, \\ \pi/2, & n = m. \end{cases}$$

Thus the U_n form an orthogonal system in $L^2((-1,1); (1-x^2)^{\frac{1}{2}}\, dx)$. Show also that this orthogonal system is complete.

4.20 Theorem (Weierstrass) Let $f \in C([a,b])$. Then there exists for each $\varepsilon > 0$ a polynomial P in one variable such that $|f(x) - P(x)| < \varepsilon$ for $x \in [a,b]$.

Proof Without loss of generality we may assume that $[a,b] = [-1,1]$ (why?). Put $g(t) := f(\cos t)$. Then $g \in \mathcal{C}_{2\pi}$ and g is even, so $\widehat{g}(n) = \widehat{g}(-n)$. Hence

$$(\sigma_N[g])(t) = \widehat{g}(0) + 2 \sum_{n=1}^{N} \frac{N+1-n}{N+1}\, \widehat{g}(n)\, \cos(nt)$$

and $\sigma_N[g] \to g$ for $N \to \infty$, uniformly on $\mathbb{R}$ (see Theorem 4.9), so certainly uniformly on $[0,\pi]$. Put

$$f_N(x) := \widehat{g}(0) + 2 \sum_{n=1}^{N} \frac{N+1-n}{N+1}\, \widehat{g}(n)\, T_n(x).$$

Then $f_N(\cos t) = (\sigma_N[g])(t)$ and $f_N(x)$ is a polynomial of degree $\leq N$ in x. Since $\sigma_N[g] \to g$ for $N \to \infty$, uniformly on $[0,\pi]$, we have that $f_N \to f$ as $N \to \infty$, uniformly on $[-1,1]$. □

Ex. 4.21 Let $f(x) := |x|$. Determine a sequence of polynomials which converges uniformly to f on $[-1,1]$.

Ex. 4.22 The analogue of Theorem 4.20 does not hold on $\mathbb{R}$. Give an example of a continuous function on $\mathbb{R}$ which cannot be uniformly approximated by polynomials on $\mathbb{R}$.

4.5 Further exercises

Ex. 4.23 For $0 \leq r < 1$ let

$$P_r(x) := \sum_{n=-\infty}^{\infty} r^{|n|}\, e^{inx} \qquad (x \in \mathbb{R}), \tag{4.12}$$

the so-called *Poisson kernel*. For $f \in \mathcal{L}^1_{2\pi}$ put

$$A_r[f] := f * P_r. \tag{4.13}$$

The functions $A_r[f]$ are known as Abel means. Let $a \in \mathbb{R}$ and suppose that f is continuous in a. Prove that

$$\lim_{r \uparrow 1} (A_r[f])(a) = f(a). \tag{4.14}$$

Moreover show that, if $f \in \mathcal{C}_{2\pi}$, the convergence in (4.14) is uniform for $a \in \mathbb{R}$.
Hint Use Remark 4.15.

Ex. 4.24 Prove that the Gibbs phenomenon cannot occur for the Cesàro and Abel means. That is, if $m < f < M$, then $m < A_r[f], C_N[f] < M$.

5 Some applications of Fourier series

5.1 The isoperimetric inequality

5.1 Theorem The area A of a region in the plane which is encircled by a closed non-selfintersecting C^1-curve of length L satisfies $A \le L^2/(4\pi)$. Equality holds iff the curve is a circle.

Proof Without loss of generality we may assume that $L = 2\pi$, and that the curve is positively oriented and parametrized by its arc length. We may also identify the plane with $\mathbb{C}$. Then the curve has the form $t \mapsto f(t)$ with $f \in \mathcal{C}^1_{2\pi}$ and with $|f'(t)| = 1$ for all t. Furthermore we may assume without loss of generality that $\widehat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = 0$. Then we have to show that $A \le \pi$ with equality iff $f(t) = e^{i(t+t_0)}$ for some $t_0 \in \mathbb{R}$. Now we have

$$\begin{aligned} A &\overset{(1)}{=} \tfrac{1}{2}\mathrm{Im} \int_0^{2\pi} f'(t)\,\overline{f(t)}\,dt = \pi\,\mathrm{Im}\,\langle f', f\rangle \le \pi\,|\langle f', f\rangle| \overset{(2)}{\le} \pi\,\|f'\|_2\,\|f\|_2 \\ &\overset{(3)}{=} \pi\,\|f\|_2 \overset{(4)}{=} \pi\,\|f - \widehat{f}(0)\|_2 \overset{(5)}{\le} \pi\,\|f'\|_2 \overset{(6)}{=} \pi. \end{aligned} \tag{5.1}$$

Equality (1) follows from:

Ex. 5.2 Let $t \mapsto f(t)$ be a positively oriented closed non-selfintersecting C^1-curve in $\mathbb{C}$. Show that the area of the encircled region equals $\frac{1}{2}\mathrm{Im}\int_0^{2\pi} f'(t)\,\overline{f(t)}\,dt$.

Inequality (2) in (5.1) is the Cauchy-Schwarz inequality. Equalities (3) and (6) use that $\|f'\|_2 = 1$ by the assumption $|f'(t)| = 1$. Equality (4) uses the assumption $\widehat{f}(0) = 0$. Equality (5) follows from:

Ex. 5.3 Let $f \in \mathcal{C}^1_{2\pi}$. Show that $\|f - \widehat{f}(0)\|_2 \le \|f'\|_2$ with equality iff $\widehat{f}(n) = 0$ for $n \ne -1, 0, 1$.

The proof of the last part of Theorem 5.1 is also left as an exercise:

Ex. 5.4 Show that equality everywhere in (5.1) implies that $f(t) = e^{i(t+t_0)}$ for some $t_0 \in \mathbb{R}$.

5.2 The heat equation

5.5 The partial differential equation

$$\frac{\partial}{\partial t} F(t,x) = \tfrac{1}{2}\,\frac{\partial^2}{\partial x^2} F(t,x) \quad ((t,x) \in (0,\infty) \times \mathbb{R}) \tag{5.2}$$

is called the *heat equation*. It is often considered with initial value

$$F(0,x) = f(x) \quad (x \in \mathbb{R}) \tag{5.3}$$

for a given function f on $\mathbb{R}$. We will assume that $f \in \mathcal{C}_{2\pi}^{\infty}$ and we are interested in a function F which is continuous on $[0,\infty) \times \mathbb{R}$ and C^{∞} on $(0,\infty) \times \mathbb{R}$, which satisfies $F(t, x+2\pi) = F(t,x)$ for all $t \geq 0$, and which is a solution of (5.2), (5.3). In a suitable normalization of the variable t we can interprete this as the evolution in time of the temperature on a ring which is idealized as the unit circle parametrized by its arc length. Then $F(t,x)$ is the temperature at position x and time t. The function f gives the initial temperature distribution at time $t = 0$.

Put $F_t(x) := F(t,x)$ and

$$\gamma_n(t) := \widehat{F_t}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t,x)\, e^{-inx}\, dx.$$

Ex. 5.6 Let $f \in \mathcal{C}_{2\pi}^{\infty}$ and let F be as above. Show the following.
(a) γ_n is continuous on $[0,\infty)$.
(b) γ_n is C^{∞} on $(0,\infty)$ with kth derivative given by

$$\gamma_n^{(k)}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^k F(t,x)}{\partial t^k}\, e^{-inx}\, dx.$$

(c) We have

$$\gamma_n'(t) = -\tfrac{1}{2}n^2\gamma_n(t), \quad \gamma_n(0) = \widehat{f}(n), \quad \gamma_n(t) = \widehat{f}(n)\, e^{-\frac{1}{2}n^2 t}.$$

Ex. 5.7 Let $f \in \mathcal{C}_{2\pi}^{\infty}$. Show that

$$F(t,x) := \sum_{n=-\infty}^{\infty} \widehat{f}(n)\, e^{inx - \frac{1}{2}n^2 t} \tag{5.4}$$

is a well-defined C^{∞} function on $(0,\infty)\times\mathbb{R}$, continuous on $[0,\infty)\times\mathbb{R}$, 2π-periodic in x and satisfying (5.2), (5.3). Conclude that $F(t,x)$ given by (5.4) is the unique solution of (5.2), (5.3) of this type.
Hint For the first part use Theorem 2.5(b).

Ex. 5.8 Let F be given by (5.4). Show that

$$\int_{-\pi}^{\pi} F_t(x)\, dx = \int_{-\pi}^{\pi} f(x)\, dx \quad (t > 0)$$

(i.e., the total heat on the ring remains constant in time), and

$$\lim_{t\to\infty} F(t,x) = \widehat{f}(0)$$

(i.e., the temperature distribution tends to the constant distribution as $t \to \infty$).

Ex. 5.9 Define the *heat kernel* by

$$H_t(x) := \sum_{n\in\mathbb{Z}} e^{inx - \frac{1}{2}n^2 t} \quad (t > 0).$$

Prove that $H_t \in \mathcal{C}_{2\pi}^{\infty}$. Show that, for F given by (5.4), we have $F_t = f * H_t$ $(t > 0)$.

Ex. 5.10 Show that $(f * H_s) * H_t = f * H_{s+t}$ $(s,t > 0)$ and give a physical explanation of this formula.

5.3 Weyl's equidistribution theorem

5.11 Theorem Let $0 < \gamma \in \mathbb{R} \setminus \mathbb{Q}$ and let $0 \le a < b \le 1$. Then

$$\lim_{n\to\infty} \frac{\#\{r \in \mathbb{N} \mid \ 0 < r < n, \ < r\gamma > \in [a,b]\}}{n} = b - a.$$

Here $\#$ denotes the number of elements of a set and $< x >$ denotes the fractional part of a positive number x.

Ex. 5.12 Prove this theorem by completing the following outline:

1) Let $f \in C_{2\pi}$ and $\gamma \in \mathbb{R} \setminus \mathbb{Q}$. Prove that

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} f(2\pi j\gamma)}{n} = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt.$$

a. In case $f \equiv 1$.
b. In case $f(t) = e^{ikt}$, $k \in \mathbb{Z}$.
c. In case f is a trigonometric polynomial.
d. For general $f \in C_{2\pi}$.

2) Prove that the formula is also valid if f is the periodic extension of the characteristic function $\chi_{[a,b]}$ of a closed interval $[a,b] \subset (0, 2\pi)$. (Look for continuous functions f_1, f_2 such that $f_1 \le f \le f_2$ and $\int_0^{2\pi}(f_2 - f_1)(t)\, dt < \epsilon$) and apply 1).

Derive the equidistribution theorem.

6 Generalities about Fourier integrals

This chapter is the beginning of Part II of the Syllabus. It deals with Fourier integrals.

6.1 In Part I we have seen that, for a 2π-periodic function f which is sufficiently smooth, for instance $f \in \mathcal{C}^2_{2\pi}$, we can write

$$f(a) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \right) e^{ina} \qquad (a \in \mathbb{R}), \tag{6.1}$$

where the doubly infinite series converges absolutely. Now we will consider functions f on $\mathbb{R}$ which are usually not periodic. The analogue of (6.1), valid for sufficiently nice functions f, is

$$f(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\, e^{-itx}\, dt \right) e^{ixa}\, dx \qquad (a \in \mathbb{R}). \tag{6.2}$$

If f is a function on $\mathbb{R}$ which is sufficiently smooth (say C^2) and which decreases sufficiently fast in absolute value to 0 as $t \to \infty$ (say $f(t) = \mathcal{O}(|t|^{-2})$ as $|t| \to \infty$), then (6.2) holds with absolutely convergent inner and outer integrals. This we will prove later. For the moment we will only prove (6.2) heuristically from (6.1).

6.2 Let f be a C^2-function on $\mathbb{R}$ with compact support and let $a \in \mathbb{R}$. We will give a heuristic proof of (6.2). For $\lambda > 0$ such that $\operatorname{supp}(f) \subset [-\lambda\pi, \lambda\pi]$ and $|a| \le \pi\lambda$ define a function $f_\lambda \in \mathcal{C}^2_{2\pi}$ such that $f_\lambda(t) := f(\lambda t)$ if $-\pi \le t \le \pi$. It follows from (6.1) that

$$\begin{aligned} f(a) = f_\lambda(a/\lambda) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-\pi}^{\pi} f(\lambda t)\, e^{-int}\, dt \right) e^{ina/\lambda} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\, e^{-int/\lambda}\, dt \right) e^{ina/\lambda}\, \lambda^{-1} \\ &= \frac{1}{2\pi} \sum_{x \in \lambda^{-1}\mathbb{Z}} \left(\int_{-\infty}^{\infty} f(t)\, e^{-ixt}\, dt \right) e^{ixa}\, \lambda^{-1} \\ &= \frac{1}{2\pi} \lim_{N \to \infty} \sum_{x \in \lambda^{-1}\mathbb{Z} \cap [-N,N]} \widehat{f}(x)\, e^{ixa}\, \lambda^{-1}, \end{aligned} \tag{6.3}$$

where we have put

$$\widehat{f}(x) := \int_{-\infty}^{\infty} f(t)\, e^{-ixt}\, dt \qquad (x \in \mathbb{R}). \tag{6.4}$$

The sum in (6.3) can be considered as a Riemann sum for the partition of $[-N, N]$ into equidistant points with distance λ^{-1}. Since the function $x \mapsto \widehat{f}(x)\, e^{ixa}$ is continuous, this Riemann sum tends to the corresponding integral. So formally the limit of (6.3) as $\lambda \to \infty$ equals

$$\frac{1}{2\pi} \lim_{N \to \infty} \int_{-N}^{N} \widehat{f}(x)\, e^{ixa}\, dx,$$

which is equal to the right hand side of (6.2). It has to be justified that we may take the limit for $\lambda \to \infty$ in (6.3) inside the limit for $N \to \infty$. We will skip this here.

6.3 Definition Let $f \in \mathcal{L}^1(\mathbb{R})$. Then the integral on the right hand side of (6.4) is absolutely convergent since $|f(t)\,e^{-ixt}| \le |f(t)|$. So (6.4) yields a well-defined function $\widehat{f}$ on $\mathbb{R}$. We call the transform $f \mapsto \widehat{f}$ given by (6.4) the *Fourier transform*. The function $\widehat{f}$ is called the *Fourier transform of* f. Note that $\widehat{f}$ only depends on the equivalence class of f. So the function $\widehat{f}$ is well-defined for $f \in L^1(\mathbb{R})$.

Ex. 6.4 Let $[a,b]$ be a closed bounded interval and let $f := \chi_{[a,b]}$ be its characteristic function. Show the following: $f \in L^1(\mathbb{R})$ and

$$\widehat{f}(x) = \begin{cases} \dfrac{e^{-iax} - e^{-ibx}}{ix} = \dfrac{2\sin(\frac{1}{2}(b-a)x)\,e^{-\frac{1}{2}i(a+b)x}}{x} & \text{if } x \neq 0, \\ b-a & \text{if } x = 0. \end{cases}$$

Observe from this result that $\widehat{f}$ is continuous and that $\lim_{x\to\pm\infty} \widehat{f}(x) = 0$.

Ex. 6.5 Let $f(x) = \frac{1}{1+x^2}$ on $\mathbb{R}$. Compute $\widehat{f}$. (Use Contour integration.)

6.6 Let $p \in [1,\infty)$, in particular we will consider $p = 1$ or 2. We write

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{1/p} \quad \text{if } f \in \mathcal{L}^p(\mathbb{R}) \text{ or } f \in L^p(\mathbb{R}),$$

and we put

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)| \quad \text{if } f \text{ is a bounded continuous function on } \mathbb{R}.$$

Let $C_c(\mathbb{R})$ denote the space of complex-valued continuous functions on $\mathbb{R}$ with compact support. The following Proposition is proved in Rudin, Theorem 3.14.

6.7 Proposition Let $p \in [1,\infty)$. Then $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Ex. 6.8 A function $f\colon \mathbb{R} \to \mathbb{C}$ is called *simple* if it is a linear combination of characteristic functions of bounded intervals. Equivalently, f is simple if there are real numbers $a_1 < a_2 < \ldots < a_n$ such that f vanishes outside $[a_1, a_n]$ and f is constant on each interval (a_i, a_{i+1}) $(i = 1, \ldots, n-1)$.
Let $p \in [1,\infty)$. Prove, by using Proposition 6.7, that the space of simple functions is dense in $L^p(\mathbb{R})$.

6.9 Theorem Let $f \in \mathcal{L}^1(\mathbb{R})$. Then

(a) $\widehat{f}$ is a bounded continuous function on $\mathbb{R}$.

(b) $\|\widehat{f}\|_\infty \le \|f\|_1$.

(c) $\lim_{x\to\pm\infty} \widehat{f}(x) = 0$ (*Riemann-Lebesgue Lemma*).

Proof For the proof of the continuity of $\widehat{f}$ note that, for $x \in \mathbb{R}$,

$$\lim_{y\to x} \int_{-\infty}^{\infty} f(t)\, e^{-iyt}\, dt = \int_{-\infty}^{\infty} \Big(\lim_{y\to x} f(t)\, e^{-iyt}\Big)\, dt = \int_{-\infty}^{\infty} f(t)\, e^{-ixt}\, dt$$

by Lebesgue's dominated convergence theorem (syll. Integratietheorie) since $|f(t)\, e^{-iyt}| \le |f(t)|$ and $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$. Clearly, it follows from (6.4) that

$$|\widehat{f}(x)| \le \int_{-\infty}^{\infty} |f(t)|\, dt = \|f\|_1 \qquad (x \in \mathbb{R}).$$

This settles (a) and (b).

For the proof of (c) first note that the result is valid if f is a simple function. This follows from Exercise 6.4. Now let $f \in \mathcal{L}^1(\mathbb{R})$. Let $\varepsilon > 0$. By Exercise 6.8 there is a simple function g such that $\|f - g\|_1 < \varepsilon/2$. By (b) we have that $|\widehat{f}(x) - \widehat{g}(x)| < \varepsilon/2$. Hence $|\widehat{f}(x)| \le \varepsilon/2 + |\widehat{g}(x)|$ and there is $M > 0$ such that $|\widehat{g}(x)| < \varepsilon/2$ if $|x| > M$. □

Ex. 6.10 Let

$$C_0(\mathbb{R}) := \{f\colon \mathbb{R} \to \mathbb{C} \mid f \text{ is continuous and } \lim_{x\to\pm\infty} f(x) = 0\}.$$

Prove that $C_0(\mathbb{R})$ is a closed linear subspace of the Banach space of bounded continuous functions on $\mathbb{R}$ with respect to the sup-norm, so $C_0(\mathbb{R})$ is a Banach space itself with respect to the sup-norm. Also observe that the map $f \mapsto \widehat{f}\colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is a bounded linear operator.

Ex. 6.11 Let $f \in \mathcal{L}^1(\mathbb{R})$, $a \in \mathbb{R}$, $c \in \mathbb{R}\backslash\{0\}$. Prove the following.

(a) If $g(t) := f(t+a)$ then $\widehat{g}(x) = e^{iax}\, \widehat{f}(x)$.

(b) If $g(t) := e^{iat}\, f(t)$ then $\widehat{g}(x) = \widehat{f}(x-a)$.

(c) If $g(t) := f(ct)$ then $\widehat{g}(x) = |c|^{-1}\, \widehat{f}(c^{-1}x)$.

(d) In particular, if $g(t) := f(-t)$ then $\widehat{g}(t) = \widehat{f}(-t)$.

6.12 Proposition Let $f, g \in \mathcal{L}^1(\mathbb{R})$. Then

$$\int_{-\infty}^{\infty} f(x)\, \widehat{g}(x)\, dx = \int_{-\infty}^{\infty} \widehat{f}(x)\, g(x)\, dx.$$

(Both integrals converge absolutely because of Theorem 6.9(a).)

Proof We have to show that

$$\int_{x=-\infty}^{\infty} \Big(\int_{y=-\infty}^{\infty} f(x)\, e^{-ixy}\, g(y)\, dy\Big)\, dx$$

is equal to

$$\int_{y=-\infty}^{\infty} \Big(\int_{x=-\infty}^{\infty} f(x)\, e^{-ixy}\, g(y)\, dx\Big)\, dy.$$

This follows from the Fubini Theorem 2.14 since

$$\int_{x=-\infty}^{\infty}\left(\int_{y=-\infty}^{\infty}|f(x)\,e^{-ixy}\,g(y)|\,dy\right)dx<\infty. \qquad \square$$

6.13 Proposition Let $f\in C^1(\mathbb{R})$ such that $f,f'\in\mathcal{L}^1(\mathbb{R})$. Then

$$(f')^\frown(x)=ix\,\widehat{f}(x), \tag{6.5}$$

and $\widehat{f}(x)=o(|x|^{-1})$ as $x\to\pm\infty$.

Proof Let $M>0$. Integration by parts shows that

$$\int_{-M}^{M}f'(t)\,e^{-ixt}\,dt=e^{-ixM}\,f(M)-e^{ixM}\,f(-M)+ix\int_{-M}^{M}f(t)\,e^{-ixt}\,dt.$$

Then (6.5) will follow by letting $M\to\infty$ in the above formula, provided we can show that $\lim_{M\to\infty}f(\pm M)=0$. These last limits are indeed valid. For instance, $f(M)=f(0)+\int_0^M f'(t)\,dt$. Hence $f(\infty):=\lim_{M\to\infty}f(M)=f(0)+\int_0^\infty f'(t)\,dt$ exists. Then necessarily $f(\infty)=0$, since otherwise $\int_{-\infty}^{\infty}|f(t)|\,dt=\infty$.

The second statement follows by combination with Theorem 6.9(c). $\square$

6.14 Proposition Let $f\in\mathcal{L}^1(\mathbb{R})$ such that the function $g\colon x\mapsto xf(x)$ is in $\mathcal{L}^1(\mathbb{R})$. Then $\widehat{f}$ is differentiable on $\mathbb{R}$ with $(\widehat{f})'=-i\widehat{g}$.

Proof Let $a,b\in\mathbb{R}$ with $a<b$. Then, by Proposition 6.12 and Exercise 1.4,

$$\begin{aligned}\int_a^b\widehat{g}(x)\,dx&=\int_{-\infty}^{\infty}\widehat{g}(x)\,\chi_{[a,b]}(x)\,dx\\&=\int_{-\infty}^{\infty}g(x)\,(\chi_{[a,b]})^\frown(x)\,dx\\&=\int_{-\infty}^{\infty}xf(x)\,\frac{e^{-iax}-e^{-ibx}}{ix}\,dx\\&=i\int_{-\infty}^{\infty}f(x)(e^{-ibx}-e^{-iax})\,dx=i(\widehat{f}(b)-\widehat{f}(a)).\end{aligned}$$

Hence

$$\frac{\widehat{f}(a+h)-\widehat{f}(a)}{h}=-ih^{-1}\int_a^{a+h}\widehat{g}(x)\,dx.$$

Now let $h\to0$ and use the fact that $\widehat{g}$ is continuous. $\square$

6.15 Corollary

(a) Let $f\in C^k(\mathbb{R})$ such that $f,f',\ldots,f^{(k)}\in\mathcal{L}^1(\mathbb{R})$. Then $(f^{(k)})^\frown(x)=(ix)^k\,\widehat{f}(x)$ and $\widehat{f}(x)=o(|x|^{-k})$ as $x\to\pm\infty$.

(b) Let f be a Lebesgue measurable function on $\mathbb{R}$ such that $\int_{-\infty}^{\infty}(1+|t|)^k\,|f(t)|\,dt<\infty$. Then $\widehat{f}\in C^k(\mathbb{R})$ and $(\widehat{f})^{(k)}(x)=\int_{-\infty}^{\infty}(-it)^k\,f(t)\,e^{-ixt}\,dt$.

(c) Let $\alpha\in\mathbb{R}$ with $\alpha>1$. Let f be a Lebesgue measurable function on $\mathbb{R}$ such that $f(x)=\mathcal{O}(|x|^{-\alpha})$ for $|x|\to\infty$. Then $\widehat{f}\in C^k(\mathbb{R})$ if $k+1<\alpha$.

Observe the general rules:

- Faster decrease of $|f(t)|$ to 0 as $|t| \to \infty$ implies more smoothness of $\widehat{f}$.
- More smoothness of f implies faster decrease of $|\widehat{f}(x)|$ to 0 as $|x| \to \infty$.

6.16 We introduce the following notation for some operators acting on functions on $\mathbb{R}$:

$$\begin{aligned}
(Df)(x) &:= f'(x) \quad (\textit{differentiation}),\\
(Mf)(x) &:= x\,f(x) \quad (\textit{multiplication}),\\
(T_a f)(x) &:= f(x+a) \quad (\textit{translation}),\\
(E_a f)(x) &:= e^{iax}\,f(x) \quad (\textit{exponential multiplication}),\\
(Pf)(x) &:= f(-x) \quad (\textit{parity operator}).
\end{aligned}$$

Some of the previous results can now be summarized in the following table:

g	$\widehat{g}$
Df	$iM\widehat{f}$
Mf	$iD\widehat{f}$
$E_a f$	$T_{-a}\widehat{f}$
$T_a f$	$E_a\widehat{f}$
$t \mapsto f(ct)$	$x \mapsto \lvert c\rvert^{-1}\widehat{f}(c^{-1}x)$
Pf	$P\widehat{f}$

6.17 Definition $\mathcal{S}$ is the set of all $f \in C^\infty(\mathbb{R})$ such that, for all nonnegative integers n, m the function $x \mapsto x^n\,f^{(m)}(x)$ is bounded.

So $\mathcal{S}$ consists of the C^∞-functions f on $\mathbb{R}$ such that f and all its derivatives are rapidly decreasing to 0 as the argument tends to $\pm\infty$. Briefly we call such functions *rapidly decreasing C^∞-functions on* $\mathbb{R}$. This class of functions was introduced by Laurent Schwartz. He used this as a class of test functions for his so-called tempered distributions.

The (immediate) proofs of the following statements are left to the reader.

6.18 Proposition $\mathcal{S}$ is a linear space. If $f \in \mathcal{S}$ then the functions Df, Mf, $E_a f$, $T_a f$, $x \mapsto f(cx)$ and Pf are also in $\mathcal{S}$. If $f \in \mathcal{S}$ then $\widehat{f} \in \mathcal{S}$.

Ex. 6.19 Let $f(t) := e^{-\frac{1}{2}t^2}$ (the *Gaussian function*). Show the following:

(a) $f \in \mathcal{S}$.

(b) $Df = -Mf$, $\quad D\widehat{f} = -M\widehat{f}$.

(c) $\widehat{f}(x) = C\,e^{-\frac{1}{2}x^2}$ with $C = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}$.

Hint Solve the differential equation for $\widehat{f}$ in (b).

Ex. 6.20 Let $f(t) := e^{-\frac{1}{2}t^2}$. Then

$$\widehat{f}(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}\,e^{-ixt}\,dt = e^{-\frac{1}{2}x^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+ix)^2}\,dt \quad (x \in \mathbb{R}).$$

Derive from this formula the result of Exercise 6.19(c), by contour integration.

Ex. 6.21 Let $f \in \mathcal{L}^1(\mathbb{R})$. Suppose that f has compact support, i.e., f vanishes outside some bounded interval. Prove the following.

(a) If $\widehat{f} = 0$ then $f = 0$ a.e..
Hint Reduce this to the case of periodic functions.

(b) The function $\widehat{f}$ is the restriction to $\mathbb{R}$ of an entire analytic function on $\mathbb{C}$ (the function given by (6.4) for $x \in \mathbb{C}$).

(c) If $\widehat{f}$ also vanishes on $\mathbb{R}$ outside some bounded interval then $f = 0$ a.e..

6.22 Definition-Theorem For $f, g \in \mathcal{L}^1(\mathbb{R})$ the *convolution product* $f * g$ is defined by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy \qquad (x \in \mathbb{R}). \tag{6.6}$$

Then the integral in (6.6) converges absolutely for almost all $x \in \mathbb{R}$ and $f * g \in L^1(\mathbb{R})$. Furthermore,

$$\|f * g\|_1 \le \|f\|_1\, \|g\|_1 \tag{6.7}$$

and

$$(f * g)\widehat{\ }(x) = \widehat{f}(x)\, \widehat{g}(x). \tag{6.8}$$

Proof Analogous to the proof of Definition-Theorem 2.19. Use the Fubini Theorem 2.14. □

Ex. 6.23 Let $p(x)$ be a polynomial with real coefficients in the real variable x. Find necessary and sufficient conditions in order that the function $x \mapsto e^{p(x)}$ belongs to $\mathcal{S}$.

7 Inversion formula

7.1 (Dirichlet kernel) Let $f \in \mathcal{L}^1(\mathbb{R})$, $\omega > 0$. Put

$$(I_\omega[f])(s) := \frac{1}{2\pi} \int_{-\omega}^{\omega} \widehat{f}(x)\, e^{ixs}\, dx \quad (s \in \mathbb{R}). \tag{7.1}$$

In view of (6.2) we expect (but still have to prove) that $I_\omega[f] \to f$ as $\omega \to \infty$. The advantage of (7.1) compared to the corresponding integral from $-\infty$ to ∞ is that it can be written as an integral transformation (of convolution type) acting on f with very simple integral kernel: the *Dirichlet kernel*

$$\frac{\sin(\omega t)}{\pi t} = \frac{1}{2\pi} \int_{-\omega}^{\omega} e^{-ixt}\, dx. \tag{7.2}$$

Proposition We have

$$(I_\omega[f])(s) = \int_{-\infty}^{\infty} f(t+s)\, \frac{\sin(\omega t)}{\pi t}\, dt, \tag{7.3}$$

$$\left|\frac{\sin(\omega t)}{\pi t}\right| \le \omega/\pi \quad (t \in \mathbb{R}), \tag{7.4}$$

$$\int_{-\infty}^{\infty} \frac{\sin(\omega t)}{\pi t}\, dt = \frac{2}{\pi} \int_0^{\infty} \frac{\sin t}{t}\, dt = 1 \tag{7.5}$$

(the last two integrals converging but not absolutely converging).

The proof of these formulas is left to the reader. For the proof of (7.3) substitute (6.4) into (7.1) and interchange the two integrals (allowed by Fubini's theorem; check this).

7.2 The following results are analogous to §3.2. We give a convergence criterium for Fourier inversion in a certain point.

Theorem Let $f \in \mathcal{L}^1(\mathbb{R})$, $a \in \mathbb{R}$. Suppose that one of the following two conditions holds:
(a) There are $M, \alpha, \delta > 0$ such that

$$|f(a+x) - f(a)| < M\, |x|^\alpha \quad \text{for } |x| < \delta.$$

(b) f is continuous in a and also right and left differentiable in a.
Then we have:

$$\lim_{\omega \to \infty} (I_\omega[f])(a) = f(a). \tag{7.6}$$

Proof Condition (b) implies condition (a), so we may assume Condition (a). We have

$$\begin{aligned} &(I_\omega[f])(a) - f(a) \int_{-1}^{1} \frac{\sin(\omega x)}{\pi x}\, dx \\ &= \int_{-1}^{1} \frac{f(a+x) - f(a)}{\pi x}\, \sin(\omega x)\, dx + \int_{|x|>1} \frac{f(a+x)}{\pi x}\, \sin(\omega x)\, dx. \end{aligned} \tag{7.7}$$

Since

$$\int_{-1}^{1} \frac{\sin(\omega x)}{\pi x}\,dx = \int_{-\omega}^{\omega} \frac{\sin y}{\pi y}\,dy \longrightarrow \frac{2}{\pi}\int_0^\infty \frac{\sin y}{y}\,dy = 1 \quad \text{as } \omega \to \infty,$$

it is sufficient to prove that the right hand side tends to 0 as $\omega \to \infty$. However, we will see that the right hand side is of the form $\pi^{-1}\int_{-\infty}^{\infty} g(x)\sin(\omega x)\,dx$ with $g \in \mathcal{L}^1(\mathbb{R})$, so it tends to 0 as $\omega \to \infty$ because of the Riemann-Lebesgue lemma (see Theorem 6.9(c)).

The claim about the right hand side is proved by splitting it up as a sum of three terms (where we assume that $\delta < 1$):

$$\begin{aligned} |x| < \delta: &\qquad |g(x)| = \left|\frac{f(a+x)-f(a)}{x}\right| < M\,|x|^{\alpha-1}, \\ \delta \le |x| < 1: &\qquad |g(x)| = \left|\frac{f(a+x)-f(a)}{x}\right| \le \delta^{-1}(|f(a+x)| + |f(a)|), \\ |x| \ge 1: &\qquad |g(x)| = \left|\frac{f(a+x)}{x}\right| \le |f(a+x)|. \end{aligned}$$

□

7.3 Theorem Let $f \in \mathcal{L}^1(\mathbb{R})$, let $a \in \mathbb{R}$, and suppose that the limits

$$f(a_+) := \lim_{x\downarrow a} f(x), \qquad f(a_-) := \lim_{x\uparrow a} f(x)$$

exist. Suppose that one of the two following conditions holds:

(a) There are $M, \alpha, \delta > 0$ such that

$$|f(a+x) - f(a_+)| \le Mx^\alpha \quad \text{and} \quad |f(a-x) - f(a_-)| \le Mx^\alpha \quad \text{for } 0 < x < \delta.$$

(b) f is right and left differentiable in a in the sense that the following two limits exist:

$$f'(a_+) := \lim_{y\downarrow 0} \frac{f(a+y) - f(a_+)}{y}, \qquad f'(a_-) := \lim_{y\downarrow 0} \frac{f(a-y) - f(a_-)}{-y}\,.$$

Then we have:

$$\lim_{\omega\to\infty} (I_\omega[f])(a) = \tfrac{1}{2}(f(a_+) + f(a_-)).$$

Ex. 7.4 Prove Theorem 7.3 as a corollary of Theorem 7.2, analogous to the proof of Theorem 3.5.

7.5 Corollary (*localization principle*) Let $f, g \in \mathcal{L}^1(\mathbb{R})$, let $a \in \mathbb{R}$, and suppose that $f(x) = g(x)$ for x in a certain neighbourhood of a. Then precisely one of the following two alternatives holds for the behaviour of $(I_\omega[f])(a)$ and $(I_\omega[g])(a)$ as $\omega \to \infty$:

(a) Both limits exist and are equal:

$$\lim_{\omega\to\infty} (I_\omega[f])(a) = \lim_{\omega\to\infty} (I_\omega[g])(a)$$

(b) Neither $(I_\omega[f])(a)$ nor $(I_\omega[g])(a)$ has a limit as $\omega \to \infty$.

Thus, if for some $f \in \mathcal{L}^1(\mathbb{R})$ and some $a \in \mathbb{R}$ the inversion formula (7.6) holds, then this inversion formula remains valid if f is perturbed within $\mathcal{L}^1(\mathbb{R})$ such that f remains unchanged on some neighbourhood of a.

Ex. 7.6 Prove Corollary 7.5.

7.7 Corollary Let $f \in \mathcal{L}^1(\mathbb{R})$ such that also $\widehat{f} \in \mathcal{L}^1(\mathbb{R})$ and f is differentiable on $\mathbb{R}$. Then $(\widehat{f}\,)\widehat{\ }$ is well-defined and

$$(\widehat{f}\,)\widehat{\ }(s) = 2\pi\, f(-s) \qquad (s \in \mathbb{R}).$$

Ex. 7.8 Prove Corollary 7.7. Can the differentiability assumption be relaxed?

Ex. 7.9 (van Rooij, Voorbeeld 15.8) Define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

(a) Show that $f \in \mathcal{L}^1(\mathbb{R})$ and $\widehat{f}(x) = 2x^{-2}(1 - \cos x)$ $(x \ne 0)$, $\widehat{f}(0) = 1$.

(b) Show that $\widehat{f} \in \mathcal{L}^1(\mathbb{R})$ and that $(\widehat{f}\,)\widehat{\ }(a) = 2\pi\, f(-a)$ for all $a \in \mathbb{R}$.

(c) Show that

$$\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2}\, dx = \pi.$$

Ex. 7.10 Let

$$f(t) := \begin{cases} \frac{1}{2} - t & \text{if } 0 < t \le \frac{1}{2}, \\ -\frac{1}{2} - t & \text{if } -\frac{1}{2} \le t < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\widehat{f}(x)$. Examine the behaviour of $\widehat{f}(x)$ as $|x| \to \infty$.

Ex. 7.11 Let f be continuously differentiable on $\mathbb{R}\backslash\{0\}$. Suppose that $f, f' \in L^1(\mathbb{R})$ and that the following four limits exist:

$$a := \lim_{t \downarrow 0} f(t), \quad b := \lim_{t \uparrow 0} f(t), \quad \lim_{t \downarrow 0} \frac{f(t) - a}{t}, \quad \lim_{t \uparrow 0} \frac{f(t) - b}{t}.$$

(a) Show that $\widehat{f}(x) = \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$.

(b) Show that $\widehat{f}(x) = o(|x|^{-1})$ as $|x| \to \infty$ iff $a = b$.

Ex. 7.12 Let

$$f(t) := \begin{cases} (1 - t^2)^{\alpha - \frac{1}{2}} & \text{if } |t| < 1, \\ 0 & \text{if } |t| \ge 1. \end{cases}$$

Assume that $\operatorname{Re} \alpha > -\frac{1}{2}$. Show that

$$\widehat{f}(x) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + 1)} \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k!\,(\alpha + 1)_k},$$

where $(\alpha + 1)_k := (\alpha + 1)(\alpha + 2) \ldots (\alpha + k)$.
What can be concluded about the speed with which $|\widehat{f}(x)|$ tends to 0 as $|x| \to \infty$?

8 L^2 theory

We will work now with the space $\mathcal{L}^2(\mathbb{R})$ of square integrable functions on $\mathbb{R}$ and with the Hilbert space $L^2(\mathbb{R})$ of equivalence classes of such functions. Inner product and norm are given by

$$< f, g >:= \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx, \qquad \|f\|_2 := (\langle f, f\rangle)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} |f(x)|^2\,dx\right)^{\frac{1}{2}}. \tag{8.1}$$

The space $\mathcal{S}$ (see Definition 6.17) is a linear subspace of $L^2(\mathbb{R})$.

We will now denote by $\mathcal{F}$ a slightly renormalized version of the Fourier transform $f \mapsto \widehat{f}$:

$$(\mathcal{F}f)(x) := \frac{1}{\sqrt{2\pi}}\,\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\,e^{-ixt}\,dt. \tag{8.2}$$

Here $f \in \mathcal{L}^1(\mathbb{R})$, in particular we will take $f \in \mathcal{S}$.

8.1 Proposition The map $\mathcal{F}\colon \mathcal{S} \to \mathcal{S}$ is a linear bijection, with inverse

$$(\mathcal{F}^{-1}g)(s) = (\mathcal{F}g)(-s) \qquad (g \in \mathcal{S}). \tag{8.3}$$

Furthermore $\mathcal{F}$ preserves the inner product (8.1) on $\mathcal{S}$:

$$\langle \mathcal{F}f, \mathcal{F}g\rangle = \langle f, g\rangle \qquad (f, g \in \mathcal{S}). \tag{8.4}$$

For convenience we give also more extended versions of (8.3) and (8.4):

$$(\mathcal{F}^{-1}g)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\,e^{isx}\,dx, \tag{8.5}$$

$$\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt = \int_{-\infty}^{\infty} (\mathcal{F}f)(x)\,\overline{(\mathcal{F}g)(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(x)\,\overline{\widehat{g}(x)}\,dx. \tag{8.6}$$

The proof of this Proposition is immediate, in view of Corollary 7.7, Proposition 6.18 and Proposition 6.12.

8.2 We will now introduce some special C^∞ functions, which we will need as tools (see also van Rooij, §18.1).

Let $a < b$ and put

$$f(x) := \begin{cases} \exp\left(\dfrac{1}{x-b} - \dfrac{1}{x-a}\right) & \text{if } a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in C^\infty(\mathbb{R})$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$.

Let

$$F(x) := \Big(\int_a^x f(t)\,dt\Big) \Big/ \Big(\int_a^b f(t)\,dt\Big).$$

Then $F \in C^\infty(\mathbb{R})$, $F(x) = 0$ if $x \leq a$, $F(x) = 1$ if $x \geq b$, and F is strictly increasing on $[a, b]$.

Ex. 8.3 Let $[a,b]$ be a closed bounded interval and $\chi_{[a,b]}$ its characteristic function. Show that $\chi_{[a,b]}$ can be approximated arbitrarily closely in L^2 norm by functions in $\mathcal{S}$.

8.4 Proposition The space $\mathcal{S}$ is dense in $L^2(\mathbb{R})$.

Proof By Exercise 6.8 the simple functions are dense in $L^2(\mathbb{R})$. Next use Exercise 8.3. □

8.5 As a corollary of the previous propositions we obtain:

Theorem (Parseval-Plancherel) The map $\mathcal{F}\colon \mathcal{S} \to \mathcal{S}$ has a unique extension to a continuous linear map $\mathcal{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$. This extension is an isometry of Hilbert spaces:

$$\langle \mathcal{F}f, \mathcal{F}g\rangle = \langle f, g\rangle \qquad (f, g \in L^2(\mathbb{R})),$$

it is bijective, and it satisfies

$$(\mathcal{F}^2 f)(s) = f(-s) \quad \text{a.e.}$$

We have obtained two definitions of the Fourier transform, which we will denote by $\mathcal{F}_1$ resp. $\mathcal{F}_2$ for the moment:

$$\begin{aligned} (\mathcal{F}_1 f)(x) :=& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ixt}\, dt \qquad (f \in L^1(\mathbb{R})),\\ (\mathcal{F}_2 f)(x) :=& (L^2) \lim_{n\to\infty} \mathcal{F}_1 f_n, \quad \text{where } F_n \in \mathcal{S} \text{ and } (L^2) \lim_{n\to\infty} f_n = f. \end{aligned}$$

For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ it will turn out that $\mathcal{F}_1 f = \mathcal{F}_2 f$.

Lemma Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then there is a sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{S}$ such that $\|f - f_n\|_1 \to 0$ and $\|f - f_n\|_2 \to 0$ as $n \to \infty$.

Proof Let $\varepsilon > 0$. There exists $M > 1$ such that $\|f\chi_{[-M,M]} - f\|_p < \varepsilon$ for $p = 1, 2$. Write $g := f\chi_{[-M,M]}$. Then there exists $h \in \mathcal{S}$ such that $\|g - h\|_2 < \varepsilon/(2\sqrt{M}) < \varepsilon$. We can even choose h in this way such that it has support within $[-2M, 2M]$ (why?). Thus

$$\|g - h\|_1 = \int_{-2M}^{2M} |g(x) - h(x)|\, dx \le 2\sqrt{M}\, \|g - h\|_2 < \varepsilon.$$

Hence $\|f - h\|_1 < 2\varepsilon$ and $\|f - h\|_2 < 2\varepsilon$. □

8.6 Theorem If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\mathcal{F}_1 f = \mathcal{F}_2 f$.

Proof Take a sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{S}$ such that $\|f - f_n\|_1 \to 0$ and $\|f - f_n\|_2 \to 0$ if $n \to \infty$. Then $\mathcal{F}_2 f = \lim_{n\to\infty} \mathcal{F}_1 f_n$ in $L^2(\mathbb{R})$. Since $\sqrt{2\pi}\, \|\mathcal{F}_1 f - \mathcal{F}_1 f_n\|_\infty \le \|f - f_n\|_1$, it follows that $(\mathcal{F}_1 f)(x) = \lim_{n\to\infty} (\mathcal{F}_1 f_n)(x)$ pointwise. From integration theory we know that a convergent sequence in $L^2(\mathbb{R})$ always has a subsequence which converges almost everywhere to the limit function. Hence the sequence $(\mathcal{F}_1 f_n)_{n=1}^{\infty}$ has a subsequence $(\mathcal{F}_1 f_{n_k})_{k=1}^{\infty}$ such that

$$(\mathcal{F}_2 f)(x) = \lim_{k\to\infty} (\mathcal{F}_1 f_{n_k})(x) = (\mathcal{F}_1 f)(x) \quad \text{a.e.}$$

□

8.7 Theorem If $f \in L^2(\mathbb{R})$ then

$$(\mathcal{F}f)(x) = (L^2) \lim_{M\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-M}^{M} f(t)\, e^{-ixt}\, dt,$$

which can be rewritten as

$$\lim_{M\to\infty} \int_{-\infty}^{\infty} \Big|(\mathcal{F}f)(x) - \frac{1}{\sqrt{2\pi}} \int_{-M}^{M} f(t)\, e^{-ixt}\, dt\Big|^2\, dx = 0.$$

Proof $\|f - f\chi_{[-M,M]}\|_2 \to 0$ if $M \to \infty$. Hence $\|\mathcal{F}_2 f - \mathcal{F}_2(f\chi_{[-M,M]})\|_2 \to 0$ as $M \to \infty$. Because $f\chi_{[-M,M]} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, it follows that $\mathcal{F}_2(f\chi_{[-M,M]}) = \mathcal{F}_1(f\chi_{[-M,M]})$. □

Ex. 8.8 (van Rooij, Opgave 19.A) Show that not $\mathcal{L}^1(\mathbb{R}) \subset \mathcal{L}^2(\mathbb{R})$. But show that all bounded functions in $\mathcal{L}^1(\mathbb{R})$ lie in $\mathcal{L}^2(\mathbb{R})$.

Ex. 8.9 (van Rooij, Voorbeeld 19.9) Use the Fourier transform computed in Exercise 7.9 in order to show that

$$\int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{2\pi}{3}.$$

Ex. 8.10 (van Rooij, Opgave 19.B) Let $f, g \in \mathcal{L}^2(\mathbb{R})$. Then $fg \in \mathcal{L}^1(\mathbb{R})$, thus $\widehat{fg}$ exists. Take representatives in $\mathcal{L}^2(\mathbb{R})$ of $\widehat{f}, \widehat{g} \in L^2(\mathbb{R})$ and denote these representatives also by $\widehat{f}, \widehat{g}$. Show that

$$2\pi\, \widehat{fg}(x) = \int_{-\infty}^{\infty} \widehat{f}(t)\, \widehat{g}(x-t)\, dt.$$

Ex. 8.11 (van Rooij, Opgave 19.E) For n a nonnegative integer define the function $h_n \in \mathcal{S}$ and the *Hermite polynomial* H_n by

$$h_n(x) := (x - d/dx)^n\, e^{-x^2/2}, \qquad H_n(x) := e^{x^2/2}\, h_n(x).$$

Prove the following.

(a) We have

$$H_{n+1}(x) = 2x\, H_n(x) - H_n'(x) \quad \text{and} \quad H_n(x) = (-1)^n\, e^{x^2}\, (d/dx)^n\, e^{-x^2}.$$

H_n is a polynomial of degree n with real coefficients; the term of degree n in H_n has coefficient 2^n.

(b) $\widehat{h_n} = (-i)^n \sqrt{2\pi}\, h_n$. (Thus h_n is an eigenfunction of the Fourier transform $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.)

(c) If $n \neq m$ then $\langle h_n, h_m \rangle = 0$. (So the functions h_n form an orthogonal system in $L^2(\mathbb{R})$.)
Hint Let $n > m$. Substitute $h_n(x) = (-1)^n\, e^{x^2/2}\, (d/dx)^n\, e^{-x^2}$, $h_m(x) = e^{-x^2/2}\, H_m(x)$, and perform integration by parts.

(d) If $f \in \mathcal{L}^2(\mathbb{R})$ and $\langle f, h_n \rangle = 0$ for all n, then $f(x) = 0$ a.e.. (So the functions h_n form a complete orthogonal system in $L^2(\mathbb{R})$.)
Hint For each polynomial P we have $\int_{-\infty}^{\infty} f(x)\, e^{-x^2/2}\, P(x)\, dx = 0$. By the dominated convergence theorem this implies that

$$\int_{-\infty}^{\infty} f(x)\, e^{-x^2/2} e^{-ixy}\, dx = 0.$$

Ex. 8.12 (For this exercise use results from both Fourier series and Fourier integrals.) Below define $x^{-1} \sin x$ for $x = 0$ by continuity.

(a) Let $t \in \mathbb{R}$. Show that for each $x \in (-\pi, \pi)$ we have

$$\sum_{n=-\infty}^{\infty} \frac{\sin(\pi(t-n))}{\pi(t-n)}\, e^{inx} = e^{ixt}$$

with pointwise convergence. What is the evaluation of the sum on the left hand side for other real values of x?

(b) Show that, for all $n, m \in \mathbb{Z}$, we have

$$\int_{-\infty}^{\infty} \frac{\sin(\pi(t-n))}{\pi(t-n)}\, \frac{\sin(\pi(t-m))}{\pi(t-m)}\, dt = \delta_{n,m},$$

where the integral converges absolutely.

(c) Does there exist $f \in L^2(\mathbb{R})$ with $f \neq 0$ such that

$$\int_{-\infty}^{\infty} f(t)\, \frac{\sin(\pi(t-n))}{\pi(t-n)}\, dt = 0 \quad \text{for all } n \in \mathbb{Z}\,?$$

(d) Let $f \in L^2([-\pi, \pi])$. Extend $\widehat{f}$, defined as a function on $\mathbb{Z}$ by (1.8), to a function on $\mathbb{R}$ by

$$\widehat{f}(t) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-ixt}\, dx \qquad (t \in \mathbb{R}).$$

Show that

$$\widehat{f}(t) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\, \frac{\sin(\pi(t-n))}{\pi(t-n)} \qquad (t \in \mathbb{R})$$

with absolutely convergent sum.

9 Poisson summation formula

The Poisson summation formula is an equality of two doubly infinite sums over sets of equidistant points in $\mathbb{R}$. The first sum involves a function f on $\mathbb{R}$, the second one its Fourier transform $\widehat{f}$.

9.1 Theorem Let $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ such that $f(t) = \mathcal{O}(|t|^{-2})$ as $|t| \to \infty$ and $\widehat{f}(x) = \mathcal{O}(|x|^{-2})$ as $|x| \to \infty$. (Take for instance $f \in \mathcal{S}$.) Then

$$\sum_{l=-\infty}^{\infty} f(x+2\pi l) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \widehat{f}(k)\, e^{ikx} \qquad (x \in \mathbb{R}), \tag{9.1}$$

where the series on both sides pointwise and absolutely converge to a function in $C_{2\pi}$ (i.e., to a 2π-periodic continuous function).

Proof Put

$$g_N(x) := \sum_{l=-N}^{N} f(x+2\pi l), \qquad g(x) := \lim_{N\to\infty} g_N(x) = \sum_{l=-\infty}^{\infty} f(x+2\pi l).$$

The last series converges pointwise and absolutely on $\mathbb{R}$ because $f(x) = \mathcal{O}(|x|^{-2})$ as $|x| \to \infty$. This series even converges uniformly for x in a compact subset of $\mathbb{R}$. Hence g is continuous on $\mathbb{R}$. Clearly, g is also 2π-periodic. Next observe that

$$|g_N(x)| \le \sum_{l=-\infty}^{\infty} |f(x+2\pi l)| \quad \text{and} \quad \int_{-\pi}^{\pi} \left(\sum_{l=-\infty}^{\infty} |f(x+2\pi l)| \right) dx = \int_{-\infty}^{\infty} |f(y)|\, dy < \infty.$$

This justifies an application of the dominated convergence theorem as follows ($k \in \mathbb{Z}$):

$$\int_{-\pi}^{\pi} g(x)\, e^{-ikx}\, dx = \lim_{N\to\infty} \int_{-\pi}^{\pi} g_N(x)\, e^{-ikx}\, dx$$
$$= \lim_{N\to\infty} \int_{-(2N+1)\pi}^{(2N+1)\pi} f(y)\, e^{-iky}\, dy = \int_{-\infty}^{\infty} f(y)\, e^{-iky}\, dy = \widehat{f}(k).$$

Hence, because $\widehat{f}(k) = \mathcal{O}(|k|^{-2})$ as $|k| \to \infty$, we conclude that $g(x)$ equals the right hand side of (9.1). □

Specialization of (9.1) yields

$$\sum_{l=-\infty}^{\infty} f(2\pi l) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \widehat{f}(k). \tag{9.2}$$

Here f is as in Theorem 9.1 and there is absolute convergence on both sides.

Ex. 9.2 Let f be as in Theorem 9.1. Assume $c>0$. Prove the following identities.

$$(a)\qquad \sum_{l=-\infty}^{\infty} f(x+2\pi c\,l)=\frac{1}{2\pi c}\sum_{k=-\infty}^{\infty}\widehat{f}(c^{-1}k)\,e^{ikc^{-1}x},$$

$$(b)\qquad \sum_{l=-\infty}^{\infty} f(2\pi c\,l)=\frac{1}{2\pi c}\sum_{k=-\infty}^{\infty}\widehat{f}(c^{-1}k),$$

$$(c)\qquad \sum_{l=-\infty}^{\infty} e^{-\frac{1}{2}(x+2\pi c\,l)^2}=\frac{1}{c\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} e^{-\frac{1}{2}c^{-2}k^2}e^{ikc^{-1}x},$$

$$(d)\qquad \sum_{l=-\infty}^{\infty} e^{-\pi(x+c\,l)^2}=c^{-1}\sum_{k=-\infty}^{\infty} e^{-\pi c^{-2}k^2}e^{2\pi ikc^{-1}x},$$

$$(e)\qquad \sum_{l=-\infty}^{\infty} e^{-\pi c^2l^2}=c^{-1}\sum_{k=-\infty}^{\infty} e^{-\pi c^{-2}k^2}.$$

Ex. 9.3 Show that (d) and (e) of Exercise 9.2 remain valid if $c,x\in\mathbb{C}$ with $|\arg c^2|<\pi/2$. One of the *theta functions* is defined by

$$\vartheta_3(z\mid\tau):=\sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2}\,e^{2inz}\qquad(\operatorname{Im}\tau>0,\ z\in\mathbb{C}).$$

Use (d) of Exercise 9.2 in order to show that

$$\tau^{-\frac{1}{2}}\exp\big(-z^2/(\pi\tau)\big)\,\vartheta_3(\tau^{-1}z\mid\tau^{-1}i)=\vartheta_3(z\mid i\tau)\qquad(\operatorname{Re}\tau>0,\ z\in\mathbb{C}).$$

Ex. 9.4 (van Rooij, Opgave 14.A) Show that

$$\sum_{n\in\mathbb{Z}}\frac{1}{n^2+\beta^2}=\frac{\pi}{\beta}\,\frac{\cosh\pi\beta}{\sinh\pi\beta}\qquad(\beta>0).$$

Hint Apply (9.2) with $f(x):=e^{-\beta|x|}$.

Ex. 9.5 Apply (9.2) with f as in Exercise 7.10 in order to show that

$$\sum_{m=-\infty}^{\infty}\frac{1}{(x+m)^2}=\frac{\pi^2}{(\sin\pi x)^2}\qquad(x\in\mathbb{R}\backslash\mathbb{Z}).$$

10 Some applications of Fourier integrals

10.1 Heisenberg's inequality

10.1 Lemma Let V be a complex vector space with hermitian inner product $\langle\,.\,,\,.\,\rangle$ and with linear operators $A, B\colon V \to V$ such that $\langle Af, g\rangle = -\langle f, Ag\rangle$ and $\langle Bf, g\rangle = -\langle f, Bg\rangle$ for all $f, g \in V$. Then

$$\left|\langle \tfrac{1}{2i}\,(AB - BA)f, f\rangle\right| \le \|Af\|\,\|Bf\| \qquad (f \in V). \tag{10.1}$$

For given $f \in V$ equality holds in (10.1) iff $Af = i\lambda Bf$ for some $\lambda \in \mathbb{R}$ or $Bf = 0$.

Proof Observe that

$$\langle \tfrac{1}{2i}\,(AB - BA)f, f\rangle = \tfrac{-1}{2i}\big(\langle Bf, Af\rangle - \langle Af, Bf\rangle\big) = \mathrm{Im}\,\big(\langle Af, Bf\rangle\big)$$

and

$$\left|\mathrm{Im}\,\big(\langle Af, Bf\rangle\big)\right| \le \left|\langle Af, Bf\rangle\right| \le \|Af\|\,\|Bf\|. \qquad \square$$

10.2 We now discuss a special case of the above lemma. Take $V := \mathcal{S}$, $Af := f'$, $(Bf)(x) := ixf(x)$. Then

$$\tfrac{1}{2i}\,\big((AB - BA)f\big)(x) = \tfrac{1}{2}f(x).$$

Hence

$$\tfrac{1}{2} \le \frac{1}{\|f\|_2}\left(\int_{-\infty}^{\infty} x^2\,|f(x)|^2\,dx\right)^{\frac{1}{2}} \frac{1}{\|\widehat{f}\,\|_2}\left(\int_{-\infty}^{\infty} y^2\,|\widehat{f}(y)|^2\,dy\right)^{\frac{1}{2}} \qquad (f \in \mathcal{S}). \tag{10.2}$$

For given $f \in \mathcal{S}$ equality holds in (10.2) iff $f'(x) = -\lambda\,x\,f(x)$ for some $\lambda \in \mathbb{R}$, i.e., iff $f(x) = \text{const.}\,e^{-\frac{1}{2}\lambda x^2}$ for some $\lambda > 0$. (For $\lambda \le 0$ the function f is not in $\mathcal{S}$.)

As a corollary of (10.2) we find that for $f \in \mathcal{S}$, $x_0, y_0 \in \mathbb{R}$ the following inequality holds:

$$\tfrac{1}{2} \le \frac{1}{\|f\|_2}\left(\int_{-\infty}^{\infty} (x - x_0)^2\,|f(x)|^2\,dx\right)^{\frac{1}{2}} \frac{1}{\|\widehat{f}\,\|_2}\left(\int_{-\infty}^{\infty} (y - y_0)^2\,|\widehat{f}(y)|^2\,dy\right)^{\frac{1}{2}}. \tag{10.3}$$

For fixed $f \in \mathcal{S}$ the right hand side of (10.3) attains its absolute minimum for

$$x_0 = \mu(f) := \frac{\int_{-\infty}^{\infty} x\,|f(x)|^2\,dx}{\int_{-\infty}^{\infty} |f(x)|^2\,dx}, \qquad y_0 = \mu(\widehat{f}\,) := \frac{\int_{-\infty}^{\infty} y\,|\widehat{f}(y)|^2\,dy}{\int_{-\infty}^{\infty} |\widehat{f}(y)|^2\,dy}.$$

Here $\mu(f)$ is the *mean value* of the stochastic variable x with respect to the probability distribution $\big(\int_{-\infty}^{\infty} |f(x)|^2\,dx\big)^{-1}\,|f(x)|^2\,dx$ on $\mathbb{R}$ and $\mu(\widehat{f}\,)$ is the mean value of the stochastic variable y with respect to the probability distribution $\big(\int_{-\infty}^{\infty} |\widehat{f}(y)|^2\,dy\big)^{-1}\,|\widehat{f}(y)|^2\,dy$ on $\mathbb{R}$. Denote the corresponding *variances* by

$$\sigma(f) := \frac{1}{\|f\|_2}\left(\int_{-\infty}^{\infty} \big(x - \mu(f)\big)^2\,|f(x)|^2\,dx\right)^{\frac{1}{2}},$$

$$\sigma(\widehat{f}\,) := \frac{1}{\|\widehat{f}\,\|_2}\left(\int_{-\infty}^{\infty} \big(y - \mu(\widehat{f}\,)\big)^2\,|\widehat{f}(y)|^2\,dy\right)^{\frac{1}{2}}.$$

Then we finally obtain *Heisenberg's inequality*

$$\sigma(f)\,\sigma(\widehat{f}\,) \geq \tfrac{1}{2}. \tag{10.4}$$

Thus the two stochastic variables x and y cannot be concentrated both on arbitrarily small intervals.

Heisenberg's inequality is better known as *Heisenberg's uncertainty relation* when interpreted in quantum mechanics or in signal analysis. In quantum mechanics the variables x and y are the position and impulse operator, respectively, for a one-dimensional particle. Then position and impulse cannot be both measured with arbitrary accuracy. In signal analysis x is the time and y is the frequency of a signal. Here time and frequency cannot both be measured with arbitrary accuracy.

Ex. 10.3 Let $f \in \mathcal{S}$, $\|f\|_2 = 1$, $\alpha, \beta, a, b > 0$, $(\alpha + a^2)(\beta + b^2) < \frac{1}{4}$. Show that it is impossible that the following two inequalities hold together:

$$\int_{|x|>a} x^2\,|f(x)|^2\,dx < \alpha \quad \text{and} \quad \int_{|y|>b} y^2\,|\widehat{f}(y)|^2 < 2\pi\beta.$$

10.2 Fourier Transform on finite cyclic groups

The impact of the seemingly pure mathematical topic of this section on numerical and applied mathematics cannot be overestimated. In the present section all relevant groups will be finite. We can endow such group with the discrete topology (all subsets are open) and obtain what is called a *topological group*.

For completeness, we mention here that a topological group is a group endowed with a topology such that the group operations *multiplication* and *inversion* are *continuous*. In particular the notion of a continuous function on a topological group is available. Important examples of topological groups are $\mathbb{R}^n$ and $\mathbb{C}^n$ with the Euclidean topology. We will not pursue the implications of this notion at all, but we will use the terminology in the following sense. The definitions will always be for a topological group G, in the results the group will always be finite. As a consequence, we will speak of continuous mappings of our finite groups, although this is strictly speaking superfluous: all mappings of a discrete topological space are continuous. However, using the terminology as we do, it will fit perfectly in the general framework of topological and Lie groups.

10.4 Let $N \in \mathbb{N}$, set $\omega_N = e^{2\pi i/N}$. We introduce

$$G_N = \langle \omega_N \rangle = \{\omega_N^k, k = 1, \ldots, N\}.$$

Thus G_N is the multiplicative cyclic subgroup of order N of the circle group $T = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$. The relative topology on G_N, inherated from the Euclidean topology on $\mathbb{C}$ is discrete.

Next $C(G)$ denotes the vector space of the continuous complex valued functions on a topological group G. Of course $C(G_N)$ may be identified with $\mathbb{C}^N$.

A *character* on a group G is a continuous homomorphism from G to T. The set of characters on G form a group, denoted by $\hat{G}$ with multiplication

$$e_1 e_2(g) = e_1(g) e_2(g), \quad e_i \in \hat{G},\ g \in G.$$

Ex. 10.5 Show that $\hat{G}$ is indeed a group. What is its unit element? Can you describe the inversion?

10.6 Lemma

$$\hat{G}_N \cong G_N.$$

Proof Define $\alpha : G_N \to \hat{G}_N$ by

$$(\alpha(\omega_N^j))(\omega_N^k) = \omega_N^{kj}.$$

One can check easily that α is an injective homomorphism. It is also surjective: If $e \in \hat{G}_N$, then $e(\omega_N^k) = (e(\omega_N))^k \in T$ for all integers k. Since $(e(\omega_N))^N = e(\text{id}) = 1$, we conclude that $e(\omega_N) = \omega_N^j$ for some $j \in 1, .., N$. It then follows that $e = \alpha(\omega_N^j)$. □

10.7 As $C(G_N) \cong \mathbb{C}^N$, it can be given the inner product of $\mathbb{C}^N$. We want the function 1 to have norm 1, therefore we normalize:

$$\langle F, G\rangle = \frac{1}{N} \sum_{g \in G_N} F(g)\overline{G(g)}.$$

We set $e_j = \alpha(\omega_N^j) \in \hat{G}_N$ and sometimes we will identify e_j and ω_N^j. We note that $\bar{\omega}_N = \omega_N^{-1}$.

10.8 Proposition $\hat{G}_N$ is an orthonormal basis of $C(G_N)$.

Proof Since $\hat{G}_N$ has N elements, which equals the dimension of $C(G_N)$, it suffices to show that $\hat{G}_N$ is an orthonormal system:

$$\begin{aligned}
\langle e_l, e_j\rangle &= \frac{1}{N}\sum_{k=1}^{N} \omega_N^{lk}\omega_N^{-jk} = \frac{1}{N}\sum_{k=1}^{N} \omega_N^{(l-j)k} \\
&= \begin{cases} 1 & \text{if } l = j; \\ \omega_N^{l-j}\frac{1-\omega_N^{(l-j)N}}{1-\omega_N^{l-j}} = 0 & \text{if } l \neq j. \end{cases}
\end{aligned}$$

□

The proposition implies the following representation formula for $f \in C(G_N)$:

$$f(\omega_N^l) = \sum_{e \in \hat{G}} \langle f, e\rangle e(\omega_N^l) = \sum_{e \in \hat{G}} \hat{f}(e)e(\omega_N^l) = \sum_{j=1}^{N} \hat{f}(\omega_N^j)\omega_N^{lj}. \tag{10.5}$$

Here $\hat{f}$ is the *finite Fourier transform* of f. It is as expected, the function on $\hat{G}_N$ defined by

$$\hat{f}(e_j) = (\mathcal{F}_N f)(e_j) = \frac{1}{N}\sum_{k=1}^{N} f(\omega_N^k)\omega_N^{-jk}. \tag{10.6}$$

10.9 For numerical purposes we are interested in the number of computations that are necessary to compute $\hat{f}(e_j)$. Formula ((10.6)) gives $2N$ computations to compute $\hat{f}(e_j)$, assuming knowledge of f and G_N. Thus we need atmost cN^2 operations to compute $\mathcal{F}_N f$. In fact we can do much better.

10.3 Fast Fourier Transform

Theorem Suppose $N = nm$, $n, m \in \mathbb{N}$ and let $f \in C(G_N)$. Let $F_j \in C(G_m)$ be defined by

$$F_j(\omega_m^k) = f(\omega_N^{nk+j}), \quad j = 1, \dots, n, \quad k = 1, \dots, m,$$

with $\omega_m = \omega_N^n$. Then we have

$$\hat{f}(\omega_N^l) = \frac{1}{n} \sum_{j=1}^{n} \mathcal{F}_m(\omega_N^{-lj} F_j)(\omega_m^{l \bmod m}).$$

Proof In the next computation we set $p = nk + j$, hence $\omega_N^p = \omega_m^k \omega_N^j$. If p runs from 1 to N, then j runs from 1 to n and k from 1 to m.

$$\begin{aligned} \hat{f}(\omega_N^l) &= \frac{1}{N} \sum_{p=1}^{N} f(\omega_N^p)\omega_N^{-pl} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{m} \sum_{k=1}^{m} F_j(\omega_m^k)(\omega_m^{-k}\omega_N^{-j})^l \\ &= \frac{1}{n} \sum_{j=1}^{n} \frac{1}{m} \left(\sum_{k=1}^{m} \omega_N^{-jl} F_j(\omega_m^k)\omega_m^{-kl} \right) = \\ &= \frac{1}{n} \sum_{j=1}^{n} (\mathcal{F}_m\, \omega_N^{-jl} F_j)(\omega_m^{l \bmod m}). \end{aligned}$$

□

10.10 Thus, to compute $\hat{f}(\omega_N^l)$ we need to compute $\mathcal{F}_m F_j$. Suppose it takes $A(m)$ computations to compute the Fourier transform in G_m (given $F \in C(G_m)$). To compute $\omega_N^{-jl} F_j$ takes m computations, to compute $\mathcal{F}_m(\omega_N^{-jl} F_j)$ takes $A(m) + m$ computations. We need all n of these, yielding $n(A(m) + m)$ computations, the result of which we store. Finally, for each $\hat{f}(\omega_N^l)$ we need n additions of our stored results. This gives the total of computations

$$A(N) = Nn + n(A(m) + m) = N(n + 1) + A(m)n.$$

10.11 Theorem If $N = n_1 \cdots n_p$, $n_p = 1$, then $A(N) \leq N(1 + \sum_i (n_i + 1))$.

Proof Using the previous calculation we compute

$$\begin{aligned} A(n_1 \cdots n_p) &\leq N(n_1 + 1) + A(n_2 \cdots n_p)n_1 \leq N(n_1 + 1) + N(n_2 + 1) + A(n_3 \cdots n_p)n_2 n_1 \\ &\leq \ldots \leq N \sum (n_i + 1) + N. \end{aligned}$$

□

10.12 Corollary If $N = q^p$ we need at most $O(N \log N)$ computations to compute Fourier transform in G_N. (Here q is fixed and $p \to \infty$.)

Proof

$$A(n) \leq N(2 \sum_{j=1}^{p} (q + 1) + N) = N(2p(q + 1) + 1).$$

Note that $p = \log N / \log q$, so that

$$A(N) \leq 2N \frac{\log N}{\log q}(q + 1) + N = O(N \log N).$$

□

10.13 Next we indicate the relation of this finite Fourier transform with ordinary Fourier transform on the line. Suppose f has compact support on $[-N, N]$ and the values of f at the points j/K are known, $j \in J = \{-KN, -KN+1, \ldots, KN-1\}$. Then we can approximate $\hat{f}(x)$ as follows:

$$\hat{f}(x) = \int_{-N}^{N} f(t)e^{ixt}dt = \frac{1}{K}\int_{-NK}^{NK} f(\frac{s}{K})e^{-i\frac{x}{K}s}ds \approx \frac{1}{K}\sum_{j\in J} f(\frac{j}{K})e^{\frac{-ixj}{K}}.$$

We substitute $x = \frac{\pi p}{N}$ and find

$$\hat{f}(\frac{\pi p}{N}) \approx \frac{1}{K}\sum_{j\in J} f(\frac{j}{K})e^{-i\frac{2\pi pj}{2NK}} = \frac{2N}{2KN}\sum_{j} \in J f(\frac{j}{K})\omega_M^{-pj} = 2N(\mathcal{F}_M g)(\omega_M^p).$$

Here we have written $M = 2KN$ and defined $g \in C(G_M)$ by $g(\omega_M^j) = f(j/K)$, $(j \in J)$.

Thus we have found a way to compute the (approximate) Fourier transform of a function of which the value is known only in finitely many points. Using the "fast way" of computing it via Fourier transform on finite groups as we described here is called *Fast Fourier Transform*. The paper of Cooley and Tuckey on this topic is in fact in the top ten of most cited mathematical papers.

Ex. 10.14 Convolution on G_N For $f, g \in C(G_N)$ the convolution $f * g$ is defined as follows:

$$f * g(\omega_N^j) = \frac{1}{N}\sum_{k=1}^{N} f(\omega_N^k)g(\omega_N^{j-k}).$$

Show that $\mathcal{F}_N(f * g)(\omega_N^l) = \mathcal{F}_N f(\omega_N^l)\mathcal{F}_N g(\omega_N^l)$.
Also show that $f * g = g * f$ and $f * (g * h) = (f * g) * h$.

Ex. 10.15 Fast multiplication Let F, G be polynomials of degree, p, q respectively. View F and G as elements of $C(G_N)$ with $N = p+q+1$, by declaring $F(\omega_N^k)$ equal to the coefficient of z^k of F; the same goes for G.

Give a rough estimate for the number of computations needed to compute $F \cdot G$ directly, and next, by observing that $F \cdot G$ as element of $C(G_N)$ is the convolution of F and G and using fast Fourier transform.
Apply this and describe a fast way for computing the product of two very large (10000-digit) numbers.